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Global well-posedness and analyticity for the 3D fractional magneto-hydrodynamics equations in the Besov-Morrey spaces characterized by Semi-group

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ABSTRACT: Our paper establishes the existence and uniqueness of solutions, the analyticity, and the decay estimates of the solutions of the 3D fractional magneto-hydrodynamics equations (FMHD) in the Besov-Morrey spaces characterized by the semigroup $\mathcal{L}_\alpha := e^{-t(-\Delta)^\alpha}$, noted by $\mathrm{N}^s_{p,\lambda}$. Assuming that the initial data (a_0, m_0) are small and belong to $\mathrm{N}^s_{p,\lambda}$, we prove the global well-posedness of the (FMHD) equation.

Key Words: Fractional magneto-hydrodynamic (FMHD) equations, Global well-posedness, Analytic solution, Besov-Morrey spaces characterized by semigroup.

Contents

1. Introduction

In this paper, we are concerned with the the generalized fractional magneto-hydrodynamic equations (FMHD) in $\mathbb{R}^3 \times \mathbb{R}^+$:

$$\begin{cases} a_t + (a\cdot\nabla)a + \mu(-\Delta)^\alpha a + \nabla P = (m\cdot\nabla)m & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ m_t + (a\cdot\nabla)m + \nu(-\Delta)^\alpha m = (m\cdot\nabla)a & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \nabla\cdot a = 0, \quad \nabla\cdot m = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ a_{t=0} = a_0,\, m|_{t=0} = m_0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $a = (a_1, a_2, a_2)$ represents the velocity field of the flow, $m = (m_1, m_2, m_3)$ denotes the magnetic field, P denotes the pressure function, $\mu > 0$ denotes the viscosity coefficient, and ν represents the diffusivity coefficient, a_0 and m_0 are respectively the initial velocity and the initial magnetic field with free divergence (i.e $\nabla\cdot a_0 = 0$ and $\nabla\cdot m_0 = 0$). The operator $(-\Delta)^\alpha$ is the Fourier multiplier of symbol $|\xi|^{2\alpha}$ given by

$$\mathcal{F}\big((-\Delta)^\alpha v\big) = |\xi|^{2\alpha}\mathcal{F}(v),$$

where $\mathcal{F}$ is the Fourier transform. To simplify and without loss of generality, we consider only the case where $\mu = \nu = 1$.

Fractional magnetohydrodynamic (FMHD) equations are a mathematical model used to describe the behavior of conductive fluids subject to magnetic fields, such as plasmas or liquid metals. They combine fluid dynamics and electromagnetism, and incorporate fractional calculus to account for non-integer order derivatives.

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FMHD equations have diverse applications, from modeling magnetic confinement fusion devices and magnetohydrodynamic turbulence, to studying geophysical and astrophysical phenomena such as the Earth's magnetic field and plasma behavior in stars.

At $\alpha = 1$, the equations can explain mathematically why the Earth has a non-zero large-scale magnetic field that changes polarity every few hundred centuries. For further details, please consult [7] and its associated references. It should be noted that the first equation in the system (1.1) describes momentum conservation, the second equation depicts magnetic induction, and the third equation reflects the conservation of mass.

The system (1.1) governing fluid motion can be simplified to the fractional Navier-Stokes equations if the function $m(y;t) = m_0(y) = 0$. These equations are widely studied in various functional spaces and can be viewed as the equivalent of Newton's second law for fluid motion.

The system (1.1) exhibits a scaling property, indicating that if (a, m, P) represents a solution to the system, then the triplet $(a_\kappa, m_\kappa, P_\kappa)$ can also be a solution, where

$$a_\kappa(y,t) := \kappa^{1-2\alpha} a\left(\kappa^{-1}y, \kappa^{-2\alpha}t\right), \quad m_\kappa(y,t) := \kappa^{1-2\alpha} m\left(\kappa^{-1}y, \kappa^{-2\alpha}t\right),$$
$$P_\kappa := \theta^{2-4\alpha} P\left(\kappa^{-1}x, \kappa^{-2\alpha}t\right) \quad \text{for} \quad \kappa \in \mathbb{R},$$

is also a solution of the system (1.1) with the initial data

$$a_{0,\kappa} := \kappa^{1-2\alpha} a_0\left(\kappa^{-1}x\right), \quad m_{0,\kappa} := \kappa^{1-2\alpha} m_0\left(\kappa^{-1}x\right).$$

There exists an extensive body of literature on the global well-posedness of (1.1) in various critical spaces under certain scalings. For instance, Wang [17] established the global well-posedness in Sobolev space $\mathcal{H}^s$ with $s \geq 3$, Lui and Zhao [13] get the global existence with small initial data (u_0, b_0) belongs to the critical Fourier–Herz spaces $\mathbb{B}_p^{1-2\alpha}$ where $1 \leq p \leq 2$.many authors [3,4,5,6,8,9,10] have established the global well-posedness results with small initial data belonging to the critical Fourier-Besov-Morrey spaces $\mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}(\mathbb{R}^3)$. Recently, Wang and Weihua in [18] showed the global well-posedness and analyticity in the critical variable Fourier–Besov spaces.

