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Bernoulli wavelet operational matrix of fractional derivative through wavelet-polynomial transformation and its applications in solving fractional order differential equations

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ABSTRACT: In this paper, a new numerical method based on Bernoulli wavelets operational matrix for solving fractional differential equations (FDEs) is presented. The fractional derivative is described in the Caputo sense. The Bernoulli wavelets are first presented, then an operational matrix of fractional order derivative is derived through a wavelet-polynomial transformation matrix. This matrix is then used with collocation methods to reduce the linear and non-linear FDEs to a system of algebraic equations. Some numerical examples are illustrated to demonstrate the validity and applicability of the technique.

Key Words: Bernoulli wavelets, operational matrix of fractional derivatives, Caputo derivative, fractional order differential equations.

Contents

1. Introduction

The computation of fractional order is a generalization of ordinary differentiation and integration into an arbitrary order. The notion of fractional calculation began with some speculations of G.W. Leibniz (1695-1697) and L. Euler (1730) [15]. For the historical development of fractional differential operators see [30], [32]. Fractional derivatives provide more accurate models of real-world problems than integer derivatives. Fractional differential equations provide an excellent mathematical description for modeling many complex phenomena in various fields such as mechanics [36], biology [9], [2], chemistry [13], economics [9], control theory [36] and others. Due to the extensive applications of FDEs in engineering and science, research in this area has grown significantly, and there has been considerable interest in developing numerical schemes for their solution. Generally, most FDEs do not have exact solutions, and

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no known mechanism exists for obtaining accurate FDE solutions. In order to investigate the difficulty in obtaining an analytical solution to these equations, numerous authors have proposed an approximative method for the numerical solution of FDEs, such as Adomians decomposition methods [41], homotopy analysis method [18], [31] and Variational iteration method [46].
One of these numerical techniques is the wavelet method. The wavelets are a special kind of function with compact support that provides the basis for many important spaces. Therefore, it has a variety of areas of science and engineering such as mathematics, signal processing, etc. During the last three decades, various type of wavelet basis for solving differential or integral equations has gained more attention [24], [33], [42] and [37]. This numerical method reduces the solution of fractional order differential equations to the solution of algebraic equations using operational matrices. There are two approaches to achieving this, the initial approach involves using a fractional order integration matrix to turn FDEs into fractional integral equations, eliminating the need for integral operations, for example, Bernoulli polynomials [20], Bernoulli wavelets [26], Legendre polynomials [1], Chebyshev polynomials [8], Chebyshev wavelets [28], and Bernstein polynomials [40]. The second helpful approach for solving FDEs is to use an operational matrix of fractional derivatives to eliminate differential operators and simplify the problem into a system of algebraic equations.

In this paper, a new numerical approach method for solving the initial and boundary value problems of fractional order differential equations is presented. The method is based on Bernoulli wavelet approximation. we considered Bernoulli wavelets because of their numerous characteristics and ease of application to solving differential equations in general, and fractional differential equations in particular, as compared to other wavelets. First, this family is generated from orthogonal polynomials, which provide the wavelets with orthogonal properties as well as polynomial advantages. Second, their properties and derivative formula make it simple to generate the operational matrix of derivative and the operational matrix of derivative of fractional order. The Bernoulli wavelets are first constructed. Then, the operational matrix of the fractional order of derivative for Bernoulli wavelets is obtained by expanding these wavelets into Bernoulli polynomials. The method presented represents a simpler technique of obtaining the operational matrix with straightforward applicability to the FDEs in comparison to the existing operational matrices, most of which are obtained either by direct integration of the wavelet vector or through block pulse functions. This operational matrix of fractional order of derivative is used to reduce the fractional order differential equations to algebraic equations.
The outline of this paper is as follows: In Section 2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus. Section 3 is devoted to the basic formulation of wavelets and the Bernoulli wavelets. In Section 4, we derive the Bernoulli operational matrix of fractional order derivative in the Caputo sense. Section 5 is devoted to the numerical method for solving the initial and boundary value problems for fractional order differential equations, and in Section 6 we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering five numerical examples.

2. Preliminaries and notations

2.1. The fractional derivative in the Caputo sense

There are various definitions of fractional integration and derivatives of order μ . The widely used definition of a fractional integration is the Riemann–Liouville definition and of a fractional derivative is the Caputo definition. The Caputo fractional derivative uses initial and boundary conditions of integer order derivatives having some physical interpretations. For this exact reason, we'll employ the Caputo fractional derivative in this work ([35]).

Definition 2.1 *The fractional-order derivative is defined by Caputo as*

$$D^{\mu}f(x)=\frac{1}{\Gamma(n-\mu)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\mu+1-n}}dt,\quad n-1<\mu\le n,\quad n\in\mathbb{N}, \tag{2.1}$$

where $\mu > 0$ is the order of the derivative, n is the smallest integer greater than μ and Γ is the gamma function. . For the Caputo derivative we have

$$D^{\mu}C = 0, C \ \ is \ \ constant, \tag{2.2}$$

$$D^{\mu}x^{q} = \begin{cases} 0 \ , \quad for \ \ q \in \mathbb{N}_0 \ and \ \ q < \lfloor \mu \rfloor, \\ \dfrac{\Gamma(q+1)}{\Gamma(q+1-\mu)}x^{q-\mu}, \quad for \ q \in \mathbb{N}_0 \ and \ q \geq \lfloor \mu \rfloor \ or \quad q \notin \mathbb{N}_0 \ and \ q < \lfloor \mu \rfloor. \end{cases} \tag{2.3}$$

The ceiling function $\lfloor \mu \rfloor$ is used to indicate the least integer greater than or equal to μ, and the floor function $\lfloor \mu \rfloor$ to denote the largest integer less than or equal to μ. Moreover $\mathbb{N} = 1, 2, ...$ and $\mathbb{N}_0 = 0, 1, 2, ...$ Recall that for $\mu \in \mathbb{N}$, the typical integer order differential operator and the Caputo differential operator are identical. The Caputo fractional differentiation is a linear operation, namely:

$$D^{\mu}(\lambda f(x) + \gamma g(x)) = \lambda D^{\mu} f(x) + \gamma D^{\mu} g(x),$$

where λ and γ are constants.

