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Algebra of generalized (k, n)-Fibonacci Toeplitz and Hankel matrices

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ABSTRACT: The aim of this paper is to investigate the generalized (k, n)-Fibonacci Toeplitz and Hankel matrices formed with the entries of the generalized Fibonacci sequence of order k. We obtain the determinant, trace, inverse, spread, and some algebraic properties for these matrices in closed form. Moreover, we obtain the $||.||_1, ||.||_\infty$, Euclidean norm and bounds (both lower and upper) for the spectral norm.

Key Words: k-step Fibonacci sequence, Toeplitz matrix, Hankel matrix, norms, determinant, spread.

Contents

1. Introduction

In recent years, there has been much interest in the study of the generalization of existing number sequences (like Fibonacci, Lucas, Pell, Mersenne, etc.), obtaining identities and their application in matrix theory and other branches of science [4,10,11,17].

Recently, Akbulak and Bozkurt [1] studied the bounds for the spectral norms of Toeplitz matrices formed with Fibonacci and Lucas numbers of order 2. Later, Shou Q. Shen [20] extended this study for k-Fibonacci and k-Lucas numbers of order 2 and studied the bounds for norms of the corresponding Toeplitz matrices. Karpuz et al. [9] studied Hankel and Toeplitz matrices involving Pell numbers and obtained both lower and upper bounds for the spectral norms. Halıcı [7] presented a study on lower and upper bounds for the spectral norms of Hankel matrices with Pell numbers. Soykan and Gocen [23] and Kumari et al. [13] studied some algebraic properties of circulant matrices with generalized 3-primes and Mersenne numbers, respectively. These types of special matrices are used in various areas like signal analysis, image processing, coding theory, vibration analysis, etc. [2,12,15,18,21,24,26] and many other problems. Some recent studies on circulant, Toeplitz and Hankel matrices, their norms and bounds for spectral norms can be seen in [3,5,6,19,22]. Most of the works in this direction are done for a special number sequence of orders two and three.

In this study, we consider the generalized Fibonacci sequence of order $k \geq 2$. First, we establish a recursive matrix with entries from the generalized Fibonacci sequence, called the (k, n)-Fibonacci Toeplitz matrix and investigate its algebraic properties and different norms on it. Furthermore, we extend our study to the associated (k, n)-Hankel matrix.

The generalized Fibonacci sequence $\{f_{k,n}\}$ of order $k(\geq 2) \in \mathbb{N}$ is defined recursively as

$$f_{k,k+n} = f_{k,k+n-1} + f_{k,k+n-2} + f_{k,k+n-3} + ... + f_{k,n+1} + f_{k,n}, \quad n \geq 0, \tag{1.1}$$

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where $f_{k,0} = f_{k,1} = ... = f_{k,k-2} = 0$ and $f_{k,k-1} = 1$.

It is also known as k-step Fibonacci sequence. The associated generalized k-Fibonacci matrix denoted by Q_k^n is defined [16] as

$$Q_k^n = \begin{bmatrix} f_{k,n+k-1} & f_{k,n+k-2} + f_{k,n+k-3} + ... + f_{k,n} & f_{k,n+k-2} + ... + f_{k,n+1} & \cdots & f_{k,n+k-2} \\ f_{k,n+k-2} & f_{k,n+k-3} + f_{k,n+k-4} + ... + f_{k,n-1} & f_{k,n+k-3} + ... + f_{k,n} & \cdots & f_{k,n+k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{k,n+1} & f_{k,n} + f_{k,n-1} + ... + f_{k,-k+n+2} & f_{k,n} + ... + f_{k,-k+n+3} & \cdots & f_{k,n} \\ f_{k,n} & f_{k,n-1} + f_{k,n-2} + ... + f_{k,-k+n+1} & f_{k,n-1} + ... + f_{k,-k+n+2} & \cdots & f_{k,n-1} \end{bmatrix} \quad (1.2)$$

where the generator (initial) Fibonacci matrix Q_k is given by

$$Q_k = Q_k^1 = \begin{bmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & ... & 1 & 0 \end{bmatrix}.$$