In order to solve the equation (1.1), we consider the following equivalent integral equation coming from Duhamel's principle

$$\begin{cases} a = \mathcal{L}_\alpha(t)a_0 - \displaystyle\int_0^t \mathcal{L}_\alpha(t-\varrho)\mathbb{P}\nabla\cdot(a\otimes a - m\otimes m)(\cdot,\varrho)d\varrho := \mathcal{A}_1^\alpha(a,m), \\ m = \mathcal{L}_\alpha(t)m_0 - \displaystyle\int_0^t \mathcal{L}_\alpha(t-\varrho)\mathbb{P}\nabla\cdot(a\otimes m - m\otimes a)(\cdot,\varrho)d\varrho := \mathcal{A}_2^\alpha(a,m), \end{cases} \tag{1.2}$$

where $\mathcal{L}_\alpha := e^{-t(-\Delta)^\alpha}$ denotes the fractional heat semigroup operator, which can be regarded as the convolution operator with the kernel $k_t(x) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\alpha}})$, and $\mathbb{P} = Id - \nabla\Delta^{-1}div$ is the Leray-Hopf projector, which is a pseudo differential operator of order 0.

Throughout this work, we use $(a,m) \in \mathcal{B}$ to denote $(a,m) \in \mathcal{B}\times\mathcal{B}$ for a Banach space $\mathcal{B}$ $\Big($the product $\mathcal{B}\times\mathcal{B}$ will be endowed with the usual norm $\|(a,m)\|_{\mathcal{B}\times\mathcal{B}} := \|a\|_{\mathcal{B}} + \|m\|_{\mathcal{B}}$ $\Big)$.

By utilizing the equivalent integral equation (1.2) and applying the contraction principle, we can demonstrate the existence of global solutions for (1.1), provided that the initial data is sufficiently small and belongs to the space $\mathrm{N}_{p,\lambda}^s$. Then we show the analyticity of this solution by Gevrey estimates.

Allow us to present the function space $\mathrm{N}_{p,\lambda}^s$ of the Besov Morrey type, which is defined by the linear semigroup $\mathcal{L}_\alpha$.

Definition 1.1 *Let $\alpha > 0$, $1 \leq p \leq \infty$, $s > 0$ and $0 \leq \lambda < 3$, the function space $\mathrm{N}_{p,\lambda}^s(\mathbb{R}^3)$ is defined as follows:*

$$\mathrm{N}_{p,\lambda}^s(\mathbb{R}^3) = \{\varphi \in \mathcal{S}', \|\varphi\|_{\mathrm{N}_{p,\lambda}^s} < \infty\},$$
$$\|\varphi\|_{\mathrm{N}_{p,\lambda}^s} = \sup_{t>0} t^s \|\mathcal{L}_\alpha(t)\varphi\|_{\mathcal{M}_p^\lambda}.$$

The structure of this paper is as follows. Section 2 revisits the Morrey space definition and outlines some of its properties that will be applied in the sequel. Our findings on global solutions are presented in Section 3, and Section 4 contains the proof of the solutions analyticity and the decay in time estimate.

2. Preliminaries

In this section, we recall the definition of the Morrey space and some of their properties which will be used throughout the paper, and then we give some lemmas concerning the Gevrey estimate.

We start by recalling the functional spaces $\mathcal{M}_q^\lambda$.

Definition 2.1 *Let $1 \le q \le \infty$ and $0 \le \lambda < d$. The homogeneous Morrey space $\mathcal{M}_q^\lambda$ are defined by*

$$\mathcal{M}_q^\lambda(\mathbb{R}^d) := \{\varphi \in L^1_{loc}(\mathbb{R}^d), \|\varphi\|_{\mathcal{M}_q^\lambda} < \infty\},$$

with

$$\|\varphi\|_{\mathcal{M}_q^\lambda} := \sup_{x_0 \in \mathbb{R}^d} \sup_{R>0} R^{-\frac{\lambda}{q}} \|\varphi\|_{L^q(B(x_0,R))}, \tag{2.1}$$

where $B(y,R)$ is the open ball in $\mathbb{R}^d$ centered at y and with radius $R > 0$.

