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Novel Properties for Intuitionistic Bipolar Fuzzy Matrices

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ABSTRACT: An intuitionistic bipolar fuzzy matrix is a matrix whose entries are real numbers in $[-1, 1]$. In this paper, we study intuitionistic bipolar fuzzy algebra, intuitionistic bipolar fuzzy relation and then introduce the intuitionistic bipolar fuzzy matrices. On the other hand, an order relation is defined. Some properties and results of intuitionistic bipolar fuzzy matrix are investigated.

Key Words: Intuitionistic bipolar fuzzy set, intuitionistic bipolar fuzzy relation, intuitionistic bipolar fuzzy matrix.

Contents

Introduction

Thomason M. G. [1] in 1977 first introduced the concept of fuzzy matrix. Fuzzy matrix have an important role in the development of various fields of science. Many researchers have studied this concept and presented many results in it, and examples of that. Ragab at al [2] in 1994 gave some properties on adjoint and determinaant of square fuzzy matrix, while L. J. Xin [3] studied powers convergent of controllable fuzzy matrix. A. K. shymal el al [4,5] discussed two new operators on fuzzy matrices and distance between fuzzy matrices in 2004 and 2005. P. Priya el al [6] in 2019 introduced basic concepts of fuzzy matrices. In 1986, K. T. Atanassov [7] introduced the intuitionistic fuzzy set concept as a generalization of fuzzy set where he add a new component which determines a non membership degree in the fuzzy set definition. Many application have been made on this concept in many field. After that and in 1998, Zhang [8] gave the notion of bipolar fuzzy sets as also a generlization of fuzzy set. In bipolar fuzzy set, a membership degree range in $[-1, 1]$. This concept has also been used in many fields and many scientific papers have been published on it.
Recently, in 2015 Ezhilmaran et al [9] worked on integrating the concept of bipolar fuzzy set with the concept of intuitionistic fuzzy set and obtained a bipolar intuitionistic fuzzy sets, this concept is an extension of intuitionistic fuzzy set where have positive and negative values. Many researchers have used this concept and of them. In 2019, Sonia. M. et al [10] studied the producer of bipolar intuitionistic fuzzy graphs. In 2020, Alnaser et al [11] discussed a bipolar intuitionistic fuzzy graphs and its matrices, Massa'deh. M. et al [12] in 2021 studied a bipolar intuitionistic fuzzy graph over Cayley groups and Deva. N. et al [13] in 2022 introduced a bipolar intuitionistic fuzzy competition graphs. In this article, we introduce and study some basic properties of an intuitionistic bipolar fuzzy element by using min-max composition. Also, intuitionistic bipolar fuzzy relation, intuitionistic bipolar fuzzy matrix and its basic properties are given here. Some theorems and exa-ples related to this topic and presented in this article.

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1. Preliminaries

Definition 1.1 [2] A fuzzy matrix A is a matrix $[a_{ij}]_{m\times n}$ where $a_{ij}\in[0,1], 1\le i\le m$ and $1\le j\le n$.

Definition 1.2 [9] Let A be a non empty set. An intuitionistic bipolar fuzzy set $\delta=\{a,\lambda^+(a),\lambda^-(a),\mu^+(a),\mu^-(a)\}$ such that $\lambda^+:A\to[0,1]$, $\lambda^-:A\to[-1,0],\mu^+:A\to[0,1]$ and $\mu^-:A\to[-1,0]$ are the mappings; $0\le\lambda^+(a)-\mu^+(a)\le1,-1\le\lambda^-(a)+\mu^-(a)\le0$.

Definition 1.3 [14] If $(a,a'),(b,b')$ are two intuitionistic fuzzy values then:
1) $\max\{(a,a'),(b,b')\}=(\max\{a,b\},\min\{a',b'\})$.
2) $\min\{(a,a'),(b,b')\}=(\min\{a,b\},\max\{a',b'\})$.

Definition 1.4 [14] Let $A=<a_{ij}>$ be a matrix with order $m\times n$ if all a_{ij} such that $i=1,\cdots,m$ and $j=1,\cdots,n$ are intuitionistic fuzzy values, then A is called an intuitionistic fuzzy matrix.