Theorem 1.1 *[16] Let $k(\geq 2) \in \mathbb{N}$. Then, for each $m, n \in \mathbb{Z}$ the following relations hold*

$$Q_k^0 = I_k, \quad (Q_k^1)^n = Q_k^n, \quad (Q_k^n)^{-1} = Q_k^{-n}, \quad Q_k^m Q_k^n = Q_k^{m+n} \quad \text{and} \quad \det(Q_k^n) = (-1)^{(k-1)n}.$$

1.1. Norms and bounds

Definition 1.1 *[25] Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ be a rectangular matrix. Then, for matrix A, norms are defined as*

$$||A||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \quad \text{and} \quad ||A||_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|. \quad (1.3)$$

$$||A||_M = \max(m,n) \max_{i,j} |a_{ij}| \qquad (\text{maximum norm}) \quad (1.4)$$

$$||A||_G = \sqrt{mn} \max_{i,j} |a_{ij}| \qquad (\text{G-norm/geometric mean norm}) \quad (1.5)$$

Definition 1.2 *[8] For a matrix $A = [a_{ij}]$ of size $m \times n$, the Euclidean (Frobenius) norm and the spectral norm of A are defined by*

$$||A||_E = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad ||A||_2 = \sqrt{\max_{1 \leq i \leq n} |\lambda_i|}, \qquad \text{respectively.}$$

where λ_i's are the eigenvalues of $A^\theta A$ and the matrix A^θ is tranjugate of the matrix A.

A inequalities between norms are given as

$$\frac{1}{\sqrt{n}} ||A||_E \leq ||A||_2 \leq ||A||_E. \quad (1.6)$$

2. Fibonacci Toeplitz Matrices

Definition 2.1 *A Toeplitz matrix of order k is a square matrix of the form*

$$T = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-k+2} & t_{-k+1} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-k+3} & t_{-k+2} \\ t_2 & t_1 & t_0 & \cdots & t_{-k+4} & t_{-k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k-2} & t_{k-3} & t_{k-4} & \cdots & t_0 & t_{-1} \\ t_{k-1} & t_{k-2} & t_{k-3} & \cdots & t_1 & t_0 \end{bmatrix}_{k \times k}, \quad (2.1)$$

where its entries are $t_{ij} = t_{i-j},\ 1 \leq i,j \leq k$ *for an infinite sequence* $\{t_n\}_{n\in\mathbb{Z}}$. *In other words, a square matrix of order* k *is a Toeplitz matrix if all the entries along each of the* $2k-1$ *diagonals are the same and the elements in the first row and the first column are successive terms of a sequence.*

Transformation of generalized k-Fibonacci matrices Q_k^n into Toeplitz matrices: Consider the generalized k-Fibonacci matrix Q_k^n as defined in (1.2) whose elements are linear combinations of terms of the generalized Fibonacci sequence.

Now, using the column operations $C_1' \leftarrow C_1$, $C_2' \leftarrow C_2 + C_1'$ on Q_k^n and Eqn.(1.1), we get

$$Q_k^n \sim \begin{bmatrix} f_{k,k+n-1} & f_{k,k+n} & f_{k,k+n-2}+\ldots+f_{k,n+1} & \cdots & f_{k,k+n-2} \\ f_{k,k+n-2} & f_{k,k+n-1} & f_{k,k+n-3}+\ldots+f_{k,n} & \cdots & f_{k,k+n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{k,k+n-(k-1)} & f_{k,n+2} & f_{k,n}+\ldots+f_{k,-k+n+3} & \cdots & f_{k,n} \\ f_{k,n} & f_{k,n+1} & f_{k,n-1}+\ldots+f_{k,-k+n+2} & \cdots & f_{k,n-1} \end{bmatrix}.$$