The space $\mathcal{M}_q^\lambda$ endowed with the norm $\|f\|_{\mathcal{M}_q^\lambda}$ is a Banach space and has the following scaling property

$$\|f(\beta x)\|_{\mathcal{M}_q^\lambda} = \beta^{-\frac{n-\lambda}{q}} \|f(x)\|_{\mathcal{M}_q^\lambda} \qquad for \qquad \mu > 0.$$

In the case of $p = 1$, the norm $|\cdot|_{L^1}$ in equation (2.1) corresponds to the total variation of the measure f on the ball $B(y,R)$, and the space $\mathcal{M}_q^\lambda$ is regarded as a subset of Radon measures. When $\lambda = 0$, $\mathcal{M}_q^\lambda$ is equal to L^q.

Lemma 2.1 *[11] (**Hölder's inequality**)*
Let $0 \le \lambda_1,\ \lambda_2,\ \lambda_3 < n$ and $1 \le r_1, r_2, r_3 < \infty$, such that $\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}$ and $\frac{\lambda_3}{r_3} = \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2}$, then we have

$$\|fg\|_{\mathcal{M}_{r_3}^{\lambda_3}} \le \|f\|_{\mathcal{M}_{r_1}^{\lambda_1}} \|g\|_{\mathcal{M}_{r_2}^{\lambda_2}}. \tag{2.2}$$

Lemma 2.2 *[11] Let $1 \le r_1 \le r_2 \le \infty$ and $\frac{d-\lambda_1}{r_1} \le \frac{d-\lambda_2}{r_2}$, then*

$$\mathcal{M}_{r_2}^{\lambda_2}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{r_1}^{\lambda_1}(\mathbb{R}^d).$$

By combining [15, Lemma 3] with [14, Lemma 2.3], we can derive the subsequent lemma

Lemma 2.3 *Let $1 \le q_1 \le q_2 \le \infty$, $0 \le \lambda < d$ and $\beta = (\beta_1, \beta_2) \in (\mathbb{N} \cup \{0\})^2$.*
If $f \in \mathcal{S}'(\mathbb{R}^d)$, then there exists a constant C depending only on d such that

$$\|e^{-t(-\Delta)^\alpha} f\ \|_{\mathcal{M}_{q_2}^\lambda} \le C t^{-\frac{1}{2\alpha}(\frac{d-\lambda}{q_1} - \frac{d-\lambda}{q_2})} \|f\|_{\mathcal{M}_{q_1}^\lambda}.$$

$$\|\partial^\beta e^{-t(-\Delta)^\alpha} f\|_{\mathcal{M}_{q_2}^\lambda} \le C t^{-\frac{|\beta|}{2\alpha} - \frac{1}{2\alpha}(\frac{d-\lambda}{q_1} - \frac{d-\lambda}{q_2})} \|f\|_{\mathcal{M}_{q_1}^\lambda}.$$

Finally, we will use the following three lemmas to obtain the Gevrey estimates.

Lemma 2.4 *[16] If the operator $\mathcal{O} = e^{-[\sqrt{\varepsilon_1 - \varepsilon_2} + \sqrt{\varepsilon_2} - \sqrt{\varepsilon_1}]|\Lambda|^\alpha}$ for $0 \le \varepsilon_2 \le \varepsilon_1$, then $\mathcal{O}$ is either the identity operator or an $L^1(\mathbb{R}^d)$ kernel whose $L^1(\mathbb{R}^d)$ norm is bounded independent of $\varepsilon_2, \varepsilon_1$.*

Lemma 2.5 *[1] The operator $\mathcal{O} = e^{\frac{1}{2}b(-\Delta)^\alpha + \sqrt{b}|\Lambda|^\alpha}$ is a Fourier multiplier which maps boundedly $\mathcal{M}_q^\lambda(\mathbb{R}^d) \to \mathcal{M}_q^\lambda(\mathbb{R}^d)$, $1 < q < \infty$, and its operator norm is uniformly bounded with respect to $b \ge 0$.*

At the end of this section, We will introduce a bounded estimate that involves the bilinear operator $L_t(\Phi_1,\Phi_2)$ in the following form:

$$\begin{aligned} L_t(\Phi_1,\Phi_2) :&= e^{\sqrt{t}|\Lambda|^\alpha}\left(e^{-\sqrt{t}|\Lambda|^\alpha}\Phi_1 e^{-\sqrt{t}|\Lambda|^\alpha}\Phi_2\right) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix\zeta}e^{\sqrt{t}(|\zeta|_1-|\zeta-\nu|_1-|\nu|_1)}\widehat{\Phi}_1(\zeta-\nu)\widehat{\Phi}_2(\eta)d\nu d\zeta. \end{aligned}$$

Lemma 2.6 *[14] Let $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$ and $1<q<\infty$. Then we have*

$$\|L_t(\Phi_1,\Phi_2)\|_{\mathcal{M}^\lambda_q(\mathbb{R}^d)} \le C\|\Phi_1\|_{\mathcal{M}^\lambda_{q_1}(\mathbb{R}^d)}\|\Phi_2\|_{\mathcal{M}^\lambda_{q_2}(\mathbb{R}^d)}.$$

Where C is a positive constant independent of Φ_1 and Φ_2.

