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Cyclic Cohomology Group and Cyclic Amenability of Induced Semigroup Algebras

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ABSTRACT: Let S be a discrete semigroup with idempotent set E, and T be a left multiplier operator on S, which induces a new semigroup denoted by S_T with idempotent set E_T. In this paper, we examine the properties of the induced semigroup algebra $\ell^1(S_T)$, where we show that, under certain conditions on T, the first cyclic cohomology groups $\mathcal{HC}^1(\ell^1(S), \ell^\infty(S))$ and $\mathcal{HC}^1(\ell^1(S_T), \ell^\infty(S_T))$ are equal, where S is assumed to be a monoid semigroup. Additionally, we demonstrate that when S is a completely regular semigroup, the semigroup algebra $\ell^1(S_T)$ is cyclic amenable. Finally, we provide examples at the end of each section to illustrate the conditions discussed.

Key Words: Cyclic cohomology group, monoid semigroup, induced semigroup, semigroup algebra.

Contents

1. Introduction

The concept of an amenable Banach algebra was first defined by Johnson in [5]. Next, specific concepts of amenability of Banach algebras such as weak amenability and cyclic amenability on Banach algebras were introduced and examined. On the other hand, research on these concepts continued on algebras that were of interest, such as group algebras and semigroup algebras. The study of the weak amenability of semigroup algebras was conducted by Blackmore in [1], and interesting results emerged. The concept of cyclic amenable Banach algebras was introduced by Gronbaek in [4]. He studied the hereditary properties of this concept, also found some relations between the cyclic amenability of a Banach algebra and the trace extension property of its ideals, and showed that the free product of cyclic amenable Banach algebras is a cyclic amenable Banach algebra, while this result is not true for weak amenability. After that, Ghahramani and Loy in [3] and Shojaee and Bodaghi in [11] developed these concepts.

On the other hand, Laali in [6], with the help of a left multiplier operator T on Banach algebra A, expressed the concept of induced algebra with the symbol A_T and examined the differences and similarities between A and A_T. For example, he showed that there is a Banach algebra that is not Arens regular, but the Banach algebra A_T is Arens regular. Let S be a discrete semigroup and T be a left multiplier operator on S with idempotent set E, which makes it a newly induced semigroup S_T with idempotent set E_T. Influenced by Nasrabadi's study on the concept of module cohomology groups of semigroup algebras in [9], the authors of this paper, in collaboration with Miri [7], examined the first module cohomology groups for $\ell^1(S)$ and $\ell^1(S_T)$. Their analysis demonstrated that $\ell^1(S)$ is weakly $\ell^1(E)$-module amenable if and only if $\ell^1(S_T)$ is weakly $\ell^1(E_T)$-module amenable, where E and E_T are sets of idempotent elements in S and S_T, respectively. Also in [8], by examining the second module cohomology groups, they proved that for every odd $n \in \mathbb{N}$, $\mathcal{H}^2_{\ell^1(E_T)}(\ell^1(S_T), \ell^1(S_T)^{(n)})$ is a Banach space, when S is a commutative inverse semigroup.

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In this paper, the first part is devoted to the concept of induced semigroups and their associated semigroup algebras, where we establish their properties through a series of rules and theorems. In the second part, we investigate the first cyclic cohomology groups

$$\mathcal{HC}^1(\ell^1(S), \ell^\infty(S)) \quad \text{and} \quad \mathcal{HC}^1(\ell^1(S_T), \ell^\infty(S_T)),$$

and examine the relationship between them. In the final section, we determine sufficient conditions for the cyclic amenability of the semigroup algebra $\ell^1(S_T)$. At the end of each section, these conditions are illustrated and evaluated through specific examples.

In what follows, we recall some basic notions and definitions that will be needed throughout the paper.

Let A be a Banach algebra and X a Banach A-bimodule. The space X is called a *commutative Banach A-module* if

$$ax = xa \qquad (a \in A,\, x \in X).$$

If X is a (commutative) Banach A-module, then its dual space X^* is also a (commutative) Banach A-module, where the actions of A on X^* are defined by

$$(af)(x) = f(xa), \qquad (fa)(x) = f(ax) \qquad (a \in A,\, x \in X,\, f \in X^*).$$

The dual space A^* is also a commutative Banach A-module when A is a commutative Banach algebra.

