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On properties of s-convex functions on the co-ordinates in three and higher dimensions

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ABSTRACT: In the present paper, we discuss the class of s-convex functions on the co-ordinates for three variables and prove certain new Hermite-Hadamard type inequalities for such mappings. Using geometric reasoning, we postulate how such results appear in higher dimensions. Furthermore, we delve into various intriguing aspects of the associated H function.

Key Words: Convexity, s-Convexity on the co-ordinates, Hermite-Hadamard type inequalities.

Contents

1. Introduction

A function $\chi : J \to R$ is said to be a convex function on J if

$$\chi(\sigma\zeta + (1-\sigma)\mu) \leq \sigma\chi(\zeta) + (1-\sigma)\chi(\mu) \tag{1.1}$$

is true for every $\zeta, \mu \in J$, where J is an interval in $\mathbb{R}$ and $\sigma \in [0,1]$.
It has been an intriguing area of research to study convexity in relation to integral inequalities. In convex analysis, Hermite-Hadamard inequality is one of the most remarkable inequalities for the class of convex functions. This twofold inequality gives an estimate from both sides of the mean value of a convex function and also ensures the integrability of a convex function and is expressed as follows:

$$\chi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} \chi(\zeta)d\zeta \leq \frac{\chi(\zeta_1)+\chi(\zeta_2)}{2}, \tag{1.2}$$

where $\chi : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a convex function.

An s-convex function is a generalization of a convex function which was first introduced by Breckner [5]. A real valued function on an interval $J \subset [0,\infty)$ is s-convex in the second sense provided $\chi(\sigma u + \delta v) \leq \sigma^s\chi(u) + \delta^s\chi(v)$ for all $u, v \in J$ and $\sigma, \delta \geq 0$ with $\sigma + \delta = 1$ and some fixed $s \in (0,1]$. If we replace the condition $\sigma + \delta = 1$ with $\sigma^s + \delta^s = 1$, then we have s-convexity in the first sense. Obviously, s-convexity reduces to convexity when $s = 1$.

For s-convex functions, Dragomir and Fitzpatrick [6] established the following variant of the Hermite-Hadamard inequality, known as s-Hadamard's inequality, which is true for s-convex functions in the second sense.:

$$2^{s-1}\chi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \int_{\zeta_1}^{\zeta_2} \chi(\zeta)d\zeta \leq \frac{\chi(\zeta_1)+\chi(\zeta_2)}{s+1}. \tag{1.3}$$

If we put $s = 1$ in the above inequality, we revert back to earlier inequality.

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Later Dragomir [7] also suggested a variation for convex functions called co-ordinated convex functions, which is as follows. Consider a bidimensional rectangle $\Omega = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2]$ in $\mathbb{R}^2$ with $\zeta_1 < \zeta_2$ and $\mu_1 < \mu_2$. A mapping $\chi : \Omega \to \mathbb{R}$ is said to be convex on the co-ordinates on Ω if the partial mappings $\chi_\mu : [\zeta_1, \zeta_2] \to \mathbb{R}, \chi_\mu(u) = \chi(u, \mu)$ and $\chi_\zeta : [\mu_1, \mu_2] \to \mathbb{R}, \chi_\zeta(v) = \chi(\zeta, v)$ are convex for all $\zeta \in [\zeta_1, \zeta_2]$ and $\mu \in [\mu_1, \mu_2]$.

Dragomir [7] also proved the following Hadamard-type inequality for convex functions on the co-ordinates.

$$\begin{aligned}\chi\left(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}\right) &\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\chi(\zeta,\mu)d\zeta d\mu \\ &\leq \frac{\chi(\zeta_1,\mu_1)+\chi(\zeta_2,\mu_1))+\chi(\zeta_1,\mu_2)+\chi(\zeta_2,\mu_2)}{4}.\end{aligned} \tag{1.4}$$

Alomari and Darus [1-3] proposed a natural extension of convex functions on the co-ordinates to the concept of s-convex functions on the co-ordinates. We define such a function in the following manner: The mapping $f : \Omega \to \mathbb{R}$ is s-convex in the second sense if the partial mappings $\chi_\mu : [\zeta_1, \zeta_2] \to \mathbb{R}$ and $\chi_\zeta : [\mu_1, \mu_2] \to \mathbb{R}$ are s-convex in the second sense.
They also proved the following inequalities for an s-convex function on the co-ordinates.