Similarly, applying the column operations $C_3' \leftarrow C_3 + C_2' + C_1'$, $C_4' \leftarrow C_4 + C_3' + C_2' + C_1'$, and $C_k' \leftarrow C_k + C_{k-1}' + \ldots + C_2' + C_1'$, successively on Q_k^n along with Eqn. (1.1), we obtain

$$Q_k^n \sim \begin{bmatrix} f_{k,n+k-1} & f_{k,n+k} & f_{k,n+k+1} & \cdots & f_{k,n+2k-3} & f_{k,n+2k-2} \\ f_{k,n+k-2} & f_{k,n+k-1} & f_{k,n+k} & \cdots & f_{k,n+2k-4} & f_{k,n+2k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{k,n+1} & f_{k,n+2} & f_{k,n+1} & \cdots & f_{k,n+k-1} & f_{k,n+k} \\ f_{k,n} & f_{k,n+1} & f_{k,n+2} & \cdots & f_{k,n+k-2} & f_{k,n+k-1} \end{bmatrix}.$$

We denote this last matrix by the symbol $FT_k^{(n)}$. Note that the transpose of $FT_k^{(n)}$ satisfies the definition of the Toeplitz matrix and we call it the Fibonacci Toeplitz matrix whose entries are terms of the generalized Fibonacci sequence of order k.

2.1. Construction, determinant and inverse

Definition 2.2 **((k,n)-Fibonacci Toeplitz matrix)** *A square matrix* $T = [t_{ij}]_{1\leq i,j\leq k}$ *is called* (k,n)-*Fibonacci Toeplitz matrix if* $t_{ij} = f_{k,n+(k-1)+(j-i)}$.

Remark 2.1 *Usually a general Toeplitz matrix of order* k *has a degree of freedom (number of independent entries)* $2k-1$, *but here Fibonacci Toeplitz matrix* $FT_k^{(n)}$ *has a degree of freedom* k.

Note that the matrix $FT_k^{(n)}$ is constructed with terms of the k-step Fibonacci sequence. So once we have k consecutive terms of the sequence as elements, the remaining elements can be obtained easily using the corresponding recurrence relation. For simplicity, we use FT to represent the Fibonacci Toeplitz matrices.

Lemma 2.1 *Let* FT *be any Fibonacci Toeplitz matrix and let* $R_i(FT)$ *and* $C_j(FT)$ *denote the* i^{th} *row sum and* j^{th} *column sum of* FT, *respectively. Then*

$$R_1(FT) > R_2(FT) > \ldots > R_k(FT)$$
$$\textit{and} \qquad C_1(FT) < C_2(FT) < \ldots < C_k(FT).$$

Moreover,

$$R_1(FT) = C_k(FT), R_2(FT) = C_{k-1}(FT), \ldots, R_k(FT) = C_1(FT).$$

The following example shows the nth Fibonacci Toeplitz matrices of order four and five.

Example 2.1 *The* (k,n)-*Fibonacci Toeplitz matrices of order* $k=4$ *and* $k=5$ *are*

$$FT_4^{(n)} = \begin{bmatrix} f_{4,n+3} & f_{4,n+4} & f_{4,n+5} & f_{4,n+6} \\ f_{4,n+2} & f_{4,n+3} & f_{4,n+4} & f_{4,n+5} \\ f_{4,n+1} & f_{4,n+2} & f_{4,n+3} & f_{4,n+4} \\ f_{4,n} & f_{4,n+1} & f_{4,n+2} & f_{4,n+3} \end{bmatrix} \quad \textit{and} \quad FT_5^{(n)} = \begin{bmatrix} f_{5,n+4} & f_{5,n+5} & f_{5,n+6} & f_{5,n+7} & f_{5,n+8} \\ f_{5,n+3} & f_{5,n+4} & f_{5,n+5} & f_{5,n+6} & f_{5,n+7} \\ f_{5,n+2} & f_{5,n+3} & f_{5,n+4} & f_{5,n+5} & f_{5,n+6} \\ f_{5,n+1} & f_{5,n+2} & f_{5,n+3} & f_{5,n+4} & f_{5,n+5} \\ f_{5,n} & f_{5,n+1} & f_{5,n+2} & f_{5,n+3} & f_{5,n+4} \end{bmatrix}.$$