Let A be a Banach algebra and X a Banach A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that

$$D(ab) = D(a)b + aD(b) \qquad (a, b \in A).$$

A derivation D is said to be *inner* if there exists $x \in X$ such that

$$D(a) = \mathrm{ad}_x(a) = ax - xa \qquad (a \in A).$$

A derivation $D : A \to A^*$ is called *cyclic* if

$$[D(a)](b) + [D(b)](a) = 0 \qquad (a, b \in A).$$

Definition 1.1. *A Banach algebra A is called* weakly amenable *if every bounded derivation $D : A \to A^*$ is inner. The algebra A is called* cyclic amenable *if every bounded cyclic derivation $D : A \to A^*$ is inner.*

We use the notation $\mathcal{Z}^1(A, A^*)$ to denote the set of derivations $D : A \to A^*$, and $\mathcal{B}^1(A, A^*)$ to denote the subset of inner derivations. The first cohomology group with coefficients in A^*, denoted $\mathcal{H}^1(A, A^*)$, is the quotient $\mathcal{Z}^1(A, A^*)/\mathcal{B}^1(A, A^*)$. Therefore, A is weakly amenable if and only if

$$\mathcal{H}^1(A, A^*) = 0.$$

Similarly, we use the notation $\mathcal{ZC}^1(A, A^*)$ for the set of all bounded cyclic derivations and $\mathcal{HC}^1(A, A^*)$ for the first cyclic cohomology group, which is the quotient $\mathcal{ZC}^1(A, A^*)/\mathcal{B}^1(A, A^*)$. Hence, we say that A is cyclic amenable if

$$\mathcal{HC}^1(A, A^*) = 0.$$

According to the definitions of weak amenability and cyclic amenability, weak amenability implies cyclic amenability for Banach algebras. However, there exists a cyclically amenable Banach algebra that is not weakly amenable.

Remark 1.1.

(i) We know that, while every weakly amenable Banach algebra is essential ($\overline{A^2} = A$), the same does not necessarily hold for cyclic amenable Banach algebras.

(ii) Let A be a non-zero Banach algebra with zero product, meaning $ab = 0$ for all $a, b \in A$. It is shown in Example 2.5 of [4] that the Banach algebra A is cyclic amenable if and only if its dimension is one.

Definition 1.2. *A* monoid semigroup *is a set equipped with an associative binary operation and a unique identity element. More precisely, a monoid is a semigroup S with an identity element $e \in S$ such that, for every element $a \in S$, the equalities*

$$e \cdot a = a \quad \text{and} \quad a \cdot e = a$$

hold.

An element $a \in S$ is called a *regular element* of the semigroup S if there exists $a^* \in S$ such that $a = aa^*a$. An element $a \in S$ is called *completely regular* if it is regular and satisfies $aa^* = a^*a$. We say that S is a *regular* (respectively, *completely regular*) semigroup if every element $a \in S$ is regular (respectively, completely regular).

2. Induced Semigroup S_T with the Left Multiplier Map T

Let S be a discrete semigroup. A map $T : S \to S$ is called a *left* (resp. *right*) *multiplier* on S if

$$T(st) = T(s)t \quad (\text{resp. } T(st) = sT(t)) \quad \text{for all } s, t \in S.$$

The class of left (resp. right) multiplier maps on S is denoted by $\mathrm{Mul}_l(S)$ (resp. $\mathrm{Mul}_r(S)$). An operator T is a multiplier map if $T \in \mathrm{Mul}_l(S) \cap \mathrm{Mul}_r(S)$. The class of all multiplier maps on S is denoted by $\mathrm{Mul}(S)$.

Let $T \in \mathrm{Mul}_l(S)$. We define a new operation "$\circ$" on S as follows:

$$s \circ t := sT(t) \quad \text{for every } s, t \in S.$$

The semigroup S, equipped with the operation "$\circ$", forms a new semigroup called the *induced semigroup*, denoted by S_T.

When S is a discrete semigroup, the Banach space $\ell^1(S)$ is defined as:

$$\ell^1(S) = \Big\{ f : S \to \mathbb{C} : \|f\|_1 = \sum_{x \in S} |f(x)| < \infty \Big\}.$$

So $\ell^1(S)$ is the set of all complex functions

$$f = \sum_{x \in S} f(x)\delta_x = \sum_{x \in A} f(x)\delta_x,$$

where $f \in \ell^1(S)$ is zero except at most on a countable subset $A \subset S$.