$$\begin{aligned}\chi\left(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}\right) &\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\chi(\zeta,\mu)d\zeta d\mu \\ &\leq \frac{\chi(\zeta_1,\mu_1)+s\chi(\zeta_2,\mu_1)+s\chi(\zeta_1,\mu_2)+s^2\chi(\zeta_2,\mu_2)}{(s+1)^2}.\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}4^{s-1}\chi\left(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}\right) &\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\chi(\zeta,\mu)d\zeta d\mu \\ &\leq \frac{\chi(\zeta_1,\mu_1)+\chi(\zeta_2,\mu_1))+\chi(\zeta_1,\mu_2)+\chi(\zeta_2,\mu_2)}{(s+1)^2}.\end{aligned} \tag{1.6}$$

For other approaches and results on s-convex functions on the co-ordinates and their different variants, see [4], [8-15]. In the next section, we define the class of s-convex functions on the co-ordinates in three co-ordinates and prove analogous Hermite-Hadamard type inequalities.

2. Main Results

Definition 1. Consider the tridemsional interval $\Omega = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2] \times [\eta_1, \eta_2]$ in $[0, \infty)^3$ with $\zeta_1 < \zeta_2, \mu_1 < \mu_2$ and $\eta_1 < \eta_2$. The mapping $\chi : \Omega \to \mathbb{R}$ is s-convex on Ω if $\chi(\sigma\zeta + \delta u, \sigma\mu + \delta v, \sigma\eta + \delta w) \leq \sigma^s\chi(\zeta, \mu, \eta) + \delta^s\chi(u, v, w)$, holds for all $(\zeta, \mu, \eta), (u, v, w) \in \Omega$ with $\sigma^s + \delta^s = 1$ and for some fixed $s \in (0, 1]$.

Theorem 1. Suppose that $\chi : \Omega = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2] \times [\eta_1, \eta_2] \subseteq [0, \infty)^3 \to \mathbb{R}$ is a s-convex function on the co-ordinates on Ω. Then one has the inequalities:

$$\begin{aligned}
&\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big) \leq \frac{1}{3}\Big\{\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\zeta,\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)d\zeta+\\
&\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\mu,\frac{\eta_1+\eta_2}{2}\Big)d\mu+\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\eta\Big)d\eta\Big\}\\
&\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta\\
&\leq \frac{1}{3(s+1)^2}\Big\{\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_1,\eta_1)d\zeta+\frac{s}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_1,\eta_2)d\zeta\\
&+\frac{s}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_2,\eta_1)dx+\frac{s^2}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_2,\eta_2)d\zeta\\
&+\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_1,\mu,\eta_1)d\mu+\frac{s}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_2,\mu,\eta_1)d\mu\\
&+\frac{s^2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_2,\mu,\eta_2)d\mu+\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_2,\mu_1,\eta)d\eta\\
&+\frac{s}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_1,\mu_2,\eta_2)d\eta+\frac{s^2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_2,\mu_2,\eta)d\eta\Big\}\\
&\leq \frac{1}{(s+1)^3}\Big\{\chi(\zeta_1,\mu_1,\eta_1)+s\chi(\zeta_2,\mu_1,\eta_1)+s\chi(\zeta_1,\mu_1,\eta_2)+s\chi(\zeta_1,\mu_1,\eta_1)\\
&+s^2\chi(\zeta_2,\mu_1,\eta_2)+s^2\chi(\zeta_2,\mu_2,\eta_1)+s^2\chi(\zeta_1,\mu_2,\eta_2)+s^3\chi(\zeta_2,\mu_2,\eta_2)\Big\}.
\end{aligned}$$

Proof: Since $\chi:\Omega\to R$ is co-ordinated convex on Ω, it follows that the mappings $\phi_\zeta:[\mu_1,\mu_2]\times[\eta_1,\eta_2]\to[0,\infty), \phi_\zeta(\mu,\eta)=\chi(\zeta,\mu,\eta)$ is s-convex on $[\zeta_1,\zeta_2]\times[\mu_1,\mu_2]$ for all $\zeta\in[\zeta_1,\zeta_2]$, $\phi_\mu:[\zeta_1,\zeta_2]\times[\eta_1,\eta_2]\to[0,\infty), \phi_\mu(\zeta,\eta)=\chi(\zeta,\mu,\eta)$ is s-convex on $[\zeta_1,\zeta_2]\times[\eta_1,\eta_2]$ for all $\mu\in[\mu_1,\mu_2]$ and $\phi_\eta:[\zeta_1,\zeta_2]\times[\mu_1,\mu_2]\to[0,\infty), \phi_\eta(\zeta,\mu)=\chi(\zeta,\mu,\eta)$ is s-convex on $[\zeta_1,\zeta_2]\times[\mu_1,\mu_2]$ for all $\eta\in[\eta_1,\eta_2]$.
Thus we have, we have