The initial FT-matrix $FT_k^{(0)}$ is an upper triangular matrix whose diagonal and super-diagonal entries are 1 and the rest non-zero entries are some powers of 2, therefore its determinant is 1. It is of the form.

$$FT_k^{(0)} = \begin{bmatrix} 1 & 1 & 2 & \dots & 2^{k-3} & 2^{k-2} \\ 0 & 1 & 1 & \dots & 2^{k-4} & 2^{k-3} \\ 0 & 0 & 1 & \dots & 2^{k-5} & 2^{k-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 2^1 \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}. \tag{2.2}$$

Lemma 2.2 *Let R be a $k \times k$ upper triangular matrix whose diagonal entries are 1 and other non-zero entries are -1, i.e. matrix of the form*

$$R = \begin{bmatrix} 1 & -1 & -1 & \dots & -1 & -1 \\ 0 & 1 & -1 & \dots & -1 & -1 \\ 0 & 0 & 1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

then R is the inverse of initial FT-matrix $FT_k^{(0)}$.

Proof: Since $FT_k^{(0)} R = I_k = RFT_k^{(0)}$ for all k, so proof is trivial. □

For instance, let $k = 4,\ 5$, then initial FT-matrices $FT_4^{(0)}$, $FT_5^{(0)}$ and their respective inverse are given in the following examples:

$$FT_4^{(0)} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$FT_5^{(0)} = \begin{bmatrix} 1 & 1 & 2 & 4 & 8 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Corollary 2.1 *For $k \geq 2$, determinant of the initial Fibonacci Toeplitz matrices $FT_k^{(0)}$ is 1.*

It is worth to note the following result from [17] (Corollary 3.4) which we use later.

Lemma 2.3 *Let Q_k^1 be the Fibonacci matrix and A be any matrix of the same size, then first row of $Q_k^1 A$ is the sum of the corresponding column of A and row-2 to row-(k) is row-1 to row-(k-1) of A.*

Consider the initial Fibonacci Toeplitz matrix $FT_k^{(0)}$ and Lemma 2.3, then inductive hypothesis on n proves the following theorem concern with the direct generalized formula for Fibonacci Toeplitz matrices.

Theorem 2.1 *Let $n \in \mathbb{N}$ and $k \geq 2$, then we have*

$$FT_k^{(n)} = Q_k^n FT_k^{(0)}.$$

Theorem 2.2 *Let $n \in \mathbb{N}$, then for $k \geq 2$, the determinant of Fibonacci Toeplitz matrices is*

$$\det(FT_k^{(n)}) = (-1)^{n(k-1)}.$$

Proof: From Theorem 2.1, Theorem 1.1 and Corollary 2.1, we have

$$\det(FT_k^{(n)}) = \det(Q_k^n FT_k^{(0)}) = \det(Q_k^n)\det(FT_k^{(0)}) = (-1)^{n(k-1)}.$$

□

Theorem 2.3 *The trace of Fibonacci Toeplitz matrices $FT_k^{(n)}$ is $kf_{k,n+k-1}$.*

Proof: Note that the Fibonacci Toeplitz matrix is a diagonal constant matrix and here the diagonal elements are $f_{k,n+k-1}$. So by using the fact that the trace of a square matrix can be achieved by taking sum of its diagonal entries, the proof is trivial. □

From Theorem 2.2, it is clear that (k, n)-Fibonacci Toeplitz matrices are non-singular irrespective of those n and k. Therefore, inverse of $FT_k^{(n)}$ exists for all $n \in \mathbb{Z}$ and $k \geq 2$. Hence the following theorem.