2. Intuitionistic bipolar fuzzy matrices

Definition 2.1 $\delta=\{a=(-a^-,a^+,-a^-,a^+),a\in A\}$ where $-a^-,-a'^-\in[-1,0]$ this means that $a^-,a'^-\in[0,1]$ and $a^+,-a'^+\in[0,1]$ such that $0\le a^++a'^+\le1$ and $-1\le-a^-+(-a'^-)\le0$.

Definition 2.2 Let a,b be two intuitionistic bipolar fuzzy values such that $a=(-a^-,a^+,-a'^-,a'^+)$ and $b=(-b^-,b^+,-b'^-,b'^+)$, then the equality of two elements a and b is denoted by $a=b$ and is defined by $a=b$ if and only if $-a^-=-b^-,a^+=b^+,-a'^-=-b'^-$ and $a'^+=b'^+$.

Definition 2.3 Let a,b be two intuitionistic bipolar fuzzy values where $a=(-a^-,a^+,-a'^-,a'^+)$ and $b=(-b^-,b^+,-b'^-,b'^+)$ and $a^-,a^+,a'^-,a'^+,b^-,b^+,b'^-,b'^+\in[0,1]$ then the following operations are defined as:

1. The disjunction of a and b is denoted by $a+b$ and is defined by $a+b=(-a^-,a^+,-a'^-,a'^+)+(-b^-,b^+,-b'^-,b'^+)\equiv(-\max\{a^-,b^-\},\max\{a^+,b^+\},-\max\{a'^-,b'^-\},\max\{a'^+,b'^+\})$.
2. The parallel conjunction of a and b is denoted by $a\cdot b$ and is defined by $a\cdot b=(-a^-,a^+,-a'^-,a'^+)\cdot(-b^-,b^+,-b'^-,b'^+)\equiv(-\min\{a^-,b^-\},\max\{a^+,b^+\},-\max\{a'^-,b'^-\},\min\{a'^+,b'^+\})$.
3. The serial conjunction of a and b is denoted by $a\times b$ and is defined by $a\times b=(-a^-,a^+,-a'^-,a'^+)\times(-b^-,b^+,-b'^-,b'^+)\equiv(-\max\{\min\{a^-,b^+\},\min\{a^+,b^-\}\},\max\{\min\{a^-,b^-\},\min\{a^+,b^+\}\}-\max\{\min\{a'^-,b'^+\},\min\{a'^+,b'^-\}\},\max\{\min\{a'^-,b'^-\},\min\{a'^+,b'^+\}\})$.
4. The negation of a is denoted by $-a$ and is defined by $-a=(-a^-,a^+,-a'^-,a'^+)=(-a^+,a^-,-a'^+,a'^-)$.
5. The complement of a is denoted by a^c and is defined by $a^c=(-a^-,a^+,-a'^-,a'^+)^c=((-a^-)^c,(a^+)^c,(-a'^-)^c,(a'^+)^c)=(-1+a^-,1-a^+,-1+a'^-,1-a'^+)$.
6. The implication of a to b is denoted by $a\Rightarrow b$ and is defined by $a\Rightarrow b=a^c+b$.

Now we introduced some baisc definitions related to intuitionistic bipolar fuzzy value, and then we prove some simple properties on it.

Definition 2.4 The zero element of an intuitionistic bipolar fuzzy value is denoted by 0_{IB} and is defined by $0_{IB}=(0,0,0,0)$.

Definition 2.5 The unit element of an intuitionistic bipolar fuzzy value is denoted by I_{IB} and is defined by $I_{IB}=(-1,1,-1,1)$.

Definition 2.6 The identity element of an intuitionistic bipolar fuzzy value in respect to seral conjunction is denoted by e_{IB} and is defined by $e_{IB}=(0,1,0,1)$.