$$\begin{aligned}
\phi_\eta\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2}\Big) &\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\phi_\eta(x,y)dxdy\\
&\leq \frac{\phi_\eta(\zeta_1,\mu_1)+s\phi_\eta(\zeta_2,\mu_1)+s\phi_\eta(\zeta_1,\mu_2)+s^2\phi_\eta(\zeta_2,\mu_2)}{(s+1)^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\eta\Big)d\eta\\
&\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta\\
&\leq \frac{1}{(s+1)^2}\Big\{\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_1,\mu_1,\eta)d\eta+\frac{s}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_2,\mu_1,\eta)d\eta\\
&+\frac{s}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_1,\mu_2,\eta)d\eta+\frac{s^2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_2,\mu_2,\eta)d\eta\Big\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\mu,\frac{\eta_1+\eta_2}{2}\Big)d\eta\\
&\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta\\
&\leq \frac{1}{(s+1)^2}\Big\{\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_1,\mu_1,\eta)d\eta+\frac{s}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_1,\mu,\eta_2)d\mu\\
&\quad+\frac{s}{d-c}\int_{\mu_1}^{\mu_2}\chi(\zeta_2,\mu,\eta_1)d\mu+\frac{s^2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_2,\mu,\eta_2)d\mu\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\zeta,\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)d\zeta\\
&\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta\\
&\leq \frac{1}{(s+1)^2}\Big\{\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_1,\eta_1)d\zeta+\frac{s}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_1,\eta_2)d\zeta\\
&\quad+\frac{s}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_2,\eta_1)d\zeta+\frac{s^2}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_2,\eta_2)d\zeta\Big\}.
\end{aligned}$$

Adding the above inequalities, we have the required second and third inequalities.

Again by Hadamard's inequality, we have

$$\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)\leq\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\zeta,\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)d\zeta$$

$$\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)\leq\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi\Big(\frac{\mu_1+\mu_2}{2},\mu,\frac{\eta_1+\eta_2}{2}\Big)d\mu$$

$$\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)\leq\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\eta\Big)d\eta$$

Adding these inequalities, we have our first required inequality.

Finally by s-Hadamard's inequality, we have

$$\begin{aligned}
\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(x,\mu_1,\eta_1)d\zeta &\leq \frac{\chi(\zeta_1,\mu_1,\eta_1)+s\chi(\zeta_2,\mu_1,\eta_1)}{s+1}\\
\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(x,\mu_1,\eta_2)d\zeta &\leq \frac{\chi(\zeta_1,\mu_1,\eta_2)+s\chi(\zeta_2,\mu_1,\eta_2)}{s+1}\\
\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(x,\mu_2,\eta_1)d\zeta &\leq \frac{\chi(\zeta_1,\mu_2,\eta_1)+s\chi(\zeta_2,\mu_2,\eta_1)}{s+1}\\
\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(x,\mu_2,\eta_2)d\zeta &\leq \frac{\chi(\zeta_1,\mu_2,\eta_2)+s\chi(\zeta_2,\mu_2,\eta_2)}{s+1}
\end{aligned}$$

Similarly,

$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_1,y,\eta_1)d\mu\le\frac{\chi(\zeta_1,\mu_1,\eta_1)+s\chi(\zeta_1,\mu_2,\eta_1)}{s+1}$$
$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_1,y,\eta_2)d\mu\le\frac{\chi(\zeta_1,\mu_1,\eta_2)+s\chi(\zeta_1,\mu_2,\eta_2)}{s+1}$$
$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_2,y,\eta_1)d\mu\le\frac{\chi(\zeta_2,\mu_1,\eta_1)+s\chi(\zeta_2,\mu_2,\eta_1)}{s+1}$$
$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_2,y,\eta_2)d\mu\le\frac{\chi(\zeta_2,\mu_1,\eta_2)+s\chi(\zeta_2,\mu_2,\eta_2)}{s+1}$$