Theorem 2.4 (Inverse of $FT_k^{(n)}$) *For non-negative integer n and integer $k \geq 2$, we have*

$$(FT_k^{(n)})^{-1} = RQ_k^{-n}.$$

Proof: From Theorem 2.1 and Theorem 1.1, we write

$$\begin{aligned}(FT_k^{(n)})^{-1} &= (Q_k^n FT_k^{(0)})^{-1}\\ &= (FT_k^{(0)})^{-1}(Q_k^n)^{-1}\\ &= RQ_k^{-n}.\end{aligned}$$

□

2.2. Matrix norms

Theorem 2.5 *For (k, n)-Fibonacci Toeplitz matrices, we have*

$$||FT_k^{(n)}||_1 = ||FT_k^{(n)}||_\infty = f_{k,n+2k-1}.$$

Proof: The proof is trivial by Definition 1.1 along with Eqn. (1.1) and Lemma 2.1. □

Euclidean norm. The Euclidean norm for (k, n)-Fibonacci Toeplitz matrices is given by

$$||FT_k^{(n)}||_E = \sqrt{\sum_{i,j=0}^{n-1} |t_{ij}|^2},$$

which gives

$$||FT_k^{(n)}||_E^2 = kf_{k,n+k-1}^2 + \sum_{i=1}^{k-1}(k-i)f_{k,n+k-1-i}^2 + \sum_{i=1}^{k-1}(k-i)f_{k,n+k-1+i}^2. \tag{2.3}$$

Theorem 2.6 *The lower and upper bounds for spectral norm of matrices $FT_k^{(n)}$ are given by*

$$\begin{aligned}||FT_k^{(n)}||_2 &\geq \frac{1}{\sqrt{n}}\sqrt{kf_{k,n+k-1}^2 + \sum_{i=1}^{k-1}(k-i)f_{k,n+k-1-i}^2 + \sum_{i=1}^{k-1}(k-i)f_{k,n+k-1+i}^2},\\ ||FT_k^{(n)}||_2 &\leq \sqrt{\Big(1 + \sum_{r=0}^{k-2} f_{k,n+k+(r-1)}^2\Big)\Big(\sum_{r=0}^{k-1} f_{k,n+k+(r-1)}^2\Big)}.\end{aligned}$$

Proof: The lower bound for spectral norm follows using Eqn. (2.3) in relation (1.6), i.e

$$||FT_k^{(n)}||_2 \geq \frac{1}{\sqrt{n}}\sqrt{kf_{k,n+k-1}^2 + \sum_{i=1}^{k-1}(k-i)f_{k,n+k-1-i}^2 + \sum_{i=1}^{k-1}(k-i)f_{k,n+k-1+i}^2}.$$

Now to obtain the upper bound for the spectral norm, we use the following result. Let us first define the maximum row and column length norm on square matrices $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ and $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ as

$$r(A) = \max_i \sqrt{\sum_{j=1}^{n}|a_{ij}|^2} \quad \text{and} \quad c(B) = \max_j \sqrt{\sum_{i=1}^{n}|b_{ij}|^2}, \quad \text{respectively.}$$

If $C = A \circ B$ be the Hadamard product of A and B (entry-wise product), then we have

$$||C||_2 \leq r(A)c(B). \tag{2.4}$$

Now, let A and B be two matrices as defined below such that they satisfy $FT_k^{(n)} = A \circ B$,

$$A = (a_{ij}) = \begin{cases} a_{ij} = 1 & j = k, \\ a_{ij} = f_{k,n+(k-1)+(j-i)} & j \neq k, \end{cases} \quad \text{and} \quad B = (b_{ij}) = \begin{cases} b_{ij} = 1 & j \neq k, \\ b_{ij} = f_{k,n+(k-1)+(j-i)} & j = k. \end{cases}$$

Then, clearly

$$r(A) = \max_i \sqrt{\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{1 + \sum_{r=0}^{k-2} f_{k,n+k+(r-1)}^2}$$

and

$$c(B) = \max_j \sqrt{\sum_{i=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{r=0}^{k-1} f_{k,n+k+(r-1)}^2}.$$

Hence from Eqn. (2.4), the upper bound for the spectral norm is given by

$$||FT_k^{(n)}||_2 \leq r(A)c(B) = \sqrt{\Big(1 + \sum_{r=0}^{k-2} f_{k,n+k+(r-1)}^2\Big)\Big(\sum_{r=0}^{k-1} f_{k,n+k+(r-1)}^2\Big)}.$$

Thus, this completes the proof. □

3. Fibonacci Hankel Matrices

Let $FT_k^{(n)}$ be a (k,n)-Fibonacci Toeplitz matrix and J be the exchange matrix of same order then the nth Fibonacci Hankel matrix of order k denoted by $FH_k^{(n)}$ is defined as $FH_k^{(n)} = FT_k^{(n)}J$, (the usual matrix multiplication) and it is called (k,n)-Fibonacci Hankel matrix. Thus, we have the following remark.