Proposition 2.1 *Let a, b, c be intuitionistic bipolar fuzzy values where* $a = (-a^-, a^+, -a'^-, a'^+)$, $b = (-b^-, b^+, -b'^-, b'^+)$ *and* $c = (-c^-, c^+, -c'^-, c'^+)$, *then the following properties are satisfied:*

1. $a + b = b + a, a \cdot b = b \cdot a$ *and* $a \times b = b \times a$.
2. $a + (b + c) = (a + b) + c, a \cdot (b \cdot c) = (a \cdot b) \cdot c$ *and* $a \times (b \times c) = (a \times b) \times c$.
3. $a + 0_{IB} = 0_{IB} + a = a, a \cdot 0_{IB} = 0_{IB} \cdot a = a$ *and* $a \times 0_{IB} = 0_{IB} \times a = a$.
4. *Inverse element does not exist except the identity in respect to the operations.*
5. $a \cdot (b + c) = a \cdot b + a \cdot c$ *and* $a \times (b + c) = a \times b + a \times c$.
6. $a - b = -(b - a)$ *such that* $a - b = a + (-b)$.
7. $a \cdot (-b), (-a) \cdot b, -(a \cdot b)$ *are not equal while* $a \times (-b) = (-a) \times b = -(a \times b)$.
8. $a \cdot (b - c) \neq a \cdot b - a \cdot c$ *while* $a \times (b - c) = a \times b - a \times c$.

Proof: Given that $a = (-a^-, a^+, -a'^-, a'^+), b = (-b^-, b^+, -b'^-, b'^+)$ and $c = (-c^-, c^+, -c'^-, c'^+)$ then:

1. $a \times b = (-a^-, a^+, -a'^-, a'^+) \times (-b^-, b^+, -b'^-, b'^+)$
$= (-\max\{\min\{a^-, b^+\}, \min\{a^+, b^-\}\}, \max\{\min\{a^-, b^-\}, \min\{a^+, b^+\}\}, -\max\{\min\{a'^-, b'^+\},$
$\min\{a'^+, b'^-\}\}, \max\{\min\{a'^-, b'^-\}, \min\{a'^+, b'^+\}\})$
$= (-\max\{\min\{a^+, b^-\}, \min\{a^-, b^+\}\}, \max\{\min\{a^-, b^-\}, \min\{a^+, b^+\}\}, -\max\{\min\{a'^+, b'^-\},$
$\min\{a'^-, b'^+\}\}, \max\{\min\{a'^-, b'^-\}, \min\{a'^+, b'^+\}\})$
$= (-\max\{\min\{b^-, a^+\}, \min\{b^+, a^-\}\}, \max\{\min\{b^-, a^-\}, \min\{b^+, a^+\}\}, -\max\{\min\{b'^-, a'^+\},$
$\min\{b'^+, a'^-\}\}, \max\{\min\{b'^-, a'^-\}, \min\{b'^+, a'^+\}\})$
$= b \times a.$
The proofs are simlar for other cases.