and

$$\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_1,\mu_1,z)d\eta\le\frac{\chi(\zeta_1,\mu_1,\eta_1)+s\chi(\zeta_2,\mu_1,\eta_1)}{s+1}$$
$$\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_2,\mu_1,z)d\eta\le\frac{\chi(\zeta_1,\mu_1,\eta_1)+s\chi(\zeta_2,\mu_1,\eta_1)}{s+1}$$
$$\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_1,\mu_2,z)d\eta\le\frac{\chi(\zeta_1,\mu_2,\eta_1)+s\chi(\zeta_1,\mu_2,\eta_2)}{s+1}$$
$$\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_2,\mu_2,z)d\eta\le\frac{\chi(\zeta_2,\mu_2,\eta_1)+s\chi(\zeta_2,\mu_2,\eta_2)}{s+1}.$$

Combining the above inequalities by multiplying with coefficients s and s^2 and then adding, we have

$$\begin{aligned}&\frac{1}{3(s+1)^2}\Big\{\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_1,\eta_1)d\zeta+\frac{s}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_1,\eta_2)d\zeta\\&+\frac{s}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_2,\eta_1)d\zeta+\frac{s^2}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu_2,\eta_2)d\zeta\\&+\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_1,\mu,\eta_1)d\mu+\frac{s}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_2,\mu,\eta_1)d\mu\\&+\frac{s^2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi(\zeta_2,\mu,\eta_2)d\mu+\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_2,\mu_1,\eta)d\eta\\&+\frac{s}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_1,\mu_2,\eta_2)d\eta+\frac{s^2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi(\zeta_2,\mu_2,\eta)d\eta\Big\}\\\le&\frac{1}{3(s+1)^3}\Big\{3\chi(\zeta_1,\mu_1,\eta_1)+3s\chi(\zeta_2,\mu_1,\eta_1)+3s\chi(\zeta_1,\mu_1,\eta_2)+3s\chi(\zeta_1,\mu_2,\eta_1)\\&+3s^2\chi(\zeta_2,\mu_1,\eta_2)+3s^2\chi(\zeta_2,\mu_2,\eta_1)+3s^2\chi(\zeta_1,\mu_2,\eta_2)+3s^3\chi(\zeta_2,\mu_2,\eta_2)\Big\}.\end{aligned}$$

and thus we have our final required inequality.

□

Corollary 1. If we set $s=1$ in Theorem 1, then for a convex function for co-ordinates χ in three dimensions we have the following inequality

$$\begin{aligned}&\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)\\\le&\frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta\\\le&\frac{1}{8}\Big\{\chi(\zeta_1,\mu_1,\eta_1)+\chi(\zeta_2,\mu_1,\eta_1)+\chi(\zeta_1,\mu_1,\eta_2)+\chi(\zeta_1,\mu_2,\eta_1)\\&+\chi(\zeta_2,\mu_1,\eta_2)+\chi(\zeta_2,\mu_2,\eta_1)+\chi(\zeta_1,\mu_2,\eta_2)+\chi(\zeta_2,\mu_2,\eta_2)\Big\},\end{aligned}$$

which is a three dimensional analogoue of inequality (1.4) by Dragomir [7].

The following section delves into an examination of the geometric properties exhibited by s-convex functions in higher dimensions. Additionally, it explores the corresponding inequalities that arise in relation to these functions. By studying these geometric properties and associated inequalities, we hope to gain a deeper understanding of the behavior and characteristics of s-convex functions in higher-dimensional spaces.

3. s-Convexity in higher dimensions

We observe in the Theorem 1 that associated to every vertex of the cuboid $\Omega = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2] \times [\eta_1, \eta_2]$, there are coefficients $1, s, s^2, s^3$ in the corresponding inequality (see Figure 1).

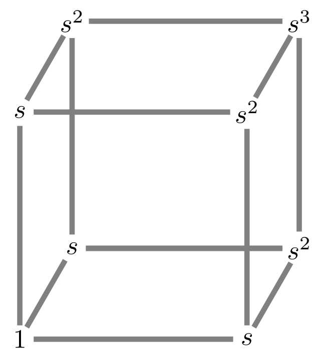

Figure 1: Figure 1. The cuboid $\Omega = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2] \times [\eta_1, \eta_2]$.