3. The properties of Bernoulli wavelets

3.1. Properties of the Bernoulli wavelets

Constructing wavelets using orthogonal Bernoulli polynomials offers the advantage of optimal approximation properties. These polynomials, which are orthogonal with respect to a specific inner product, allow for the construction of wavelets that accurately represent signals. Orthogonal Bernoulli polynomials are particularly useful in areas such as signal compression, numerical solutions to differential equations, and numerical integration. The orthogonality property enhances the efficiency and accuracy of numerical methods, making them effective for solving complex problems with reduced computational effort ([16]). Bernoulli wavelets are derived from the Bernoulli polynomials by dilation and translation, can assume any positive integer m is the order of Bernoulli polynomials. They are defined on the interval $[0, 1)$ as follows:

$$\psi_{n,m}(x) = \begin{cases} \dfrac{2^{\frac{k+1}{2}}}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{2m!}\alpha_{2m}}}\beta_m(2^{k+1}x - 2n + 1) \ , \ \frac{n-1}{2^k} \leq x \leq \frac{n}{2^k}, \\ 0 \qquad\qquad otherwise, \end{cases} \tag{3.1}$$

where $n = 1, ..., 2^k$ represents the number of decomposition levels, $m = 0, ..., M$ is the degree of the Bernoulli polynomials $M \in \mathbb{N}^*$. The coefficient $\dfrac{2^{\frac{k+1}{2}}}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{2m!}\alpha_{2m}}}$ is for normality. Here, β_m are Bernoulli polynomials of order m which can be defined by :

$$\beta_m = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-1} t^i, \tag{3.2}$$

where α_i; $i = 0, 1, ..., m$ are Bernoulli numbers. These are a set of signed rational numbers that are generated by the series expansion of trigonometric functions [36] and can be defined by the identity

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.$$

The first Bernoulli numbers are
$\alpha_0 = 1$, $\alpha_1 = -\dfrac{1}{2}$, $\alpha_2 = -\dfrac{1}{6}$, ...,
with $\alpha_{2i+1} = 0, i = 1, 2, 3, ...$, and the first Bernoulli polynomials are

$$\beta_0(t) = 1, \beta_1(t) = t - \frac{1}{2}, \beta_2(t) = t^2 - t + \frac{1}{6}, \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, ... \tag{3.3}$$

The following relations are satisfied by the Bernoulli polynomials [19].

$$\beta_n(0) = \beta_n(1) = \alpha_n, \quad n \neq 1, \tag{3.4}$$

$$\beta'_n(t) = n\beta_{n-1}(t), n \geq 1, \tag{3.5}$$

$$\int_0^1 \beta_n(t)dt = 0, n \geq 1, \tag{3.6}$$

They also satisfy the following interesting criteria, given by [5]

$$\int_0^1 \beta_n(t)\beta_m(t)dt = (-1)^{n-1}\frac{m!n!}{(m+n)!}\alpha_{n+m}; \quad m; n \geq 1 \tag{3.7}$$

Bernoulli polynomials form a complete basis ([27]) over the interval $[0, 1]$.

3.2. Function approximation in terms of Bernoulli wavelets

Suppose that $\{\psi_{1,0},, ..., \psi_{1,M}, ..., \psi_{2^k,0}, ..., \psi_{2^k,M}\} \subset L^2([0, 1])$ be the set of Bernoulli wavelets and $Y = span\{\psi_{1,0},, ..., \psi_{1,M}, ..., \psi_{2^k,0}, ..., \psi_{2^k,M}\}$. Since Y is a finite dimensional vector space subspace, consequently, there exists a unique best approximation $f_0(t) \in Y$ for any element $f(t)$ in $L^2([0, 1])$ such that

$$\forall y(t) \in Y; \|f(t) - f_0(t)\| \leq \|f(t) - y(t)\|.$$

Since $f_0(t) \in Y$, there exist the unique coefficients $c_{10}; c_{11}; ...; c_{2^{k-1}(M-1)}$ such that

$$f(t) \simeq f_0(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = C^T\Psi(t), \tag{3.8}$$

where T indicates transposition, C and Ψ are 2^kM vectors given by

$$C = (c_{1,0}, ..., c_{1,M}, ..., c_{2^k,0}, ..., c_{2^k,M}),$$

and

$$\Psi(t) = (\psi_{1,0}(t),, ..., \psi_{1,M}(t), ..., \psi_{2^k,0}(t), ..., \psi_{2^k,M}(t)). \tag{3.9}$$

4. Bernoulli Operational Matrix of Fractional Order Derivative

In this section, the fractional derivative operational Bernoulli wavelet matrix is obtained by first converting these wavelets to Bernoulli polynomials. After obtaining the Bernoulli polynomials operational matrix of the fractional derivative, we calculate the fractional derivative's Bernoulli wavelet operational matrix.

4.1. Transformation matrix of the Bernoulli wavelet to the Bernoulli polynomials

Any arbitrary function $y(t) \in L^2([0, 1])$, can be expanded into Bernoulli functions as

$$y(t) \simeq \sum_{i=0}^{M-1} a_i\beta_i(t) = A^TB(t), \tag{4.1}$$

where

$$A^T = (a_0, ..., a_{M-1}),$$

and

$$B^T_{1\times M}(t) = (\beta_0(t), ..., \beta_{M-1}(t)). \tag{4.2}$$

Hence, A^T is given by

$$A^T = Y^TH^{-1}, \tag{4.3}$$

where

$$Y^T = \begin{bmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_{M-1} \end{bmatrix}, H = \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0M-1} \\ h_{10} & h_{11} & \cdots & h_{1M-1} \\ \vdots & \vdots & \cdots & \vdots \\ h_{M-10} & h_{M-11} & \cdots & h_{M-1M-1} \end{bmatrix},$$

with

$$y_i = \int_0^1 y(t)\beta_i(t)dt, \quad h_{ij} = \int_0^1 \beta_i(t)\beta_j(t)dt \quad i; j = 0, 1, ..., M-1.$$