2. $a \times (b \times c) = (-a^-, a^+, -a'^-, a'^+) \times ((-b^-, b^+, -b'^-, b'^+) \times (-c^-, c^+, -c'^-, c'^+))$
$= (-a^-, a^+, -a'^-, a'^+) \times (-\max\{\min\{b^-, c^+\}, \min\{b^+, c^-\}\}, \{\max\{b^-, c^-\}, \min\{b^+, c^+\}\},$
$-\max\{\min\{b'^-, c'^+\}, \min\{b'^+, c'^-\}\}, \max\{\min\{b'^-, c'^-\}, \min\{b'^+, c'^+\}\})$
$= (-\max\{\min[a^-, \max\{\min\{b^-, c^-\}, \min\{b^+, c^+\}\}]\}, \min[a^+, \max\{\min\{b^-, c^+\}, \min\{b^+, c^-\}\}],$
$\max\{\min[a^-, \max\{\min\{b^-, c^+\}, \min\{b^+, c^-\}\}]\}, \min[a^-, \max\{\min\{b^-, c^-\}, \min\{b^+, c^+\}\}],$
$-\max\{\min[a'^-, \max\{\min\{b'^-, c'^-\}, \min\{b'^+, c'^+\}\}]\}, \min[a'^+, \max\{\min\{b'^-, c'^+\}, \min\{b'^+, c'^-\}\}],$
$\max\{\min[a'^-, \max\{\min\{b'^-, c'^+\}, \min\{b'^+, c'^-\}\}]\}, \min[a'^-, \max\{\min\{b'^-, c'^-\}, \min\{b'^+, c'^+\}\}])$
$= (-x^-, x^+, -x'^-, x'^+)$ where
$x^- = [\max\{\min\{a^-, \max\{\min\{b^-, c^-\}, \min\{b^+, c^+\}\}\}, \min\{a^+, \max\{\min\{b^-, c^+\}, \min\{b^+, c^-\}\}\}\}]$
$= \max[\min\{a^-, \max\{\min\{b^+, c^+\}, \min\{b^-, c^-\}\}\}, \{\max\{\min\{a^+, \min\{b^-, c^+\}\}\}, \min\{a^-,$
$\min\{b^+, c^-\}\}\}]$
$= \max[\max\{\min\{a^-, \min\{b^+, c^+\}\}, \min\{a^-, \min\{b^-, c^-\}\}\}, \max\{\min\{\min\{a^+, b^-\}, c^+\}, \min$
$\{\min\{a^+, b^+\}, c^-\}\}]$
$= \max[\max\{\min\{\min\{a^-, b^+\}, c^+\}\}, \min\{\min\{a^-, b^-\}, c^-\}, \max\{\min\{\min\{a^+, b^-\}, c^+\}, \min$
$\{\min\{a^+, b^+\}, c^-\}\}]$
$= \max[\max\{\min\{\min\{a^-, b^+\}, c^+\}\}, \min\{\min\{a^+, b^-\}, c^+\}, \max\{\min\{\min\{a^-, b^-\}, c^-\}, \min$
$\{\min\{a^+, b^+\}, c^-\}\}]$
$= \max[\min\{\max\{\min\{a^-, b^+\}, \min\{a^+, b^-\}\}, c^+\}, \min\{\max\{\min\{a^-, b^-\}, \min\{a^+, b^+\}, c^-\}\}].$
Similarly,
$x^+ = \max[\min\{\max\{\min\{a^-, b^+\}, \min\{a^+, b^-\}\}, c^-\}, \min\{\max\{\min\{a^-, b^-\}, \min\{a^+, b^+\}\}, c^+\}]$
$x'^- = \max[\min\{\max\{\min\{a'^-, b'^+\}, \min\{a'^+, b'^-\}\}, c'^+\}, \min\{\max\{\min\{a'^-, b'^-\},$
$\min\{a'^+, b'^+\}\}, c^-\}]$
$x'^+ = \max[\min\{\max\{\min\{a'^-, b'^+\}, \min\{a'^+, b'^-\}\}, c'^-\}, \min\{\max\{\min\{a'^-, b'^-\},$
$\min\{a'^+, b'^+\}\}, c^+\}].$
Also $(a \times b) \times c = (-a^-, a^+, -a'^-, a'^+) \times (-b^-, b^+, -b'^-, b'^+)) \times (-c^-, c^+, -c'^-, c'^+)$
$= (-\max\{\min\{a^-, b^+\}, \min\{a^+, b^-\}\}, \max\{\min\{a^-, b^-\}, \min\{a^+, b^+\}\}, -\max\{\min$

$\{a'^-, b'^+\}, \min\{a'^+, b'^-\}\}, \max\{\min\{a'^-, b'^-\}, \min\{a'^+, b'^+\}\}) \times (-c^-, c^+, -c'^-, c'^+)$
$= (-\max\{\min\{\max\{\min\{a^-, b^+\}, \min\{a^+, b^-\}\}, c^+\}, \min\{\max\{\min\{a^-, b^-\}, \min\{a^+,$
$b^+\}\}, c^-\}\}, \max\{\min\{\max\{\min\{a'^-, b'^+\}, \min\{a'^+, b'^-\}\}, c^-\}, \min\{\max\{\min\{a'^-, b'^-\},$
$\min\{a'^+, b'^+\}, c^+\}\}\}, (-\max\{\min\{\max\{\min\{a'^-, b'^+\}, \min\{a'^+, b'^-\}\}, c'^+\}, \min\{\max$
$\{\min\{a'^-, b'^-\}, \min\{a'^+, b'^+\}\}, c'^-\}\}, \max\{\min\{\max\{\min\{a'^-, b'^+\}, \min\{a'^+, b'^-\}\}, c^-\},$
$\min\{\max\{\min\{a'^-, b'^-\}, \min\{a'^+, b'^+\}, c^+\}\}\}).$
Hence $a \times (b \times c) = (a \times b) \times c$. Other two proofs are similar.