$(\ell^1(S), *)$ and $(\ell^1(S_T), \circledast)$ are Banach algebras with their convolution products, defined as:

$$\delta_s * \delta_t = \delta_{st}, \quad \delta_s \circledast \delta_t = \delta_{s \circ t} = \delta_{sT(t)} \qquad (s, t \in S). \tag{2.1}$$

It should be noted that in the structure of the induced semigroup algebra, there is a one-to-one correspondence between the classes $\mathrm{Mul}_l(S)$ and $\mathrm{Mul}_l(\ell^1(S))$, which is established in Lemma 2.2.

Remark 2.1. The set of all finite linear combinations of point masses $(\delta_s; s \in S)$ is dense in $\ell^1(S)$, meaning that

$$\overline{\mathrm{span}\{\delta_s; s \in S\}} = \ell^1(S).$$

Since actions and derivations are continuous, we consider point masses as representing elements of the semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$.

Lemma 2.1. *Let S be a semigroup and $T : S \to S$ be a bijective map. Then*

(i) $T \in \mathrm{Mul}_l(S)$ if and only if $T^{-1} \in \mathrm{Mul}_l(S)$.

(ii) If $T \in \mathrm{Mul}(S)$, then for every $s, t \in S$, we have $s \circ T(t) = T(s) \circ t$ and $s \circ T^{-1}(t) = T^{-1}(s) \circ t$.

Proof: For proof, refer to [7], Lemma 2.1. □

Lemma 2.2. *Let S be a semigroup and $T \in \mathrm{Mul}_l(S)$ be injective. Then, the mapping $\overline{T} : \ell^1(S) \to \ell^1(S)$, defined by $\overline{T}(\delta_x) = \delta_{T(x)}$, is an isometry, and $\overline{T} \in \mathrm{Mul}_l(\ell^1(S))$. Furthermore, the Banach algebras $(\ell^1(S_T), \circledast)$ and $((\ell^1(S))_T, \boxtimes)$ are isomorphic, with the convolution product on $((\ell^1(S))_T)$ given by:*

$$(\delta_x \boxtimes \delta_y) = \delta_x * \overline{T}(\delta_y).$$

Proof: Let $x, y \in S$ and $\alpha \in \mathbb{C}$, and $T \in \mathrm{Mul}_l(S)$ be injective. It is clear that $\overline{T}$ is injective, and according to the following relations and Remark 2.1, $\overline{T}$ is a linear isometry:

$$\overline{T}(\delta_x + \delta_y) = \overline{\delta_x + \delta_y} = \overline{\delta_x} + \overline{\delta_y} = \delta_{T(x)} + \delta_{T(y)} = \overline{T}(\delta_x) + \overline{T}(\delta_y),$$
$$\overline{T}(\alpha\delta_x) = \overline{\alpha\delta_x} = \alpha\overline{\delta_x} = \alpha\delta_{T(x)} = \alpha\overline{T}(\delta_x).$$

It also shows the following relation: $\overline{T}$ is a left multiplier map on $\ell^1(S)$:

$$\overline{T}(\delta_x * \delta_y) = \overline{T}(\delta_{xy}) = \delta_{T(xy)} = \delta_{T(x)} * \delta_y = \overline{T}(\delta_x) * \delta_y.$$

Considering that $\overline{T} \in \mathrm{Mul}_l(\ell^1(S))$, the space $\ell^1(S)$, together with the convolution product

$$(\delta_x \boxtimes \delta_y) = \delta_x * \overline{T}(\delta_y), \quad x, y \in S$$

is a Banach algebra. Now we define the map $\phi : (\ell^1(S_T), \circledast) \to ((\ell^1(S))_T, \boxtimes)$ by $\phi(\delta_x) = \delta_x$. Then

$$\phi(\delta_x \circledast \delta_y) = \phi(\delta_{x \circ y}) = \delta_{xT(y)} = \delta_x * \delta_{T(y)} = \delta_x * \overline{T}(\delta_y) = \delta_x \boxtimes \delta_y = \phi(\delta_x) \boxtimes \phi(\delta_y).$$