In order to create a M-term Bernoulli polynomial, the Bernoulli wavelets can be expanded as

$$\Psi_{2^{k-1}M\times 1}(t) = \Phi_{2^{k-1}M\times M} B_{M\times 1}(t), \tag{4.4}$$

where Φ is the transformation matrix of the Bernoulli wavelet to the Bernoulli polynomials. For example, when $M = 3$ and $k = 2$ we have

$$\Psi(t) = (\psi_{1,0}(t), \psi_{1,1}(t), \psi_{1,2}(t), \psi_{2,0}(t)\psi_{2,1}(t), \psi_{2,2}(t))^T,$$
$$B^T_{1\times 3}(t) = (\beta_0(t), \beta_1(t), \beta_2(t))^T,$$

where, for $0 \le t < \frac{1}{2}$,

$$\begin{cases} \psi_{1,0}(t) = \sqrt{2} = \sqrt{2}\beta_0(t), \\ \psi_{1,1}(t) = \sqrt{6}(-1 + 4t) = \sqrt{6}\beta_0(t) + 4\sqrt{6}\beta_1(t), \\ \psi_{1,2}(t) = \sqrt{10}(1 - 12t + 24t^2) = 3\sqrt{10}\beta_0(t) + 12\sqrt{10}\beta_1(t) + 24\sqrt{10}\beta_2(t). \end{cases}$$

For $\frac{1}{2} \le t < 1$,

$$\begin{cases} \psi_{2,0}(t) = \sqrt{2} = \sqrt{2}\beta_0(t), \\ \psi_{2,1}(t) = \sqrt{6}(-3 + 4t) = -\sqrt{6}\beta_0(t) + 4\sqrt{6}\beta_1(t), \\ \psi_{2,2}(t) = \sqrt{10}(13 - 36t + 24t^2) = 3\sqrt{10}\beta_0(t) - 12\sqrt{10}\beta_1(t) + 24\sqrt{10}\beta_2(t). \end{cases}$$

In this case,

$$\Phi = \begin{cases} \Phi_1 = [a_{i,j}]_{6\times 3} & for \;\; 0 \le t < \frac{1}{2}, \\ \Phi_2 = [b_{i,j}]_{6\times 3} & for \;\; \frac{1}{2} \le t < 1, \end{cases}$$

where

$$\Phi_1 = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{6} & 4\sqrt{6} & 0 \\ 3\sqrt{10} & 12\sqrt{10} & 24\sqrt{10} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & 4\sqrt{6} & 0 \\ -\sqrt{6} & 4\sqrt{6} & 0 \\ 3\sqrt{10} & -12\sqrt{10} & 24\sqrt{10} \end{bmatrix},$$

In general, for arbitrary M and $k = 2$ we get

$$\Phi = \begin{cases} \Phi_1 = [a_{i,j}]_{2^{k-1}M\times M} & for \;\; 0 \le t < \dfrac{1}{2}, \\ \Phi_2 = [b_{i,j}]_{2^{k-1}M\times M} & for \;\; \dfrac{1}{2} \le t < 1, \end{cases} \tag{4.5}$$

with

$$a_{i,j} = \frac{1}{2^{\frac{k-1}{2}}} \begin{cases} 2^i \frac{1}{\lambda_{i-1}} & for \;\; ; i = j, \\ 2^{j-1}\binom{i-1}{i-j}\frac{1}{\lambda_{i-1}} & for \;\; ; j < i < M, \\ 0 & otherwise, \end{cases}$$

and

$$b_{i,j}=\begin{cases}0 & for \;\; 1\leq i\leq M,\\ 0 & for \;\; j=1,...,M,\\ (-1)^{(i+j-M)}a_{i-M,j} & for \;\; M+1\leq i\leq 2^{k-1}M,\end{cases}$$

where $\lambda_0=1$ and $\lambda_i=\sqrt{\frac{(-1)^{i-1}(i!)^2}{(2i)!}\alpha_{2i}}, \quad i=1,...,M-1.$

4.2. Bernoulli wavelets Operational Matrix of Derivative

The derivative of vector $B(t)$, with the aid of (3.4), can be expressed in the matrix form by

$$B'(t)=PB(t), \tag{4.6}$$

where

$$B'(t)=(\beta_0'(t),...,\beta_{M-1}'(t)),$$

and

$$P=\begin{bmatrix}0&0&0&\cdots&0&0&0\\ 1&0&0&\cdots&0&0&0\\ 0&2&0&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ 0&0&0&\cdots&M-2&0&0\\ 0&0&0&0&0&M-1&0\end{bmatrix},$$

The derivative of the Bernoulli wavelets vector $\Psi(t)$ can be obtained as

$$D^1\Psi(t)=\frac{d}{dt}\Psi(t)=D\Psi(t), \tag{4.7}$$

where D is a relative operational square matrix of dimension $2^{k-1}\times M$ and can be evaluated from Eqs.(4.4) and (4.6) as follows

$$\frac{d}{dt}\Psi(t)=\frac{d}{dt}(\Phi B(t))=\Phi\frac{d}{dt}B(t)=\Phi PB(t)=\Phi P\Phi^{-1}\Psi(t).$$

Therefore, we have

$$D^1\Psi(t)=D\Psi(t), \tag{4.8}$$

where $D=\Phi P\Phi^{-1}$.