3. Well-Posedness

In this section, we show our main result concerning the global well-posedness of fractional incompressible magneto-hydrodynamic (FMHD) equations given by Theorem 3.1.

Theorem 3.1 *Let $\alpha,\lambda,\ p$ satisfy*

$$0\le\lambda<3 \quad and \quad \frac{1}{2}\le\alpha<min(1,\frac{1}{2}+\frac{3-\lambda}{p}).$$

Then, there exists a positive constant $\eta=\eta(\alpha,\lambda,p)$, such that for the initial velocity $(a_0,m_0)\in \mathrm{N}^s_{p,\lambda}(\mathbb{R}^3)$ satisfying div a_0 = div m_0 = 0 and

$$\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}\le\eta.$$

The Eq (1.1) have a unique global solution (a) satisfying

$$\sup_{t>0} t^{-\frac{1}{2\alpha}(1-2\alpha+\frac{3-\lambda}{p})}\|(a(t),m(t))\|_{\mathcal{M}^\lambda_q}\le 2\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}.$$

Before starting to show this theorem we will show the bilinear estimate of (1.2) given by Lemma 3.1. We denote

$$Z:=\{\varphi\in\mathcal{S}', \|\varphi\|_Z<\infty\}$$

with

$$\|\varphi\|_Z = \quad sup \quad t^{-\frac{s}{2\alpha}}\|\varphi\|_{\mathcal{M}^\lambda_q},$$

where $s=1-2\alpha+\frac{3-\lambda}{p}$. Then, we obtain the bilinear estimate for $\mathbb{B}(a,m)$ in Z.

Lemma 3.1 *Let $\alpha,\lambda,\ p$ satisfy $0\le\lambda<3$ and $\frac{1}{2}\le\alpha<min(1,\frac{1}{2}+\frac{3-\lambda}{p})$.*

Then, there exists a positive constant C_1 such that for all $a,m\in Z$

$$\|\mathbb{B}(a,m)\|_{\mathrm{N}^s_{p,\lambda}}\le C_1\|a\|_Z\|m\|_Z, \tag{3.1}$$

where

$$\mathbb{B}(a,m):=\int_0^t \mathcal{L}_\alpha(t-\rho)\mathbb{P}\nabla\cdot(a\otimes m)(\rho,x)d\rho.$$

Proof: Since $\mathbb{P}$ is bounded in $\mathcal{M}_q^\lambda(\mathbb{R}^3)$ [15], it follows from Lemma 2.3 and Hölder's inequality that

$$\begin{aligned}\Big\|\int_0^t \mathcal{L}_\alpha(t-\rho)\mathbb{P}\nabla\cdot(a\otimes m)(\rho,x)d\rho\Big\|_{\mathcal{M}_q^\lambda} &\le \int_0^t \Big\|\mathcal{L}_\alpha(t-\rho)\mathbb{P}\nabla\cdot(a\otimes m)(\rho,x)\Big\|_{\mathcal{M}_q^\lambda}d\rho\\ &\le C\int_0^t (t-\rho)^{-\frac{1}{2\alpha}-\frac{1}{2\alpha}(\frac{6-2\lambda}{p}-\frac{3-\lambda}{p})}\Big\|(a\otimes m)(\rho,x)\Big\|_{\mathcal{M}_{\frac{q}{2}}^\lambda}d\rho\\ &\le C\int_0^t (t-\rho)^{-\frac{1}{2\alpha}(1+\frac{3-\lambda}{p})}\Big\|a\Big\|_{\mathcal{M}_q^\lambda}\Big\|m\Big\|_{\mathcal{M}_q^\lambda}d\rho\\ &\le C\int_0^t (t-\rho)^{-\frac{1}{2\alpha}(1+\frac{3-\lambda}{p})}t^{\frac{s}{\alpha}}\|a\|_Z\|m\|_Z \quad d\rho.\end{aligned}$$