This shows that $(\ell^1(S_T), \circledast)$ and $((\ell^1(S))_T, \boxtimes)$ are isomorphic. □

Lemma 2.3. *Let S be a monoid semigroup with identity element $e \in E$, and let $T \in \mathrm{Mul}(S)$ be bijective. Suppose $D : \ell^1(S) \to \ell^\infty(S)$ is a derivation. Then, by Remark 2.1, we have*

$$[D(\delta_{T(x)})](\delta_y) = [D(\delta_x)](\delta_{T(y)}), \qquad [D(\delta_{T^{-1}(x)})](\delta_y) = [D(\delta_x)](\delta_{T^{-1}(y)}).$$

Proof: Assume e is the identity element of S and $T \in \mathrm{Mul}(S)$ is bijective. Then

$$\begin{aligned} D(\delta_e)(\delta_{T(x)}) &= D(\delta_e * \delta_e)(\delta_{T(x)}) \\ &= [D(\delta_e) * \delta_e + \delta_e * D(\delta_e)](\delta_{T(x)}) \\ &= D(\delta_e)(\delta_e * \delta_{T(x)}) + D(\delta_e)(\delta_{T(x)} * \delta_e) \\ &= D(\delta_e)(\delta_{T(e)} * \delta_x) + D(\delta_e)(\delta_x * \delta_{T(e)}) \\ &= [D(\delta_e) * \delta_{T(e)} + \delta_{T(e)} * D(\delta_e)](\delta_x) \\ &= [D(\delta_e) * \delta_{T(e)} + \delta_e * D(\delta_{T(e)})](\delta_x) \\ &= D(\delta_e * \delta_{T(e)})(\delta_x) = D(\delta_{T(e)})(\delta_x). \end{aligned}$$

Thus,

$$D(\delta_e)(\delta_{T(x)}) = D(\delta_{T(e)})(\delta_x).$$

Now let $x, y \in S$. Then

$$\begin{aligned} D(\delta_{T(x)})(\delta_y) &= D(\delta_{T(ex)})(\delta_y) = D(\delta_{T(e)} * \delta_x)(\delta_y) \\ &= [D(\delta_{T(e)}) * \delta_x + \delta_{T(e)} * D(\delta_x)](\delta_y) \\ &= D(\delta_{T(e)})(\delta_x * \delta_y) + D(\delta_x)(\delta_y * \delta_{T(e)}) \\ &= D(\delta_e)(\delta_x * \delta_{T(y)}) + D(\delta_x)(\delta_{T(y)} * \delta_e) \\ &= [D(\delta_e) * \delta_x + \delta_e * D(\delta_x)](\delta_{T(y)}) \\ &= D(\delta_e * \delta_x)(\delta_{T(y)}) = D(\delta_x)(\delta_{T(y)}). \end{aligned}$$

Hence,

$$D(\delta_{T(x)})(\delta_y) = D(\delta_x)(\delta_{T(y)}).$$

Similarly, one can show that

$$[D(\delta_{T^{-1}(x)})](\delta_y) = [D(\delta_x)](\delta_{T^{-1}(y)}).$$

□

3. First Cyclic Cohomology Group of Induced Semigroup Algebras

Throughout this section, it is assumed that S is a monoid discrete semigroup, $T \in \mathrm{Mul}(S)$, and T is bijective. We examine the relationship between the first cyclic cohomology groups

$$\mathcal{HC}^1(\ell^1(S), \ell^\infty(S)) \quad \text{and} \quad \mathcal{HC}^1(\ell^1(S_T), \ell^\infty(S_T)).$$

Lemma 3.1. *Let S, S_T, and T be as above. Then $D : \ell^1(S) \to \ell^\infty(S)$ is a derivation if and only if*

$$\widetilde{D} : \ell^1(S_T) \to \ell^\infty(S_T), \quad \widetilde{D}(f) := D(f \circ T^{-1}),$$

is a derivation. Furthermore, D is cyclic if and only if $\widetilde{D}$ is cyclic. Also, D is inner if and only if $\widetilde{D}$ is inner.

Proof: Note that $\widetilde{D}(\delta_x) = D(\delta_{T(x)})$ for every $x \in S$. By Lemma 3.2 in [7], $\widetilde{D}$ is a derivation if and only if D is a derivation.