4.3. Bernoulli Operational Matrix of Fractional Order Derivative

The fractional derivative of the vector $B(t)$ given(see [11]) in Eq. (4.2) can be expressed by

$$D^\mu B(t)=F^{(\mu)}B(t), \tag{4.9}$$

where $F^{(\mu)}$ is $M\times M$ operational matrix of fractional derivative of order μ in the Caputo sense. Using Eqs.(2.3), (3.2) for $i=0;1;...;M-1$, we have

$$D^\mu\beta_i=D^\mu(\sum_{r=0}^{i}\tbinom{i}{r}\alpha_{i-1}t^r)=\sum_{r=0}^{i}\tbinom{i}{r}\alpha_{i-1}D^\mu(t^r)=\sum_{r=\lfloor\mu\rfloor}^{i}b_{i,r}^{(\mu)}t^{r-\mu}, \;\; i=\lfloor\mu\rfloor,...,M-1, \tag{4.10}$$

where

$$b_{i,r}^{(\mu)}=\frac{i!\alpha_{i-r}}{(i-r)!\Gamma(r+1-\mu)}.$$

Assuming that $t^{r-\mu}$ can be expanded in terms of Bernoulli polynomials as

$$t^{r-\mu}=\sum_{j=0}^{M-1} c_{r,j}\beta_j(x), \tag{4.11}$$

and substituting Eq.(4.11) in Eq.(4.10) for $j=0,1,...,M-1$, we get

$$D^{\mu}\beta_i=\sum_{r=\lfloor\mu\rfloor}^{i} b_{i,r}^{(\mu)}\sum_{j=0}^{M-1} c_{r,j}\beta_j(x)=\sum_{j=0}^{M-1}\left(\sum_{r=\lfloor\mu\rfloor}^{i}\theta_{i,j,r}^{(\mu)}\right)\beta_j(x),\ \ i=\lfloor\mu\rfloor,...,M-1, \tag{4.12}$$

where

$$\theta_{i,j,r}^{(\mu)}=b_{i,r}^{(\mu)}c_{r,j}.$$

Therefore, we have

$$F^{(\mu)}=\begin{bmatrix} 0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0\\ \theta_{\lfloor\mu\rfloor,0,r}^{(\mu)} & \theta_{\lfloor\mu\rfloor,1,r}^{(\mu)} & \cdots & \theta_{\lfloor\mu\rfloor,M-1,r}^{(\mu)}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{r=\lfloor\mu\rfloor}^{i}\theta_{i,0,r}^{(\mu)} & \sum_{r=\lfloor\mu\rfloor}^{i}\theta_{i,1,r}^{(\mu)} & \cdots & \sum_{r=\lfloor\mu\rfloor}^{i}\theta_{i,M-1,r}^{(\mu)}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{r=\lfloor\mu\rfloor}^{M-1}\theta_{M-1,0,r}^{(\mu)} & \sum_{r=\lfloor\mu\rfloor}^{M-1}\theta_{M-1,1,r}^{(\mu)} & \cdots & \sum_{r=\lfloor\mu\rfloor}^{M-1}\theta_{M-1,M-1,r}^{(\mu)} \end{bmatrix}, \tag{4.13}$$

where $\theta_{i,j,r}^{(\mu)}$ is given by

$$\theta_{i,j,r}^{(\mu)}=\frac{i!\alpha_{i-r}}{(i-r)!\Gamma(r+1-\mu)}c_{r,j}, \tag{4.14}$$

with α_i being the Bernoulli number and $c_{r,j}$ can be obtained from (4.1). Note that in $F^{(\mu)}$, the first dμe rows coefficients are all equal to zero.

4.4. Bernoulli wavelet operational matrix of the fractional Order Derivative

We now derive the Bernoulli wavelet operational matrix of the fractional order derivative. Let

$$D^{\mu}\Psi(t)=D^{(\mu)}\Psi(t), \tag{4.15}$$

where matrix $D^{(\mu)}$ is called the Bernoulli wavelet operational matrix of the fractional order derivative. Using Eqs. (4.4),(4.9) we get

$$D^{\mu}\Psi(t)=D^{\mu}\Phi B(t)=\Phi D^{\mu}B(t)=\Phi F^{(\mu)}B(t)=\Phi F^{(\mu)}\Phi^{-1}\Psi(t). \tag{4.16}$$

Then, the Bernoulli wavelet operational matrix of the fractional derivative $D^{(\mu)}$ is given by

$$D^{(\mu)}=\Phi F^{(\mu)}\Phi^{-1}. \tag{4.17}$$

5. The numerical method

In this part, we solve linear and non-linear fractional differential equations using the Bernoulli operational matrix of fractional derivative. For the existence, uniqueness and continuous dependence of the solution of the problem, consider the Caputo fractional differential equations of the form below.

5.1. Linear Multi-Order Fractional Differential Equation

Consider the following linear Caputo fractional differential equations

$$D^{\mu}y(t) = a_k D^{\alpha_k}y(t) + a_{k-1}D^{\alpha_{k-1}} + \cdots + a_1 D^{\alpha_1}y(t) + a_0 y(t) + g(t), t \in (0,1), \tag{5.1}$$

with the initial conditions

$$y^{(i)}(0) = y_i, i = 0, 1, ..., N-1, \tag{5.2}$$

where $a_0, a_1, ..., a_k$ are real constant coefficients and also $\mu \in]N-1, N[, 0 < \alpha_1 < \alpha_2 < ... < \alpha_1 < \mu$. Here D^{μ} is the derivative of y of order μ in the sense of Caputo fractional differential operator. Moreover, the values of $y_i, i = 0, ..., N-1$ describe the initial state of $y(t)$, and $g(t)$ is a given source function.
Now, from Eq.(3.8) we approximate $y(t)$ and $g(t)$ with the Bernoulli wavelets as

$$y(t) \simeq \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = C^T\Psi(t), \tag{5.3}$$

$$g(t) \simeq \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} z_{n,m}\psi_{n,m}(t) = G^T\Psi(t), \tag{5.4}$$

where

$$G^T = (z_{1,0}, ..., z_{1,M}, ..., z_{2^k,0}, ..., z_{2^k,M})$$

is known but $C^T = (c_{1,0}, ..., c_{1,M}, ..., c_{2^k,0}, ..., c_{2^k,M})$ is an unknown vector.
In virtue of Eq.(4.15) and Eq.(5.3), the fractional state rates $D^{\mu}y(t)$ and $D^{\alpha_j}y(t)$ can be approximated as