Remark 3.1 *The Fibonacci Hankel matrices $FH_k^{(n)}$ are permutations of columns of $FT_k^{(n)}$, where permutations take places as $C_j \leftrightarrow C_{k-j+1}$ for $1 \leq j \leq k$, i.e. if $C_1, C_2, ..., C_k$ are columns of $FT_k^{(n)}$ then $C_k, C_{k-1}, ..., C_1$ are columns of $FH_k^{(n)}$.*

Definition 3.1 **((k,n)-Fibonacci Hankel matrix)** *The (k,n)-Fibonacci Hankel matrix of order k is a square matrix of the form*

$$FH_k^{(n)} = \begin{bmatrix} f_{k,n+2k-2} & f_{k,n+2k-3} & \cdots & f_{k,n+k+1} & f_{k,n+k} & f_{k,n+k-1} \\ f_{k,n+2k-3} & f_{k,n+2k-4} & \cdots & f_{k,n+k} & f_{k,n+k-1} & f_{k,n+k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ f_{k,n+k} & f_{k,n+k-1} & \cdots & f_{k,n+1} & f_{k,n+2} & f_{k,n+1} \\ f_{k,n+k-1} & f_{k,n+k-2} & \cdots & f_{k,n+2} & f_{k,n+1} & f_{k,n} \end{bmatrix}.$$

For simplicity, we denote it by FH.

For $n = 0$, we get the initial Fibonacci Hankel matrix $FH_k^{(0)}$ which is of the form

$$FH_k^{(0)} = \begin{bmatrix} 2^{k-2} & 2^{k-3} & 2^{k-4} & \dots & 1 & 1 \\ 2^{k-3} & 2^{k-4} & 2^{k-5} & \dots & 1 & 0 \\ 2^{k-4} & 2^{k-5} & 2^{k-6} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{3.1}$$

Lemma 3.1 *Let $S = (s_{ij})_{i,j=1}^k$ be a $k \times k$ matrix with entries* $s_{ij} = \begin{cases} 0 & : i+j < k+1, \\ 1 & : i+j = k+1, \\ -1 & : i+j > k+1, \end{cases}$ *i.e. matrix of the form*

$$S = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & -1 & \dots & -1 & -1 \\ 1 & -1 & -1 & \dots & -1 & -1 \end{bmatrix},$$

then S is the inverse of initial FH-matrix $FH_k^{(0)}$.

Lemma 3.2 *For $k \geq 2$, the determinant of initial Fibonacci Hankel matrix is given by*

$$\det(FH_k^{(0)}) = (-1)^{k(k-1)/2}.$$

Example 3.1 *The nth-Fibonacci Hankel matrix of order four and five are, respectively,*

$$\begin{bmatrix} f_{4,n+6} & f_{4,n+5} & f_{4,n+4} & f_{4,n+3} \\ f_{4,n+5} & f_{4,n+4} & f_{4,n+3} & f_{4,n+2} \\ f_{4,n+4} & f_{4,n+3} & f_{4,n+2} & f_{4,n+1} \\ f_{4,n+3} & f_{4,n+2} & f_{4,n+1} & f_{4,n} \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} f_{5,n+8} & f_{5,n+7} & f_{5,n+6} & f_{5,n+5} & f_{5,n+4} \\ f_{5,n+7} & f_{5,n+6} & f_{5,n+5} & f_{5,n+4} & f_{5,n+3} \\ f_{5,n+6} & f_{5,n+5} & f_{5,n+4} & f_{5,n+3} & f_{5,n+2} \\ f_{5,n+5} & f_{5,n+4} & f_{5,n+3} & f_{5,n+2} & f_{5,n+1} \\ f_{5,n+4} & f_{5,n+3} & f_{5,n+2} & f_{5,n+1} & f_{5,n} \end{bmatrix}.$$