Multiplying $t^{-\frac{s}{2\alpha}}$ on the both sides of the above two inequalities, we get

$$t^{-\frac{s}{2\alpha}}\Big\|\int_0^t \mathcal{L}_\alpha(t-\rho)\mathbb{P}\nabla\cdot(a\otimes m)(\rho,x)d\rho\Big\|_{\mathcal{M}_q^\lambda}\le Ct^{\frac{s}{2\alpha}}\|a\|_Z\|m\|_Z\int_0^t(t-\rho)^{-\frac{1}{2\alpha}(1+\frac{3-\lambda}{p})}d\rho,$$

where $s=1-2\alpha+\frac{3-\lambda}{p}$, then

$$\|\mathbb{B}(a,m)\|_{\mathrm{N}^s_{p,\lambda}}\le C_1\|a\|_Z\|m\|_Z.$$

Since C_1 is independent of t, we have now completed the proof of Lemma 3.1. □

Proof of Theorem 3.1 It is simple to verify that the indices α,λ and p provided in the Theorem 3.1 satisfy the assumptions of Lemma 3.1 and Lemma 2.3. Suppose $(a_0,m_0)\in \mathrm{N}^s_{p,\lambda}(\mathbb{R}^3)$ with divergence free. By the definitions of $\|\cdot\|_Z$, we see that

$$\begin{aligned}\|\mathcal{L}_\alpha(t)a_0\|_{\mathcal{M}_q^\lambda} &= \|e^{-t(-\Delta)^\alpha}a_0\|_{\mathcal{M}_q^\lambda}\\ &\le C_2\|a_0\|_{\mathcal{M}_q^\lambda}.\end{aligned}$$

Multiplying $t^{-\frac{s}{2\alpha}}$ on the both sides of the above two inequalities, we get

$$\|\mathcal{L}_\alpha(t)a_0\|_Z\le C_2\|a_0\|_Z. \tag{3.2}$$

Then, Let us introduce the map Φ and the complete metric space $(\mathbf{Y},d)$, defined as follows:

$$\mathbf{Y}:=\left\{(a,m)\in \mathrm{N}^s_{p,\lambda}(\mathbb{R}^3)^3,\|(a,m)\|_Z\le 2\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}\right\},$$

$$d(\vartheta_1,\vartheta_2):=\|\vartheta_1-\vartheta_2\|_{\mathrm{Z}},$$

$$\mathit{\Theta}(a,m):=(\mathcal{A}_1^\alpha(a,m),\mathcal{A}_2^\alpha(a,m)).$$

Notation: Let X_1 , X_2 be Banach spaces, we denote $\|\cdot\|_{X_1\cap X_2}:=\|\cdot\|_{X_1}+\|\cdot\|_{X_2}$.
Applying the inequality (3.1) and (3.2), then for all $(a,m)\in\mathbf{Y}$, we have

$$\|\mathcal{A}_1^\alpha(a,m)\|_Z \quad \le C_2\|a_0\|_{\mathrm{N}^s_{p,\lambda}}+C_1(\|a\|_Z^2+\|m\|_Z^2).$$

Similary, we obtain

$$\|\mathcal{A}_2^\alpha(a,m)\|_Z \quad \le C_2\|m_0\|_{\mathrm{N}^s_{p,\lambda}}+2C_1(\|a\|_Z+\|m\|_Z).$$

Then we obtient

$$\begin{aligned}\|\mathit{\Theta}(a.m)\|_Z &\le C_2\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}+C_1\|(a,m)\|_Z^2\\ &\le C_2\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}+4C_1\|(a_0,m_0)\|^2_{\mathrm{N}^s_{p,\lambda}}.\end{aligned}$$

Then, ther exist a positif constant $\epsilon_1=\max(C_1,C_2)$ such that

$$\|\mathit{\Theta}(a.m)\|_Z\le\epsilon\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}\Big[1+4\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}\Big]. \tag{3.3}$$

On the other hand, for any $(a_1,m_1),(a_2,m_2)\in \mathrm{Y}$, ther exist a positif constant ϵ_2 such that

$$\begin{aligned}\|\mathcal{A}_1^{\alpha}(a_1,m_1)-\mathcal{A}_1^{\alpha}(a_2,m_2)\|_Z &\le \|\mathbb{B}(a_1,a_1)-\mathbb{B}(a_2,a_2)\|_Z+\|\mathbb{B}(m_1,m_1)-\mathbb{B}(m_2,m_2)\|_Z\\ &\le \|\mathbb{B}(a_1,a_1-a_2)-\mathbb{B}(a_1-a_2,a_2)\|_Z+\|\mathbb{B}(m_1-m_2,m_2)-\mathbb{B}(m_1,m_1-m_2)\|_Z\\ &\le \epsilon_2\Big\{(\|a_1\|_Z+\|a_2\|_Z)\|a_1-a_2\|_Z+(\|m_1\|_X+\|m_2\|_Z)\|m_1-m_2\|_Z\Big\}\\ &\le \epsilon_2\Big\{(\|(a_1,m_1)\|_Z+\|(a_2,m_2)\|_Z)(\|a_1-a_2\|_Z+\|m_1-m_2\|_Z)\Big\}\\ &\le 2\epsilon_2\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}\Big\{\|a_1-a_2\|_Z+\|m_1-m_2\|_Z\Big\}.\end{aligned}$$