$$D^{\mu}y(t) = C^T D^{(\mu)}\Psi(t), \tag{5.5}$$

$$D^{\alpha_j}y(t) = C^T D^{(\alpha_j)}\Psi(t), j = 1, ..., k. \tag{5.6}$$

Substituting Eqs.(5.3), (5.5), Eq.(5.6) and (5.4) in Eq. (5.1), we get a system of algebraic equations, which can be written as

$$R(t) = C^T D^{(\mu)}\Psi(t) - a_k C^T D^{(\alpha_k)}\Psi(t) - a_{k-1}C^T D^{(\alpha_{k-1})}\Psi(t) - \cdots, \tag{5.7}$$

$$-a_1 C^T D^{(\alpha_1)}\Psi(t) - a_0 C^T\Psi(t) - G^T\Psi(t) = 0,$$

similar to a typical tau method [14] we generate $2^k \times M - N + 1$ linear equations by applying

$$< R(t), \Psi_j(t) >= \int_0^1 R(t)\Psi_j(t)dt = 0, \ , j = 0, ..., 2^k \times M - N + 1. \tag{5.8}$$

Moreover, substituting initial conditions (5.2) into (5.3), (5.8) we have

$$\begin{aligned} y(0) &= C^T\Psi(0) = y_0, \\ y^{(1)}(0) &= C^T D^{(1)}\Psi(0) = y_1, \\ &\vdots \\ y^{(N-1)}(0) &= C^T D^{(N-1)}\Psi(0) = y_{N-1}, \end{aligned} \tag{5.9}$$

Eqs. (5.8)-(5.9) generate $2^k \times M$ linear algebraic equations, which can be solved for the unknown vector C using Newton's iterative method. Consequently, $y(t)$ given in Eq. (5.1) can be calculated.

5.2. Non-Linear Multi-Order Fractional Differential Equation

Consider the non-linear multi-order fractional differential equation

$$D^{\mu}y(t) = F(t, y(t), D^{\alpha_1}y(t), ..., D^{\alpha_k}y(t)), \tag{5.10}$$

with initial conditions

$$y^{(i)}(0) = y_i, i = 0, 1, ..., N-1, \tag{5.11}$$

where $N-1 < \mu \le N, 0 < \alpha_1 < \alpha_2 < ... < \alpha_k < \mu$, D^{μ} denotes the Caputo fractional derivative of order μ and F can be non-linear in general.
In order to solve this problem, we must first approximate $y(t), D^{\mu}y(t)$ and $D^{\alpha_j}y(t)$ for $j = 1, ..., k$ as (5.3), (5.5) and (5.6) respectively. And by substituting these equations in (5.10) we get

$$C^T D^{(\mu)}\Psi(t) - F(t, C^T\Psi(t), C^T D^{(\alpha_1)}\Psi(t), ..., C^T D^{(\alpha_k)}\Psi(t)) = 0, \tag{5.12}$$

where $C^T = (c_{1,0}, ..., c_{1,M}, ..., c_{2^k,0}, ..., c_{2^k,M})^T$ is an unknown vector. Also for the initial conditions we use the wavelets to approximate and we get (5.9).
Now to find the solution $y(t)$ of (5.10), we collocate Eq.(5.12) at the $2^k \times M - N + 1$ points.
For suitable collocation points we use the first $2^k \times M - N + 1$ zeros of shifted Legendre polynomials $P_{2^k \times M}$ [37]. these equations together with (5.11) generates $2^k \times M$ nonlinear algebraic equations. This system can be solved using any typical iteration technique, such as Newton's iterative technique. Hence, the approximate solution $y(t)$ can be obtained.

6. Numerical Illustrations

In this section, we solve linear and non-linear fractional differential equations by using the Bernoulli operational matrix of fractional derivative.

To assess the accuracy of the proposed method, we compare the approximate solution with the exact solution. The pointwise absolute error at each grid point (i, j) and the L_2-norm error are defined, respectively, as follows: $e_{i,j} = |(u_{ex})_{i,j} - (u_{approx})_{i,j}|$, $e = |u_{ex} - u_{approx}|$.

Example 1:

Consider the following linear fractional initial value problem. This fractional differential equation models a system with mixed dynamics, where both classical (integer-order) and anomalous (fractional-order) effects are significant. It could describe phenomena in physics, engineering, or biology where memory, nonlocality, or complex scaling laws are present ([35]).

$$D^2y(t) + D^{\frac{1}{2}}y(t) + y(t) = f(t), \tag{6.1}$$

$$y(0) = y'(0) = 0, \tag{6.2}$$

where

$$f(t) = t^2 + 2 + \frac{8}{3\sqrt{\pi}}t^{1.5}.$$

By applying the technique described in Section 4.1 for the case $k = 1$ and $M = 3$, the residual of Eq.(6.1) can be calculated by the formula

$$C^T D^{(2)}\Psi(t) + C^T D^{(\frac{1}{2})}\Psi(t) + C^T\Psi(t) = F^T\Psi(t), \tag{6.3}$$

where

$$\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 6\sqrt{5} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{bmatrix}, \quad D^{(\frac{1}{2})} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{8\sqrt{3}}{3\sqrt{\pi}} & \frac{8}{5\sqrt{\pi}} & \frac{-8\sqrt{3}}{7\sqrt{5\pi}} \\ \frac{-24\sqrt{5}}{15\sqrt{\pi}} & \frac{24\sqrt{5}}{7\sqrt{3\pi}} & \frac{8\sqrt{3}}{3\sqrt{\pi}} \end{bmatrix},$$

Table 1: Comparison of $y(t)$ for $k = 1$ and $M = 6$ with $q = 1$ and exact solution, for Example 2.