3. $a \times e_{IB} = (-a^-, a^+, -a'^-, a'^+) \times (0, 1, 0, 1) = (-a^-, a^+, -a'^-, a'^+)$ and $e_{IB} \times a = (0, 1, 0, 1) \times (-a^-, a^+, -a'^-, a'^+) = (-a^-, a^+, -a'^-, a'^+)$. Hence $a \times e_{IB} = e_{IB} \times a = a$ where $e_{IB} = (0, 1, 0, 1)$ is called the identity in respect to serial conjunction. Other proofs are similar.

4. Let $b = (-b^-, b^+, -b'^-, b'^+)$ be an intuitionistic bipolar fuzzy value be the inverse of $a = (-a^-, a^+, -a'^-, a'^+)$ in respect to the disjunction $(+)$ operation. Then $a + b = b + a = 0_{IB}$, this means that, $(-a^-, a^+, -a'^-, a'^+) + (-b^-, b^+, -b'^-, b'^+) = (-b^-, b^+, -b'^-, b'^+) + (-a^-, a^+, -a'^-, a'^+) = (0, 0, 0, 0)$.
Or $(-\max\{a^-, b^-\}, \max\{a^+, b^+\}, -\max\{a'^-, b'^-\}, \max\{a'^+, b'^+\}) = (-\max\{b^-, a^-\},$
$\max\{b^+, a^+\}, -\max\{b'^-, a'^-\}, \max\{b'^+, a'^+\}) = (0, 0, 0, 0)$.
Thus $\max\{a^-, b^-\} = 0, \max\{b^+, a^+\} = 0, \max\{b'^-, a'^-\} = 0$ and $\max\{b'^+, a'^+\} = 0$ which implies that $a^- = a^+ = b^- = b^+ = 0$ and $a'^- = a'^+ = b'^- = b'^+ = 0$. But $a = (-a^-, a^+, -a'^-, a'^+)$ be any intuitionistic bipolar fuzzy element. Hence, the inverse element does not exist in respect to the operation disjunction except zero element 0_{IB}. Other proofs can be done by similar way.

5. $a \cdot (b + c) = (-a^-, a^+, -a'^-, a'^+) \cdot [(-b^-, b^+, -b'^-, b'^+) + (-c^-, c^+, -c'^-, c'^+)]$
$= (-a^-, a^+, -a'^-, a'^+)[-\max\{b^-, c^-\}, \max\{b^+, c^+\}, -\max\{b'^-, c'^-\}, \max\{b'^+, c'^+\}]$
$= [-\min\{a^-, \max\{b^-, c^-\}\}, \max\{a^+, \max\{b^+, c^+\}\}, -\max\{a'^-, \max\{b'^-, c'^-\}\},$
$\min\{a'^+, \max\{b'^+, c'^+\}\}]$
$= [\max\{-\min\{a^-, b^-\}, -\min\{a^-, c^-\}\}, \max\{\max\{a^+, b^+\}, \max\{a^+, c^+\}\}, \max\{-\max$
$\{a'^-, b'^-\}, -\max\{a'^-, c'^-\}\}, \max\{\min\{a'^+, b'^+\}, \min\{a'^+, c'^+\}\}]$
$= [-\min\{a^-, b^-\}, \max\{a^+, b^+\}, -\max\{a'^-, b'^-\}, \min\{a'^+, b'^+\}] + [-\min\{a^-, c^-\}, \max$
$\{a^+, c^+\}, -\max\{a'^-, c'^-\}, \min\{a'^+, c'^+\}]$
$= (-a^-, a^+, -a'^-, a'^+) \cdot (-b^-, b^+, -b'^-, b'^+) + (-a^-, a^+, -a'^-, a'^+) \cdot (-c^-, c^+, -c'^-, c'^+)$
$= a \cdot b + a \cdot c$. Similarly, we can prove the other results.