Therefore,

$$\|\Theta(a_1,m_1)-\Theta(a_2,m_2)\|_Z\le 2\epsilon_2\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}\Big\{\|a_1-a_2\|_Z+\|m_1-m_2\|_Z\Big\}. \tag{3.4}$$

Now, let us assume that initial velocity $(a_0,m_0)\in \mathrm{N}^s_{p,\lambda}(\mathbb{R}^3)$ satisfies

$$\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}\le \min\{\frac{1}{4\epsilon_2},\frac{1}{4\epsilon_1}\},$$

we obtain from (3.3) and (3.4) that

$$\begin{aligned}&\|\Theta(a.m)\|_Z\le 2\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}},\\ &\|\Theta(a_1,m_1)-\Theta(a_2,m_2)\|_Z\le \frac{1}{2}\Big\{\|a_1-a_2\|_Z+\|m_1-m_2\|_Z\Big\}.\end{aligned}$$

Thus, applying the contraction mapping principle, we can conclude that there exists a unique solution $(a,m)\in \mathbf{Y}$ that satisfies (1.2) for all $t>0$. This completes the proof of Theorem 3.1. □

4. Analyticity and decay in time estimate

This section is dedicated to proving the Gevrey class regularity and the decay in time estimate for the FMHD equations in the Besov-Morrey spaces carterized by semi-group.

4.1. The Analyticity

The following Theorem 4.1, provides the result regarding analyticity.

Theorem 4.1 *Under the condition of Theorem 3.1. There exists a positive constant $\beta=\beta(\alpha,\lambda,p)$, such that for the initial velocity $(a_0,m_0)\in \mathrm{N}^s_{p,\lambda}(\mathbb{R}^3)$ satisfying*

$$\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}}\le\beta,$$

the Eq (1.1) have a unique analytic solution such that

$$\|\left(e^{\mu\sqrt{t}|\Lambda|^{\alpha}}a,e^{\mu\sqrt{t}|\Lambda|^{\alpha}}m\right)\|_{\mathrm{N}^s_{p,\lambda}}\le C\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}},$$

where C is a positive constant.

Inspired by [12,14], Setting $\overline{a}(x,\ t)=e^{\sqrt{t}|\Lambda|^{\alpha}}a(x,\ t)$ and $\overline{m}(x,\ t)=e^{\sqrt{t}|\Lambda|^{\alpha}}m(x,\ t)$. Then, we can see that $(\overline{a},\overline{m})$ satisfies the following integral system:

$$\begin{cases}\overline{a}=e^{\sqrt{t}|\Lambda|^{\alpha}}\mathcal{L}_{\alpha}(t)a_0-e^{\sqrt{t}|\Lambda|^{\alpha}}\times\displaystyle\int_0^t\mathcal{L}_{\alpha}(t-\rho)\mathbb{P}\nabla\cdot(e^{-\sqrt{\rho}|\Lambda|^{\alpha}}\overline{a}\otimes e^{-\sqrt{\rho}|\Lambda|^{\alpha}}\overline{a}-e^{-\sqrt{\rho}|\Lambda|^{\alpha}}\overline{m}\otimes e^{-\sqrt{\rho}|\Lambda|^{\alpha}}\overline{m})d\rho,\\ \overline{m}=e^{\sqrt{t}|\Lambda|^{\alpha}}\mathcal{L}_{\alpha}(t)m_0-e^{\sqrt{t}|\Lambda|^{\alpha}}\times\displaystyle\int_0^t\mathcal{L}_{\alpha}(t-\rho)\mathbb{P}\nabla\cdot(e^{-\sqrt{\rho}|\Lambda|^{\alpha}}\overline{a}\otimes e^{-\sqrt{\rho}|\Lambda|^{\alpha}}\overline{m}-e^{-\sqrt{\rho}|\Lambda|^{\alpha}}\overline{m}\otimes e^{-\sqrt{\rho}|\Lambda|^{\alpha}}\overline{a})d\rho.\end{cases}$$