In order to postulate an analogous result in four dimensions, we consider a four dimensional analogue of a cuboid, namely the hypercuboid, $\Gamma = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2] \times [\eta_1, \eta_2] \times [\omega_1, \omega_2]$, which has 16 vertices and we associate coefficients $1, s, s^2, s^3, s^4$ with them in the same pattern (see Figure 2).

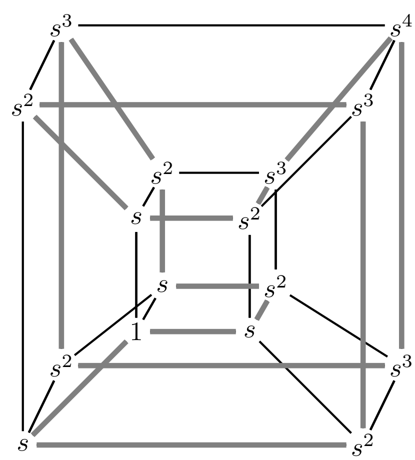

Figure 2: Figure 2. The hypercuboid, $\Gamma = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2] \times [\eta_1, \eta_2] \times [\omega_1, \omega_2]$.

In the cuboid Ω, starting with the vertex of coefficient 1, the adjoining vertices are of coefficients s, whereas their adjoining vertices are of coefficients s^2 and the far off adjoining vertex is of coefficient s^3. Following the same pattern in a hypercuboid Γ, we have the following result.

Theorem 2 Suppose that $\chi : \Gamma = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2] \times [\eta_1, \eta_2] \times [\omega_1, \omega_2] \subseteq [0, \infty)^4 \to \mathbb{R}$ is a s-convex function on the co-ordinates on Π. Then one has the inequalities:

$$\begin{aligned}
&\chi\Big(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, \frac{\eta_1+\eta_2}{2}, \frac{\omega_1+\omega_2}{2}\Big)\\
&\le \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)(\omega_2-\omega_1)} \int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\omega_1}^{\omega_2} \chi(\zeta,\mu,\eta,\omega)d\zeta d\mu d\eta d\omega\\
&\le \frac{1}{(s+1)^4}\Big\{\chi(\zeta_1,\mu_1,\eta_1,\omega_1) + s\chi(\zeta_1,\mu_2,\eta_1,\omega_1) + s\chi(\zeta_2,\mu_1,\eta_1,\omega_1) + s\chi(\zeta_1,\mu_1,\eta_2,\omega_1)\\
&+s^2\chi(\zeta_1,\mu_1,\eta_1,\omega_2) + s^2\chi(\zeta_2,\mu_2,\eta_1,\omega_1) + s^2\chi(\zeta_1,\mu_2,\eta_2,\omega_1) + s^2\chi(\zeta_2,\mu_1,\eta_2,\omega_1)\\
&+s^2\chi(\zeta_2,\mu_1,\eta_1,\omega_2) + s^2\chi(\zeta_1,\mu_2,\eta_1,\omega_2) + s^2\chi(\zeta_1,\mu_1,\eta_2,\omega_1) + s^3\chi(\zeta_2,\mu_2,\eta_2,\omega_1)\\
&+s^3\chi(\zeta_2,\mu_2,\eta_1,\omega_2) + s^3\chi(\zeta_1,\mu_2,\eta_2,\omega_2) + s^3\chi(\zeta_2,\mu_1,\eta_2,\omega_2) + s^4\chi(\zeta_2,\mu_2,\eta_2,\omega_2)\Big\}.
\end{aligned}$$

Corollary 2. If we put $s = 1$ in the above result, then for a convex function for co-ordinates χ in four dimensions, we have the following inequality