t	$y(t)$ [26]	$y(t)$ present method	Exaxt Solution
0.1	0.099684	0.099668	0.099667
0.2	0.197417	0.197373	0.197375
0.3	0.291316	0.291311	0.291312
0.4	0.379912	0.379944	0.379948
0.5	0.462083	0.462117	0.462117
0.6	0.537057	0.537054	0.537049
0.7	0.604405	0.604365	0.604367
0.8	0.664052	0.664035	0.664036
0.9	0.716264	0.716293	0.716297

$$F^T = \left[\frac{1}{3} + \frac{16}{15\sqrt{\pi}}, \frac{1}{2\sqrt{3}} + \frac{48}{35\sqrt{3\pi}}, \frac{1}{6\sqrt{5}} + \frac{16}{63\sqrt{5\pi}}\right] \quad and \quad C^T = [c_{10}, c_{11}, c_{12}].$$

Substituting D, $D^{(\frac{1}{2})}$and F^T into Eq. (6.3), we get

$$c_{10} = \frac{1}{3}, \; c_{11} = \frac{\sqrt{3}}{6}, \; c_{12} = \frac{\sqrt{5}}{30}.$$

By using Eq.(3.8), we have $y(t) = C^T \Psi(t) = t^2$.

Example 2:

The Riccati equation, particularly its fractional derivative version, has been a topic of significant interest over the past 30 years due to its broad applications across various fields. In economics, it models financial market volatility; in physics, it addresses wave processes in lossy media; and in epidemiology, it helps construct epidemic logistic models to predict key trends ([44]). Consider the following fractional Riccati equation:

$$D^q y(t) = 1 - y^2(t), 0 < q \leq 1, \tag{6.4}$$

subject to the initial condition

$$y(0) = 0. \tag{6.5}$$

When q $= 1$, it is known that the exact solution of this system [25] is

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1},$$

We applied the present method approach to solve Eq.(6.4) with $k = 2; M = 6$ and various values of q. The approximate solutions for Eq.(6.4) by Legendre wavelets method for $k = 1; M = 25$ and various values of q are also plotted in [47]. It was also considered in [26] by applying the Bernoulli wavelets method, with $k = 2$ and $M = 5$. In Tabel 1, we compare numerical results of $y(t)$ using our method with the exact solution and that obtained in [26].

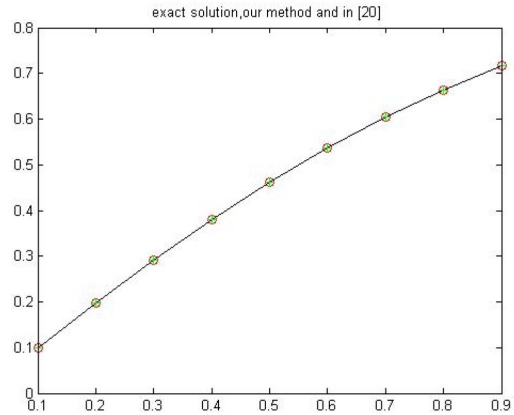

Figure 1: Comparison of $y(t)$ for $k = 1$and $M = 6$, with $q = 1$ and exact solution.

Table 2: Numerical results for various values of q for Example 2.

t	$q = 0.6$	$q = 0.9$
0.1	0.165495	0.129133
0.2	0.285617	0.238978
0.3	0.383196	0.336448
0.4	0.463523	0.422739
0.5	0.530749	0.498909
0.6	0.587967	0.565853
0.7	0.637209	0.624314
0.8	0.679446	0.674869
0.9	0.714512	0.717970

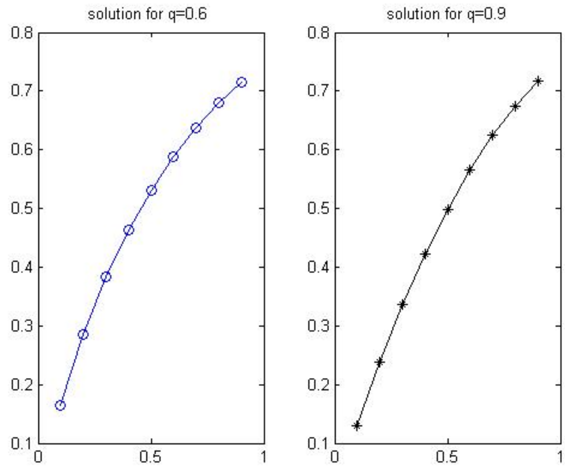

Figure 2: Numerical results for various values q, for Example 2.

Example 3:

Consider the following nonlinear initial value problem

$$D^{\nu}u(t) - u^2(t) = t + (\frac{t^{\nu+1}}{\Gamma(\nu+1)})^2 \quad 0 \leq t \leq 1 \quad ,0 < \nu \leq 1, \tag{6.6}$$

where $f(t) = t + (\frac{t^{\nu+1}}{\Gamma(\nu+1)})^2$ subject to the initial condition

$$u(0) = 0. \tag{6.7}$$

The exact solution of this problem is $u(t) = \frac{t^{\nu+1}}{\Gamma(\nu+1)}$.
By applying the method described in Section 5, the above problem is transformed as following:

$$C^T D^{(\nu)}\Psi(t) + (C^T\Psi(t))(C^T\Psi(t))^T = F^T\Psi(t), \tag{6.8}$$

where f is approximated as $f(t) = F^T\Psi(t)$. Then by collocation Eq. (6.8) at the zeros of shifted Legendre polynomials and using the initial condition (6.7) such that $C^T\Psi(0) = 0$, by using any standard iterative technique, like Newton's iterative method, we can obtain the unknown vectors C. Table 3 shows the absolute error obtained for different values of t, between the approximate solutions and the exact solution with $k = 2$, $M = 5$ for various values ν. Also, Figs. 3 and 4 show the approximate solutions obtained for $k = 2$, $M = 5$ with $\nu = 2, \nu = 0.5$ and $\nu = 0.7$ and the exact solution. From these figures, it is seen that the approximate solutions converge to the exact solution.