Corollary 3.1 *For Fibonacci Hankel matrix FH, let $R_i(FH)$ and $C_j(FH)$ denote the i^{th} row sum and j^{th} column sum of FH, respectively. Then, we have*

$$C_1(FH) > C_2(FH) > ... > C_k(FH),$$
$$R_1(FH) > R_2(FH) > ... > R_k(FH).$$

Furthermore, $R_1(FH) = C_1(FH), R_2(FH) = C_2(FH), ..., R_k(FH) = C_k(FH).$

Theorem 3.1 *The trace of Fibonacci Hankel matrix $FH_k^{(n)}$ is* $\sum_{i=0}^{k-1} f_{k,n+2i}$.

Proof: From Definition 3.1, the trace of $FH_k^{(n)}$ is given by

$$\begin{aligned} \operatorname{tr}(FH_k^{(n)}) &= f_{k,n} + f_{k,n+2} + f_{k,n+4} + ... + f_{k,n+2(k-1)} \\ &= \sum_{i=0}^{k-1} f_{k,n+2i} \end{aligned}$$

as required. □

Theorem 3.2 *Let $n \in \mathbb{N}$ and $k \geq 2$, then we have*

$$FH_k^{(n)} = Q_k^n FH_k^{(0)}.$$

Proof: Using the identity $FH_k^{(n)} = FT_k^{(n)} J$ and Theorem 2.1, we have

$$FH_k^{(n)} = FT_k^{(n)} J = Q_k^n FT_k^{(0)} J = Q_k^n FH_k^{(0)}$$

as required. □

Theorem 3.3 *Let $n \in \mathbb{N}$, then for $k \geq 2$, determinant of the (k,n)-Fibonacci Hankel matrices is*

$$\det(FH_k^{(n)}) = \begin{cases} (-1)^{k(k-1)/2} & : \text{if } n \text{ is even or } k \text{ is odd,} \\ (-1)^{k(k-1)/2+1} & : \text{otherwise.} \end{cases}$$

Proof: From Theorem 3.2 and Lemma 3.2, we write

$$\begin{aligned} \det(FH_k^{(n)}) &= \det(Q_k^n FH_k^{(0)}) \\ &= \det(Q_k^n) det(FH_k^{(0)}) \\ &= (-1)^{(k-1)n}(-1)^{k(k-1)/2} \\ &= (-1)^{(2n+k)(k-1)/2}. \end{aligned}$$

Thus, this completes the proof. □

From Theorem 3.3, the determinant being non-zero for all values of n and k implies that Fibonacci Hankel matrices $FH_k^{(n)}$ are invertible. Hence the inverse is given by

$$(FH_k^{(n)})^{-1} = SQ_k^{-n}.$$

3.1. Matrix norms

Note that from Remark 3.1, the Euclidean norm of (k,n)-Fibonacci Toeplitz matrices $FT_k^{(n)}$ and (k,n)-Fibonacci Hankel matrices $FH_k^{(n)}$ are same. Therefore from relation (2.3), we write

$$||FH_k^{(n)}||_E = \sqrt{k f_{k,n+k-1}^2 + \sum_{i=1}^{k-1}(k-i) f_{k,n+k-1-i}^2 + \sum_{i=1}^{k-1}(k-i) f_{k,n+k-1+i}^2}. \tag{3.2}$$

Theorem 3.4 *For (k,n)-Fibonacci Hankel matrices $FH_k^{(n)}$, we have*

$$||FH_k^{(n)}||_1 = ||FH_k^{(n)}||_\infty = f_{k,n+2k-1}.$$

Proof: The proof immediately follows from Corollary 3.1. □

Argument for next theorem is similar to Theorem 2.6, so we omit the proof.