To achieve the Gevrey class regularity of the solution, the initial step is to estimate the linear term $e^{\sqrt{t}|\Lambda|^\alpha}\mathcal{L}_\alpha(t)a_0$.

$$\begin{aligned}\Big\|e^{\sqrt{t}|\Lambda|^\alpha}\mathcal{L}_\alpha(t)a_0\Big\|_{\mathcal{M}_q^\lambda} &= \Big\|e^{\sqrt{t}|\Lambda|^\alpha}e^{-t(-\Delta)^\alpha}a_0\Big\|_{\mathcal{M}_q^\lambda}\\ &= \Big\|e^{\sqrt{t}|\Lambda|^\alpha}e^{\frac{1}{2}t(-\Delta)^\alpha}e^{-\frac{3}{2}t(-\Delta)^\alpha}a_0\Big\|_{\mathcal{M}_q^\lambda}\\ &\le \Big\|e^{-t(-\Delta)^\alpha}a_0\Big\|_{\mathcal{M}_q^\lambda}\\ &\le C\Big\|a_0\Big\|_{\mathcal{M}_q^\lambda}.\end{aligned}$$

Then

$$\|e^{\sqrt{t}|\Lambda|^\alpha}\mathcal{L}_\alpha(t)a_0\|_{\mathrm{N}^s_{p,\lambda}} \le C\|a_0\|_{\mathrm{N}^s_{p,\lambda}}.$$

And on the other hand applying, let

$$\overline{\mathbb{B}}(\overline{a},\overline{m}) := e^{\sqrt{t}|\Lambda|^\alpha}\int_0^t \mathcal{L}_\alpha(t-\rho)\mathbb{P}\nabla\cdot(e^{-\sqrt{\rho}|\Lambda|^\alpha}\overline{a}\otimes e^{-\sqrt{\rho}|\Lambda|^\alpha}\overline{m})d\tau.$$

Since $\mathbb{P}$ is bounded in $\mathcal{M}_q^\lambda(\mathbb{R}^3)$, it follows from Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.6, we get

$$\begin{aligned}\Big\|\overline{\mathbb{B}}(\overline{a},\overline{m})(t)\Big\|_{\mathcal{M}_q^\lambda} &= \Big\|e^{\sqrt{t}|\Lambda|^\alpha}\int_0^t \mathcal{L}_\alpha(t-\rho)\mathbb{P}\nabla\left(e^{-\sqrt{\rho}|\Lambda|^\alpha}a\otimes e^{-\sqrt{\rho}|\Lambda|^\alpha}m)\right)(\rho)d\rho\Big\|_{\mathcal{M}_q^\lambda}\\ &= \Big\|e^{\sqrt{t}|\Lambda|^\alpha}\int_0^t \mathcal{L}_\alpha(t-\rho)e^{-\sqrt{\rho}|\Lambda|^\alpha}e^{\sqrt{\rho}|\Lambda|^\alpha}\mathbb{P}\nabla\left(e^{-\sqrt{\rho}|\Lambda|^\alpha}a\otimes e^{-\sqrt{\rho}|\Lambda|^\alpha}m)\right)(\rho)d\rho\Big\|_{\mathcal{M}_q^\lambda}\\ &= \Big\|\int_0^t e^{\sqrt{t}|\Lambda|^\alpha}e^{-(t-\rho)(-\Lambda)^\alpha}e^{-\sqrt{\rho}|\Lambda|^\alpha}\mathbb{P}\nabla L_t(a,m)(\rho)d\rho\Big\|_{\mathcal{M}_q^\lambda}\\ &= \Big\|\int_0^t e^{-(\sqrt{t-\rho}+\sqrt{\rho}-\sqrt{t})|\Lambda|^\alpha}e^{\sqrt{t-\rho}|\Lambda|^\alpha}e^{-(t-\rho)(-\Delta)^\alpha}\mathbb{P}\nabla L_t(a,m)(\rho)d\rho\Big\|_{\mathcal{M}_q^\lambda}\\ &= \Big\|\int_0^t e^{-(\sqrt{t-\rho}+\sqrt{\rho}-\sqrt{t})|\Lambda|^\alpha}e^{\sqrt{t-\rho}|\Lambda|^\alpha+\frac{1}{2}(t-\rho)(-\Delta)^\alpha}e^{-\frac{3}{2}(t-\rho)(-\Delta)^\alpha}\mathbb{P}\nabla L_t(a,m)(\rho)d\rho\Big\|_{\mathcal{M}_q^\lambda}\\ &\le C\Big\|\int_0^t e^{-(t-\rho)(-\Delta)^\alpha}\nabla L_t(a,m)(\rho)d\rho\Big\|_{\mathcal{M}_q^\lambda}.\end{aligned}$$

As the remainder of the proof is similar to that of Theorem 3.1, the specifics may be excluded.