Now, by Lemma 2.3, we show that $\widetilde{D}$ is cyclic when D is cyclic. Let $x, y \in S$, then:

$$\begin{aligned}
[\widetilde{D}(\delta_x)](\delta_y) + [\widetilde{D}(\delta_y)](\delta_x) &= [D(\delta_{T(x)})](\delta_y) + [D(\delta_{T(y)})](\delta_x) \\
&= [D(\delta_x)](\delta_{T(y)}) + [D(\delta_y)](\delta_{T(x)}) \\
&= 0.
\end{aligned}$$

Conversely, if $\widetilde{D}$ is cyclic, then for $x, y \in S$:

$$\begin{aligned}
[D(\delta_x)](\delta_y) + [D(\delta_y)](\delta_x) &= [\widetilde{D}(\delta_{T^{-1}(x)})](\delta_y) + [\widetilde{D}(\delta_{T^{-1}(y)})](\delta_x) \\
&= 0.
\end{aligned}$$

Hence, D is cyclic.

Finally, let D be inner. Then there exists $\psi \in \ell^\infty(S)$ such that $D(f) = f * \psi - \psi * f$ for all $f \in \ell^1(S)$. For $x, y \in S_T$:

$$\begin{aligned}
[\widetilde{D}(\delta_x)](\delta_y) &= [D(\delta_{T(x)})](\delta_y) \\
&= [\delta_{T(x)} * \psi - \psi * \delta_{T(x)}](\delta_y) \\
&= \psi(\delta_y * \delta_{T(x)} - \delta_{T(x)} * \delta_y) \\
&= \psi(\delta_{yT(x)} - \delta_{xT(y)}) \\
&= [\delta_x \circledast \psi - \psi \circledast \delta_x](\delta_y),
\end{aligned}$$

so $\widetilde{D}$ is inner. The converse is proven similarly. □

Theorem 3.1. *Let S, T, and S_T be as above. Then*

$$\mathcal{HC}^1(\ell^1(S), \ell^\infty(S)) \simeq \mathcal{HC}^1(\ell^1(S_T), \ell^\infty(S_T)).$$

Proof: Consider the map

$$\begin{aligned}
\Gamma : \mathcal{ZC}^1(\ell^1(S), \ell^\infty(S)) &\longrightarrow \mathcal{HC}^1(\ell^1(S_T), \ell^\infty(S_T)) \\
D &\mapsto \widetilde{D} + \mathcal{B}^1(\ell^1(S_T), \ell^\infty(S_T)).
\end{aligned}$$

By Lemma 3.1, Γ is well-defined and linear. For surjectivity, let $P \in \mathcal{ZC}^1(\ell^1(S_T), \ell^\infty(S_T))$ and define $D(f) := P(f \circ T)$. Then $\Gamma(D) = \widetilde{D} = P$, and $D \in \mathcal{ZC}^1(\ell^1(S), \ell^\infty(S))$.

Moreover, Lemma 3.1 shows $\ker\Gamma = \mathcal{B}^1(\ell^1(S), \ell^\infty(S))$. Therefore,

$$\mathcal{HC}^1(\ell^1(S), \ell^\infty(S)) = \frac{\mathcal{ZC}^1(\ell^1(S), \ell^\infty(S))}{\ker\Gamma} \simeq \operatorname{Im}\Gamma = \mathcal{HC}^1(\ell^1(S_T), \ell^\infty(S_T)).$$

□

Corollary 3.1. *Vanishing of one of the cyclic cohomology groups above is equivalent to vanishing of the other. Thus, $\ell^1(S)$ is cyclic amenable if and only if $\ell^1(S_T)$ is cyclic amenable.*

Example 3.1. Let $S = \{1, -1, i, -i\}$, where $i = \sqrt{-1}$, with the complex product as the operation. Then $(S, \cdot)$ is a monoid semigroup with the set of idempotents $E = \{1\}$. Define $T = L_i$, where $L_i(x) = ix$ for all $x \in S$, and define $a \circ b = aT(b)$. Then $S_T = (S, \circ)$ has the following operation table:

∘	1	-1	i	-i
1	i	-i	-1	1
-1	-i	i	1	-1
i	-1	1	-i	i
-i	1	-1	i	-i

The set of idempotents of S_T is $E_T = \{-i\}$. According to Lemma 2.3 and Theorem 3.1, the cyclic cohomology groups $\mathcal{HC}^1(\ell^1(S), \ell^\infty(S))$ and $\mathcal{HC}^1(\ell^1(S_T), \ell^\infty(S_T))$ are equivalent.