6. $a - b = a + (-b) = (-a^-, a^+, -a'^-, a'^+) + (-b^+, b^-, -b'^+, b'^-)$
$[-\max\{a^-, b^+\}, \max\{a^+, b^-\}, -\max\{a'^-, b'^+\}, \max\{a'^+, b'^-\}]$.
Also $-[b - a] = -[b + (-a)] = -[(-b^+, b^-, -b'^+, b'^-) + (-a^+, a^-, -a'^+, a'^-)]$
$= -[-\max\{b^-, a^+\}, \max\{b^+, a^-\}, -\max\{b'^-, a'^+\}, \max\{b'^+, a'^-\}]$.
$= [-\max\{b^+, a^-\}, \max\{b^-, a^+\}, -\max\{b'^+, a'^-\}, \max\{b'^-, a'^+\}]$
$= [-\max\{a^-, b^+\}, \max\{a^+, b^-\}, -\max\{a'^-, b'^+\}, \max\{a'^+, b'^-\}]$.
Hence $a - b = -(b - a)$.

7. $a \cdot (-b) = (-a^-, a^+, -a'^-, a'^+) \cdot (-b^+, b^-, -b'^+, b'^-) = [-\min\{a^-, b^+\}, \max\{a^+, b^-\},$
$-\max\{a'^-, b'^+\}, \min\{a'^+, b'^-\}]$.
$(-a) \cdot b = (-a^+, a^-, -a'^+, a'^-) \cdot (-b^-, b^+, -b'^-, b'^+) = [-\min\{a^+, b^-\}, \max\{a^-, b^+\},$
$-\max\{a'^+, b'^-\}, \min\{a'^-, b'^+\}]$
and $-(a \cdot b) = -[(-a^-, a^+, -a'^-, a'^+) \cdot (-b^-, b^+, -b'^-, b'^+)]$
$= -[-\min\{a^-, b^-\}, \max\{a^+, b^+\}, -\max\{a'^-, b'^-\}, \min\{a'^+, b'^+\}]$
$= [-\max\{a^+, b^+\}, \min\{a^-, b^-\}, -\min\{a'^+, b'^+\}, \max\{a'^-, b'^-\}]$.
Thus $a \cdot (-b), (-a) \cdot b$ and $-(a \cdot b)$ are all equal. Actually, $a \cdot (-b) = -[(-a) \cdot b] \neq -(a \cdot b) = (-a) \cdot (-b)$.

But for serial conjunction $a \times (-b) = (-a^-, a^+, -a'^-, a'^+) \times (-b^+, b^-, -b'^+, b'^-)$
$= [-\max\{\min\{a^-, b^-\}, \min\{a^+, b^-\}\}, \max\{\min\{a^-, b^+\}, \min\{a^+, b^-\}\}, -\max\{\min$
$\{a'^-, b'^-\}, \min\{a'^+, b'^+\}\}, \max\{\min\{a'^-, b'^+\}, \min\{a'^+, b'^-\}\}]$.
$(-a) \times b = (-a^+, a^-, -a'^+, a'^-) \times (-b^-, b^+, -b'^-, b'^+)$
$= [-\max\{\min\{a^+, b^+\}, \min\{a^-, b^-\}\}, \max\{\min\{a^+, b^-\}, \min\{a^-, b^+\}\}, -\max\{\min\{a'^+, b'^+\},$

$\min\{a'^-, b'^-\}\}, \max\{\min\{a'^+, b'^-\}, \min\{a'^-, b'^+\}\}]$
$-(a \times b) = -[(-a^-, a^+, -a'^-, a'^+) \times (-b^-, b^+, -b'^-, b'^+)]$
$= -[-\max\{\min\{a^-, b^+\}, \min\{a^+, b^-\}\}, \max\{\min\{a^-, b^-\}, \min\{a^+, b^+\}\}, -\max\{\min\{a'^-, b'^+\},$
$\min\{a'^+, b'^-\}\}, \max\{\min\{a'^-, b'^-\}, \min\{a'^+, b'^+\}\}]$
$= [-\max\{\min\{a^-, b^-\}, \min\{a^+, b^+\}\}, \max\{\min\{a^-, b^+\}, \min\{a^+, b^-\}\}, -\max\{\min\{a'^-, b'^-\},$
$\min\{a'^+, b'^+\}\}, \max\{\min\{a'^-, b'^+\}, \min\{a'^+, b'^-\}\}]$. Hence $a \times (-b) = (-a) \times b = -(a \times b)$.