4.2. The Decay in time estimate

In Theorem 3.1, we proved the analyticity of the solutions, which allows us to to obtain an estimate of the time decay of the solutions.

Theorem 4.2 *Under the condition of Theorem 3.1, for any $\gamma > 0$, the global solution $(a,m) \in \mathrm{N}^s_{p,\lambda}$ and $\left(e^{\sqrt{t}|\Lambda|^\alpha}a, e^{\sqrt{t}|\Lambda|^\alpha}m\right) \in \mathrm{N}^s_{p,\lambda}$ satysfing the decay in time estimate*

$$\|\Big((-\Delta)^\gamma a(t),(-\Delta)^\gamma m(t)\Big)\|_{\mathrm{N}^s_{p,\lambda}} \le Ct^{\frac{\gamma}{2\alpha}}\|(a_0,m_0)\|_{\mathrm{N}^s_{p,\lambda}},$$

where C is a constant depend α and γ.

To show Theorem 4.2, we need the following lemma.

Lemma 4.1 *[2] The Fourier multipliers associated with the symbols $A(\xi) = |\xi|^\beta e^{-\sqrt{t}|\xi|^\alpha}$ are obtained by convolving with their respective kernels $\mathbb{K}$. These kernels are functions in L^1 with $\|\widehat{\mathbb{K}}\|_{L^1} \le Ct^{-\frac{\beta}{2\alpha}}$.*

Proof of Theoreme 4.2 Using the Lemma 4.1, Lemma 2.5 and Theorem 4.1, we obtain

$$\begin{aligned}\Big\|(-\Delta)^{\frac{\gamma}{2}}e^{-t(-\Delta)^\alpha}a(t)\Big\|_{\mathcal{M}_q^\lambda} &= \Big\|(-\Delta)^{\frac{\gamma}{2}}e^{-\sqrt{t}|\Lambda|^\alpha}e^{\sqrt{t}|\Lambda|^\alpha}e^{-t(-\Delta)^\alpha}a(t)\Big\|_{\mathcal{M}_q^\lambda}\\ &\le Ct^{-\frac{\gamma}{2\alpha}}\Big\|e^{\sqrt{t}|\Lambda|^\alpha-t(-\Delta)^\alpha}a(t)\Big\|_{\mathcal{M}_q^\lambda}\\ &\le Ct^{-\frac{\gamma}{2\alpha}}\Big\|e^{\frac{1}{2}t(-\Delta)^\alpha+\sqrt{t}|\Lambda|^\alpha e^{-\frac{3}{2}t(-\Delta)^\alpha}}a(t)\Big\|_{\mathcal{M}_q^\lambda}\\ &\le Ct^{-\frac{\gamma}{2\alpha}}\Big\|e^{-t(-\Delta)^\alpha}a(t)\Big\|_{\mathcal{M}_q^\lambda}.\end{aligned}$$

Multiplying $t^{-\frac{s}{2\alpha}}$ on the both sides of the above two inequalities, we get

$$\left\|(-\Delta)^{\frac{\gamma}{2}}a(t)\right\|_{\mathrm{N}^{s}_{p,\lambda}} \leq Ct^{-\frac{\gamma}{2\alpha}}\left\|a(t)\right\|_{\mathrm{N}^{s}_{p,\lambda}}.$$

Then, by Theorem 3.1, we have

$$\|(-\Delta)^{\frac{\gamma}{2}}a(t)\|_{\mathrm{N}^{s}_{p,\lambda}} \leq Ct^{\frac{\gamma}{2\alpha}}\|a_0\|_{\mathrm{N}^{s}_{p,\lambda}}.$$

Similary, we obtain

$$\|(-\Delta)^{\frac{\gamma}{2}}m(t)\|_{\mathrm{N}^{s}_{p,\lambda}} \leq Ct^{\frac{\gamma}{2\alpha}}\|m_0\|_{\mathrm{N}^{s}_{p,\lambda}}.$$

Then

$$\left\|\left((-\Delta)^{\frac{\gamma}{2}}a(t),(-\Delta)^{\frac{\gamma}{2}}m(t)\right)\right\|_{\mathrm{N}^{s}_{p,\lambda}} \leq Ct^{\frac{\gamma}{2\alpha}}\left\|(a_0,m_0)\right\|_{\mathrm{N}^{s}_{p,\lambda}}.$$

This finishes the proof of Theorem 4.2 .

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