$$\begin{aligned}
&\chi\Big(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, \frac{\eta_1+\eta_2}{2}, \frac{\omega_1+\omega_2}{2}\Big)\\
&\le \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)(\omega_2-\omega_1)} \int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\omega_1}^{\omega_2} \chi(\zeta,\mu,\eta,\omega)d\zeta d\mu d\eta d\omega\\
&\le \frac{1}{16}\Big\{\chi(\zeta_1,\mu_1,\eta_1,\omega_1) + \chi(\zeta_1,\mu_2,\eta_1,\omega_1) + \chi(\zeta_2,\mu_1,\eta_1,\omega_1) + \chi(\zeta_1,\mu_1,\eta_2,\omega_1)\\
&+\chi(\zeta_1,\mu_1,\eta_1,\omega_2) + \chi(\zeta_2,\mu_2,\eta_1,\omega_1) + \chi(\zeta_1,\mu_2,\eta_2,\omega_1) + \chi(\zeta_2,\mu_1,\eta_2,\omega_1)\\
&+\chi(\zeta_2,\mu_1,\eta_1,\omega_2) + \chi(\zeta_1,\mu_2,\eta_1,\omega_2) + \chi(\zeta_1,\mu_1,\eta_2,\omega_1) + \chi(\zeta_2,\mu_2,\eta_2,\omega_1)\\
&+\chi(\zeta_2,\mu_2,\eta_1,\omega_2) + \chi(\zeta_1,\mu_2,\eta_2,\omega_2) + \chi(\zeta_2,\mu_1,\eta_2,\omega_2) + \chi(\zeta_2,\mu_2,\eta_2,\omega_2)\Big\}.
\end{aligned}$$

which is analogous to inequality (1.4) for convex functions in four co-ordinates.

4. H function and its properties

In this section, we discuss a function closely related to s-convex functions. Let $\chi : \Omega = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2] \times [\eta_1, \eta_2] \to \mathbb{R}$, define a mapping $H : [0, 1] \times [0, 1] \times [0, 1] \to \mathbb{R}$ such that

$$\begin{aligned}
H(t,r,m) = \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)} \int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2} \chi\Big(&t\zeta + (1-t)\frac{\zeta_1+\zeta_2}{2},\\
&r\mu + (1-r)\frac{\mu_1+\mu_2}{2}, m\eta + (1-m)\frac{\eta_1+\eta_2}{2}\Big)d\zeta d\mu d\eta.
\end{aligned}$$

Note that,

$$H(0,0,0) = \chi\left(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, \frac{\eta_1+\eta_2}{2}\right) \tag{4.1}$$

and

$$H(1,1,1) = \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)} \int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2} \chi(\zeta,\mu,\eta)d\zeta d\mu d\eta. \tag{4.2}$$

Theorem 3. Suppose $\chi : \Omega = [\zeta_1, \zeta_2] \times [\mu_1, \mu_2] \times [\eta_1, \eta_2] \to \mathbb{R}$ is co-ordinated s-convex on Ω. Then $H(t, r, m)$ has the properties:
(a). $H(t, r, m)$ is co-ordinated s-convex on $[0, 1] \times [0, 1] \times [0.1]$.
(b). $H(t, r, m)$ has the bounds

$$\begin{aligned} \inf_{(t,r,m)\in[0,1]^3} H(t, r, m) &= H(0, 0, 0) \\ \sup_{(t,r,m)\in[0,1]^3} H(t, r, m) &= H(1, 1, 1) \end{aligned} \tag{4.3}$$

Proof: (a). Fix $r, m \in [0, 1]$. Then for all $\sigma, \delta \geq 0$ with $\sigma + \delta = 1$ and $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned} H(\sigma t_1 + \delta t_2, r, m) &= \frac{1}{(\zeta_2 - \zeta_1)(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\zeta_1}^{\zeta_2} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi((\sigma t_1 + \delta t_2)\zeta + \\ &(1 - (\sigma t_1 + \delta t_2))\frac{\zeta_1 + \zeta_2}{2}, r\mu + (1 - r)\frac{\mu_1 + \mu_2}{2}, m\eta + (1 - m)\frac{\eta_1 + \eta_2}{2})d\zeta d\mu d\eta \\ &= \frac{1}{(\zeta_2 - \zeta_1)(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\zeta_1}^{\zeta_2} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi\Big(\sigma(t_1\zeta + (1 - t_1)\frac{\zeta_1 + \zeta_2}{2}) + \\ &\delta(t_2\zeta + (1 - t_2)\frac{\zeta_1 + \zeta_2}{2}), r\mu + (1 - r)\frac{\mu_1 + \mu_2}{2}, m\eta + (1 - m)\frac{\eta_1 + \eta_2}{2}\Big)d\zeta d\mu d\eta \\ &\leq \sigma^s \cdot \frac{1}{(\zeta_2 - \zeta_1)(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\zeta_1}^{\zeta_2} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi\Big(t_1\zeta + (1 - t_1)\frac{\zeta_1 + \zeta_2}{2}, \\ &r\mu + (1 - r)\frac{\mu_1 + \mu_2}{2}, \frac{\eta_1 + \eta_2}{2}\Big)d\zeta d\mu d\eta + \\ &\delta^s \cdot \frac{1}{(\zeta_2 - \zeta_1)(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\zeta_1}^{\zeta_2} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi(t_2\zeta + (1 - t_2)\frac{\zeta_1 + \zeta_2}{2}, \\ &r\mu + (1 - r)\frac{\mu_1 + \mu_2}{2}, m\eta + (1 - m)\frac{\eta_1 + \eta_2}{2}\Big)d\zeta d\mu d\eta \\ &= \sigma^s H(t_1, r, m) + \delta^s H(t_2, r, m). \end{aligned}$$