Example 4:

Consider the following linear boundary value problem

$$D^{q_1}y(t) - D^{q_2}y(t) = g(t) \quad 0 < t < 1, 1 < q_2 \leq 2, 0 < q_1 \leq 1, \tag{6.9}$$

$$u(0) = u(1) = 0, \tag{6.10}$$

where $g(t) = -1 - e^{t-1}$. The exact solution of Eq.(6.9) with boundary condition (6.10) corresponding to $q_1 = 2$ and $q_2 = 1$ is $y(t) = t(1 - e^{t-1})$.
We approximate $y(t)$ and $g(t)$ as

$$y(t) = C^T\Psi(t), \tag{6.11}$$

Table 3: The absolute error with for $k = 2$, $M = 6$ and various values of q for Example 3.

t	$\nu = 1$	$\nu = 0.5$	$\nu = 0.7$
0.1	$3.04.10^{-5}$	$2.29.10^{-9}$	$3.78.10^{-6}$
0.2	$2.15.10^{-4}$	$2.38.10^{-8}$	$1.56.10^{-5}$
0.3	$3.08.10^{-4}$	$1.03.10^{-9}$	$2.68.10^{-6}$
0.4	$2.63.10^{-5}$	$3.42.10^{-7}$	$1.34.10^{-4}$
0.5	$1.30.10^{-4}$	$3.90.10^{-8}$	$2.42.10^{-5}$
0.6	$1.58.10^{-4}$	$2.65.10^{-7}$	$1.07.10^{-6}$
0.7	$2.63.10^{-5}$	$1.06.10^{-9}$	$1.19.10^{-5}$
0.8	$3.94.10^{-5}$	$3.48.10^{-8}$	$2.79.10^{-6}$
0.9	$1.71.10^{-4}$	$2.71.10^{-7}$	$1.08.10^{-4}$

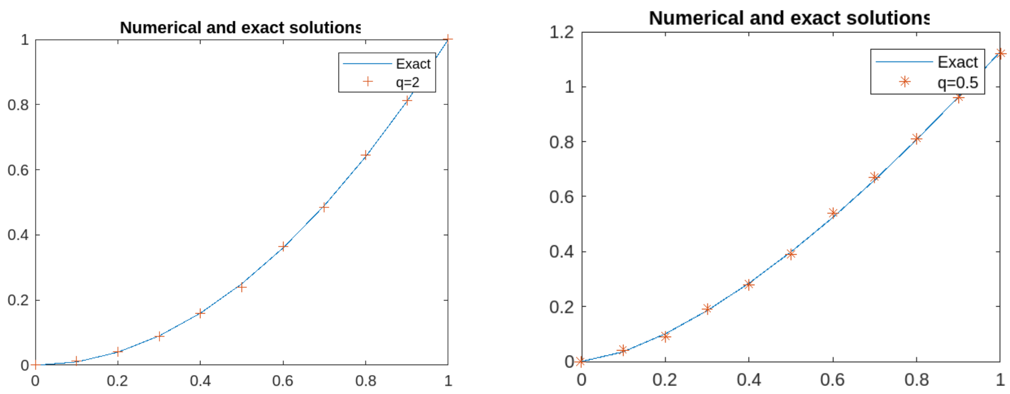

Figure 3: Numerical results for $\nu = 2$ and $\nu = 0.5$ for Example 3.

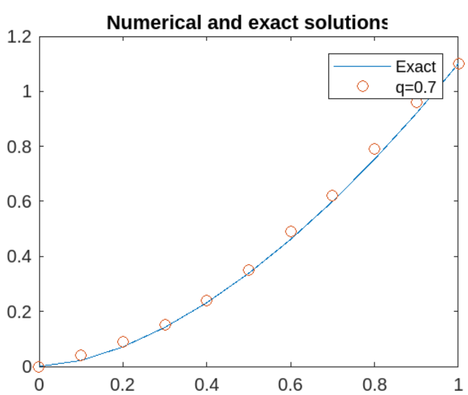

Figure 4: Numerical results for $q = 0.7$ for Example 3.

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Table 4: Numerical results for Example 4, for $k = 1$ $M = 4$ with $q_1 = 2$ and $q_2 = 1$.

t	$y(t)$ [26]	$y(t)$ [47]	$y(t)$ present method	Exaxt Solution
0.1	0.05934383	0.05934820	0.05934315	0.05934303
0.2	0.11013431	0.11014318	0.11013427	0.11013421
0.3	0.15102443	0.15103441	0.15102441	0.15102441
0.4	0.18047562	0.18048329	0.18047532	0.18047535
0.5	0.19673476	0.19673826	0.1967347	0.1967346
0.6	0.19780804	0.19780653	0.19780778	0.19780797
0.7	0.18142748	0.18142196	0.18142725	0.18142725
0.8	0.14501561	0.14500893	0.14501545	0.14501540
0.9	0.08564683	0.08564186	0.08564627	0.08564632

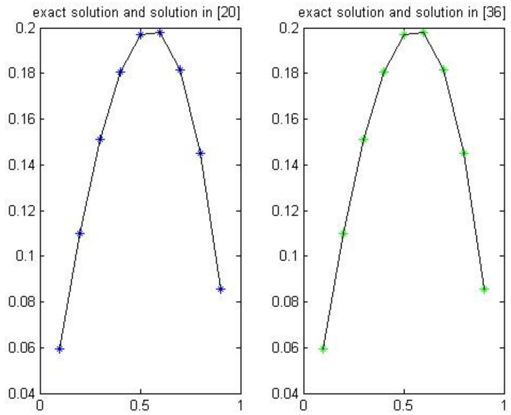

Figure 5: Numerical results for Example 4, for $k = 1$ $M = 4$, with $q_1 = 2$ and $q_2 = 1$.

and

$$g(t) = G^T \Psi(t), \tag{6.12}$$

then by using Eq.(4.15), we get the following system of algebraic equations

$$(D^{(q_1)} - D^{(q_2)})\Psi(t) = G^T \Psi(t). \tag{6.13}$$

We compare numerical results of $y(t)$ using our method with $k = 1$ and $M = 6$ with the exact solution and those obtained in [26] [47]. Table 4 shows the exact solution $y(t)$ and the approximate solution using the proposed method and the results of [26] [47].