Remark 3.1. In the previous example, $(S, \cdot)$ is a discrete semigroup with real identity $(e = 1)$, while $(S, \circ) = S_T$ is a discrete semigroup with complex identity $(e_T = -i)$. This shows that, regarding cyclic amenability, it may be easier to work with semigroup algebras having a real identity element.

4. Cyclic Amenability of Induced Semigroup Algebras

In this section, we assume S is a completely regular semigroup and $T \in \operatorname{Mul}_l(S)$ is bijective. We show that if S is completely regular, then $\ell^1(S_T)$ is cyclic amenable.

Lemma 4.1. *Let S be a semigroup and $T \in \operatorname{Mul}_l(S)$ be bijective. Then S is completely regular if and only if S_T is completely regular.*

Proof: Let $(S, \cdot)$ be completely regular and $a \in S$. Then there exists $a^* \in S$ such that

$$a = a \cdot a^* \cdot a \quad \text{and} \quad a^* \cdot a = a \cdot a^*.$$

Since T is bijective and multiplicative, we have

$$T(a) = T(a) \cdot T(a^*) \cdot T(a), \quad T(a^*) \cdot T(a) = T(a) \cdot T(a^*),$$

which implies

$$a = a \circ a^* \circ a, \quad a^* \circ a = a \circ a^*.$$

Hence, $(S_T, \circ)$ is completely regular.

Conversely, let $(S_T, \circ)$ be completely regular. For $a \in S$, since $T^{-1} \in \operatorname{Mul}_l(S)$ is bijective, there exists $a^\star \in S$ such that

$$T^{-1}(a) = T^{-1}(a) \circ T^{-1}(a^\star) \circ T^{-1}(a), \quad T^{-1}(a^\star) \circ T^{-1}(a) = T^{-1}(a) \circ T^{-1}(a^\star).$$

Applying T and using bijectivity, we get

$$a = a \cdot a^\star \cdot a, \quad a^\star \cdot a = a \cdot a^\star,$$

so $(S, \cdot)$ is completely regular. □

Lemma 4.2. *Let S be completely regular and $T \in \mathrm{Mul}_l(S)$ be bijective. Then $\ell^1(S_T)$ is cyclic amenable.*

Proof: By Lemma 4.1, S_T is completely regular. By Theorem 3.6 in [1], the semigroup algebra of a completely regular semigroup is weakly amenable. Finally, Proposition 2.4 in [4] implies that $\ell^1(S_T)$ is cyclic amenable. □

The following example shows that bijectivity of T is essential for the results above.

Example 4.1. Consider $S = \{0, 1, 2, 3, 4\}$ with $s \cdot t = \mathrm{Max}\{s, t\}$. Then S is a finite commutative idempotent semigroup and thus amenable [10, (0.18)]. It is a monoid with identity $1_S = 0$ and zero $0_S = 4$. Since every element is idempotent, S is regular, so by Proposition 2.4 of [4], $\ell^1(S)$ is amenable, weakly amenable, and cyclic amenable.

Now, let $T : S \to S$ be the non-bijective left multiplier L_4 defined by $T(s) = \mathrm{Max}\{4, s\}$. Then the induced semigroup is $S_T = \{T(s) : s \in S\} = \{4\}$, which is trivial. Although S_T is amenable, it is not regular (there is no $x \in S_T$ such that $4 = 4 \circ x \circ 4$), so by Corollary 5.3 of [2], $\ell^1(S_T)$ is not amenable. Moreover, $\ell^1(S_T)$ is not weakly amenable, since $\overline{A^2} \neq A$.

Since $S_T = \{4\}$, the space $\ell^1(S_T)$ is one-dimensional with basis δ_4, so $\dim \ell^1(S_T) = 1$. By Example 2.5 of [4], this implies $\ell^1(S_T)$ is not cyclic amenable.

Conclusion: $\ell^1(S)$ is amenable, weakly amenable, and cyclic amenable, while $\ell^1(S_T)$ is none of these.

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