Theorem 3.5 *The lower and upper bounds for the spectral norm of matrices $FH_k^{(n)}$ are*

$$\begin{aligned} ||FH_k^{(n)}||_2 &\geq \frac{1}{\sqrt{n}} \sqrt{k f_{k,n+k-1}^2 + \sum_{i=1}^{k-1}(k-i) f_{k,n+k-1-i}^2 + \sum_{i=1}^{k-1}(k-i) f_{k,n+k-1+i}^2}, \\ ||FH_k^{(n)}||_2 &\leq \sqrt{\Big(1 + \sum_{r=1}^{k-1} f_{k,n+k+(r-2)}^2\Big)\Big(\sum_{r=1}^{k} f_{k,n+k+(r-2)}^2\Big)}. \end{aligned}$$

Note that the Fibonacci Hankel matrices are the permutation of columns of Fibonacci Toeplitz matrices so the maximum and geometric mean norm for these matrices are same and given by

$$||FT_k^{(n)}||_M = ||FT_k^{(n)}||_G = kf_{k,n+2k-2},$$
$$\text{and} \qquad ||FH_k^{(n)}||_M = ||FH_k^{(n)}||_G = kf_{k,n+2k-2}.$$

i.e maximum norm and geometric mean norm for these matrices are same.

4. Spread

To solve the problem of estimation of maximum distance between two eigenvalues, L. Mirsky [14] introduced the concept of spread. The spread (S) of a complex matrix (of order n) is defined as

$$S(A) = \max_{i,j} |\alpha_i - \alpha_j|,$$

where $\alpha_1, \alpha_2, \ldots \alpha_n$ are the eigenvalues of a matrix $A \in \mathbb{M}_n(\mathbb{C})$.

If $tr(A)$ represents the trace of a matrix A, then the upper bound for the spread is given by

$$S(A) \leq \sqrt{2||A||_E^2 - \frac{2}{n}|tr(A)|^2} \tag{4.1}$$

Theorem 4.1 *Upper bound for the spread of Fibonacci Toeplitz and Fibonacci Hankel matrices, $FT_k^{(n)}$ and $FH_k^{(n)}$ are given by*

$$S(FT_k^{(n)}) \leq \sqrt{f_{k,n+k-1}^2(2k - 2k^2/n) + 2\sum_{i=1}^{k-1}(k-i)f_{k,n+k-1-i}^2 + 2\sum_{i=1}^{k-1}(k-i)f_{k,n+k-1+i}^2},$$
$$S(FH_k^{(n)}) \leq \sqrt{2kf_{k,n+k-1}^2 + 2\sum_{i=1}^{k-1}(k-i)f_{k,n+k-1-i}^2 + 2\sum_{i=1}^{k-1}(k-i)f_{k,n+k-1+i}^2 - \frac{2}{n}\Big(\sum_{i=0}^{k-1} f_{k,n+2i}\Big)^2}.$$

Proof: For Fibonacci Toeplitz matrices, using Eqn. (2.3) and Theorem 2.3 in Eqn. (4.1), we get

$$S(FT_k^{(n)}) \leq \sqrt{2kf_{k,n+k-1}^2 + 2\sum_{i=1}^{k-1}(k-i)f_{k,n+k-1-i}^2 + 2\sum_{i=1}^{k-1}(k-i)f_{k,n+k-1+i}^2 - \frac{2(kf_{k,n+k-1})^2}{n}}$$
$$\leq \sqrt{f_{k,n+k-1}^2(2k - 2k^2/n) + 2\sum_{i=1}^{k-1}(k-i)f_{k,n+k-1-i}^2 + 2\sum_{i=1}^{k-1}(k-i)f_{k,n+k-1+i}^2}.$$

Similarly, using Eqn. (3.2) and Theorem 3.1 in Eqn. (4.1), the second identity can be obtained. □

5. Conclusion

In this study, we proposed two recursive matrices namely (k, n)-Fibonacci Toeplitz matrix and (k, n)-Fibonacci Hankel matrix of higher order, whose entries are taken from the generalized Fibonacci sequence. Here, we obtained the determinant, trace, inverse, spread and some algebraic properties of these matrices in closed form. Moreover, we investigated different norms and bounds for the spectral norm of these matrices. This study can be extended for the generalized Lucas matrices given in [17,18] with the k-step Lucas sequence.

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