Likewise, if $t, m \in [0, 1]$ is fixed, then for all $r_1, r_2 \in [0, 1]$ and $\sigma, \delta \geq 0$ with $\sigma + \delta = 1$, we have $H(t, \sigma r_1 + \delta r_2, m) \leq \sigma^s H(t, r_1, m) + \delta^s H(t, r_2, m)$, and if $t, r \in [0, 1]$ is fixed, then for all $m_1, m_2 \in [0, 1]$ and $\sigma, \delta \geq 0$ with $\sigma + \delta = 1$, we have $H(t, r, \sigma m_1 + \delta m_2) \leq \sigma^s H(t, r, m_1) + \delta^s H(t, r, m_2)$.
Hence, $H(t, r, m)$ is co-ordinated s-convex on $[0, 1] \times [0, 1] \times [0, 1]$.
(b). We have

$$\begin{aligned} H(t, r, m) &= \frac{1}{(\zeta_2 - \zeta_1)(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\zeta_1}^{\zeta_2} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi\Big(t\zeta + (1 - t)\frac{\zeta_1 + \zeta_2}{2}, \\ &r\mu + (1 - r)\frac{\mu_1 + \mu_2}{2}, m\eta + (1 - m)\frac{\eta_1 + \eta_2}{2}\Big)d\zeta d\mu d\eta \\ &\geq \frac{1}{(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi\Big(\frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} (t\zeta + (1 - t)\frac{\zeta_1 + \zeta_2}{2})d\zeta, \\ &r\mu + (1 - r)\frac{\mu_1 + \mu_2}{2}, m\eta + (1 - r)\frac{\eta_1 + \eta_2}{2}\Big)d\zeta d\mu d\eta \\ &= \frac{1}{(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \chi\Big(\frac{\zeta_1 + \zeta_2}{2}, r\mu + (1 - r)\frac{\mu_1 + \mu_2}{2}, m\eta + (1 - m)\frac{\eta_1 + \eta_2}{2}\Big)d\mu d\eta \\ &\geq \frac{1}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} \chi\Big(\frac{\zeta_1 + \zeta_2}{2}, \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} (r\mu + (1 - r)\frac{\mu_1 + \mu_2}{2})d\mu, m\eta + (1 - m)\frac{\eta_1 + \eta_2}{2}\Big)d\eta \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} \chi\Big(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, m\eta+(1-m)\frac{\eta_1+\eta_2}{2}\Big)d\eta \\
&\geq \chi\Big(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}(m\eta+(1-m)\frac{\eta_1+\eta_2}{2})d\eta\Big) \\
&= \chi(\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, \frac{\eta_1+\eta_2}{2}) = H(0,0,0).
\end{aligned}
$$

Thus, we have shown that the generic function $H(t,r,m)$ has the greatest lower bound $H(0,0,0)$. Therfore we have $\inf_{(t,r,m)\in[0,1]^3} H(t,r,m) = H(0,0,0)$.