Example 5:

Consider the following nonlinear fractional differential equations

$$\begin{cases} D^q y_1(t) = -1002 y_1(t) + 1000 y_2(t), \\ D^q y_2(t) = y_1(t) - y_2(t) - y_2^2(t). \end{cases} \tag{6.14}$$

The exact solution of this system when $q = 1$ is known to be $y_1(t) = e^{2t}$ and $y_2(t) = e^t$. This example is solved by our method with $k = 1$ and $M = 5$, when $q = 0.5, 0.25$. The results obtained when $q = 0.5, 0.25$ for $y_1(t)$ and $y_2(t)$ are also shown in Table 5 and 6. We compare the results obtained by our method and those obtained [23].

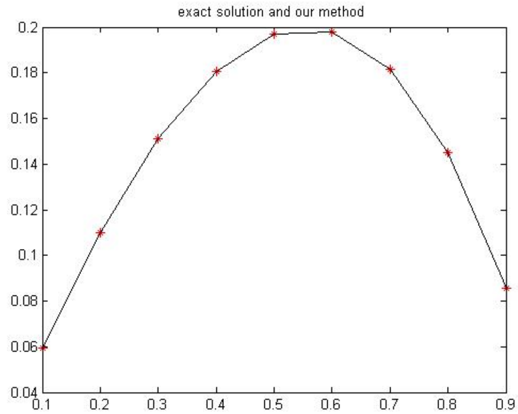

Figure 6: Numerical results for Example 4, for $k = 1$ $M = 4$ with $q_1 = 2$ and $q_2 = 1$.

Table 5: Numerical solutions $y_1(t)$ and $y_2(t)$, for $k = 1$ $M = 5$ when $q = 0.25$ obtained by the present method and that in [23].

t	$y_1(t)$ [23]	$y_1(t)$present method	$y_2(t)$ [23]	$y_2(t)$present method
0.2	0.3292530	0.3292519	0.5736073	0.5736062
0.4	0.2781248	0.2781242	0.5271530	0.5271521
0.6	0.2503713	0.2503725	0.5001338	0.5001341
0.8	0.2299841	0.2299839	0.4793216	0.4793210

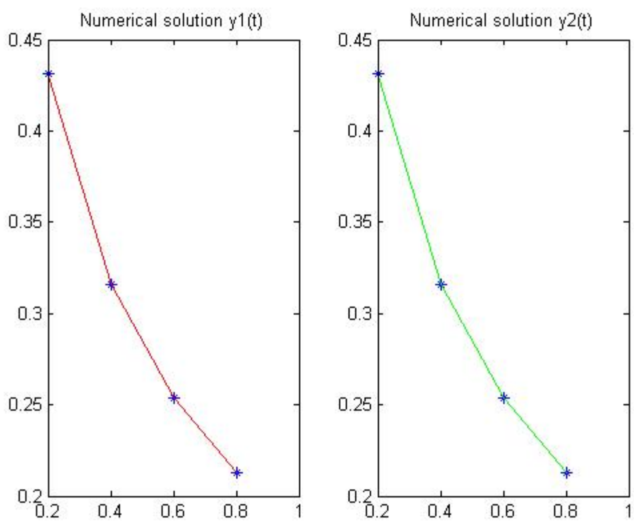

Figure 7: Comparison of $y_1(t)$ and $y_2(t)$ by our method with $k = 1$ $M = 5$ in the case corresponds to $q = 0.25$ with exact solution.

Table 6: Numerical solutions $y_1(t)$ and $y_2(t)$, for $k = 1$ $M = 5$ when $q = 0.5$ obtained by the present method and that in [23].

t	$y_1(t)$ [23]	$y_1(t)$ present method	$y_2(t)$ [23]	$y_2(t)$present method
0.2	0.4312529	0.4312533	0.6565720	0.6565728
0.4	0.3157611	0.3157602	0.5617578	0.5617572
0.6	0.2537459	0.2537464	0.5035336	0.5035341
0.8	0.2128393	0.2128381	0.4611347	0.4611354

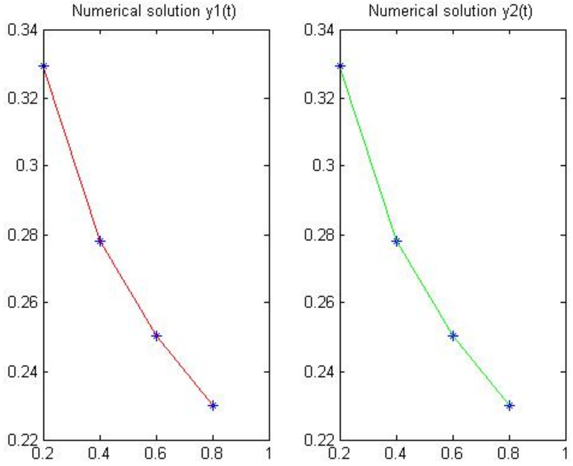

Figure 8: Comparison of $y_1(t)$ and $y_2(t)$ by our method with $k = 1$ $M = 5$ in the case corresponds to $q = 0.5$ with exact solution.

7. Conclusion

In this paper, a general formulation for the Bernoulli operational matrix of fractional derivative has been derived through wavelet-polynomial transformation. This matrix is used to approximate numerical solution of ODEs. The solutions obtained are compared with exact solutions and with the solutions obtained by some other numerical methods. Our method was based on the collocation methods and truncated Bernoulli wavelets. This operational matrix has the advantage over others in that it is less computationally complex because every operational matrix of differentiation involves more zeros, which reduces the run time and yields a highly accurate solution. The solution obtained in these examples demonstrate the effectiveness of this approach in solving these problems.

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