Next, we have

$$
\begin{aligned}
H(t,r,m) &= \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2} \chi\Big(t\zeta+(1-t)\frac{\zeta_1+\zeta_2}{2}, \\
&\qquad r\mu+(1-r)\frac{\mu_1+\mu_2}{2}, m\eta+(1-m)\frac{\eta_1+\eta_2}{2}\Big)d\zeta d\mu d\eta \\
&\leq \frac{1}{(\zeta_2-\zeta_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\eta_1}^{\eta_2}\Big\{r\cdot\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi\Big(t\zeta+(1-t)\frac{\zeta_1+\zeta_2}{2}, \mu, m\eta \\
&\quad +(1-m)\frac{\eta_1+\eta_2}{2}\Big)d\mu + (1-r)\cdot\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi\Big(t\zeta+(1-t)\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, m\eta \\
&\qquad +(1-m)\frac{\eta_1+\eta_2}{2}\Big)d\mu\Big\}d\zeta d\eta \\
&\leq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\Big[r\cdot\frac{1}{\mu_2-\mu_1}\Big\{m\cdot\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi\Big(t\zeta+(1-t)\frac{\zeta_1+\zeta_2}{2}, \mu, \eta\Big)d\eta \\
&\qquad +(1-m)\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi\Big(t\zeta+(1-t)\frac{\zeta_1+\zeta_2}{2}, \mu, \frac{\eta_1+\eta_2}{2}\Big)d\eta\Big\}d\mu \\
&\quad +(1-r)\cdot\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Big\{m\cdot\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\chi\Big(t\zeta+(1-t)\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, \eta\Big)d\eta \\
&\qquad +(1-m)\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} f\Big(t\zeta+(1-t)\frac{\zeta_1+\zeta_2}{2}, \frac{\mu_1+\mu_2}{2}, \frac{\eta_1+\eta_2}{2}\Big)d\eta\Big\}d\mu\Big]d\zeta
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
H(t,r,m) &\leq r\cdot\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Big[m\cdot\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\Big\{t\cdot\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\mu,\eta)d\zeta \\
&\qquad +(1-t)\cdot\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2}, \mu, \eta\Big)d\zeta\Big\}d\eta \\
&\quad +(1-m)\cdot\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\Big\{t\cdot\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\zeta, \mu, \frac{\eta_1+\eta_2}{2}\Big)d\zeta \\
&\qquad +(1-t)\cdot\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2}, \mu, \frac{\eta_1+\eta_2}{2}\Big)d\zeta\Big\}d\eta\Big]d\mu
\end{aligned}
$$

$$+(1-r)\cdot\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Big[m\cdot\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\Big\{t\cdot\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi(\zeta,\frac{\mu_1+\mu_2}{2},\eta)d\zeta$$
$$+(1-t)\cdot\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\eta\Big)d\zeta\Big\}d\eta$$
$$+(1-m)\cdot\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\Big\{t\cdot\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\zeta,\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)d\zeta$$
$$+(1-t)\cdot\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)d\zeta\Big\}d\eta\Big]d\mu.$$

Further simplying the above inequality, we have

$$H(t,r,m)\le trm\cdot\frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta$$
$$+(1-t)rm\cdot\frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\mu,\eta\Big)d\zeta d\eta$$
$$+tr(1-m)\cdot\frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\chi\Big(\zeta,\mu,\frac{\eta_1+\eta_2}{2}\Big)d\zeta d\mu$$
$$+(1-t)r(1-m)\cdot\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\mu,\frac{\eta_1+\eta_2}{2}\Big)d\mu$$
$$+t(1-r)m\cdot\frac{1}{(\zeta_2-\zeta_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\eta_1}^{\eta_2}\chi\Big(\zeta,\frac{\mu_1+\mu_2}{2},\eta\Big)d\zeta d\eta$$
$$+(1-t)(1-r)m\cdot\frac{1}{(\eta_2-\eta_1)}\int_{\eta_1}^{\eta_2}\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\eta\Big)d\eta$$
$$+t(1-r)(1-m)\cdot\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\chi\Big(\zeta,\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)d\zeta$$
$$+(1-t)(1-r)(1-m)\chi\Big(\frac{\zeta_1+\zeta_2}{2},\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2}\Big)$$
$$\le\Big\{trm+(1-t)rm+tr(1-m)+(1-t)r(1-m)+t(1-r)m+$$
$$(1-t)(1-r)m+t(1-r)(1-m)+(1-t)(1-r)(1-m)\Big\}\cdot$$
$$\frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\chi(\zeta,\mu,\eta)d\zeta d\mu d\eta=H(1,1,1).$$

(Note: The terms in the curly braces add up to 1).
Therefore, $H(t,r,m)$ has the upper bound $H(1,1,1)$, which clearly also makes it the least upper bound of $H(t,r,m)$. □

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