8. Using (6) and (7), we can easly prove that:
$a \cdot (b - c) = a \cdot (b + (-c)) = a \cdot b + a \cdot (-c) \neq a \cdot b - a \cdot c$ (since $a \cdot (-c) \neq -(a \cdot c)$).
But $a \times (b - c) = a \times (b + (-c)) = a \times b + a \times (-c) = a \times b - a \times c$.

□

3. Intuitionistic bipolar fuzzy relation

In this section, we introduce Cartesion product of two intuitionistic bipolar fuzzy sets and relation. Also, several basic properties are investigated.

Definition 3.1 Let A and B be two universe of discorses and let $X = \{a = (-a^-, a^+, -a'^-, a'^+);$ $a \in A\}$ and $Y = \{b = (-b^-, b^+, -b'^-, b'^+); b \in B\}$ be two intuitionistic bipolar fuzzy sets. The Cartesion product of X and Y is denoted by $X \times Y$ and is defined by $X \times Y = \{(a, b); a \in A$ of $b \in B\}$.

Definition 3.2 An intuitionistic bipolar fuzzy relation between two intuitionistic bipolar fuzzy sets X and Y as an intuitionistic bipolar fuzzy set in $X \times Y$. If R is a relation between X and $Y, a \in X$ and $b \in Y$, if $-r^-(a, b), -r'^-(a, b), r^+(a, b), r'^+(a, b)$ denote the negative and positive membership values of which a is in relation R with b, then $r = (-r^-, r^+, -r'^-, r'^+) \in R$.
Now we define an order relation " $\leq$ " below:

Definition 3.3 (Inclusion) If IBFS is an intuitionistic bipolar fuzzy set over A and if $a, b \in$ IBFS where $a = (-a^-, a^+, -a'^-, a'^+)$ and $b = (-b^-, b^+, -b'^-, b'^+)$, then $a \leq b$ if and only if $a^- \leq b^-, a^+ \leq b^+, a'^- \leq b'^-, a'^+ \leq b'^+$, that is, $a \leq b$ if and only if $a + b = b$.

Definition 3.4 Let IBFS be an intuitionistic bipolar fuzzy set over A and let $a, b \in$ IBFS where $a = (-a^-, a^+, -a'^-, a'^+)$ and $b = (-b^-, b^+, -b'^-, b'^+)$, then $a < b$ if and only if $a \leq b$ and $a \neq b$.

Lemma 3.1 *The relation " $\leq$ " is partial order relation in an intuitionistic bipolar fuzzy set.*

Proof:

1. Since $a^- \leq a^-, a^+ \leq a^+, a'^- \leq a'^-$, and $a'^+ \leq a'^+$, so we write $a \leq a$ for all a. That is, the relation " $\leq$ " is reflexive.

2. Let $a \leq b$ and $b \leq a$ for any intuitionistic bipolar fuzzy values. Then $a^- \leq b^-, a^+ \leq b^+, a'^- \leq b'^-$, and $a'^+ \leq b'^+$ or $a^- = b^-, a^+ = b^+, a'^- = b'^-$ and $a'^+ = b'^+$ or $a = b$ for any intuitionistic bipolar fuzzy values a, b. That is, the relation " $\leq$ " is anti symmetric.

3. Let $a \leq b$ and $b \leq c$ for any intuitionistic bipolar fuzzy values a, b, c. Then $a^- \leq b^-, a^+ \leq b^+, a'^- \leq b'^-, a'^+ \leq b'^+$ and $b^- \leq c^-, b^+ \leq c^+, b'^- \leq c'^-, b'^+ \leq c'^+$ or $a^- \leq b^- \leq c^-, a^+ \leq b^+ \leq c^+, a'^- \leq b'^- \leq c'^-$ and $a'^+ \leq b'^+ \leq c'^+$ or $a^- \leq c^-, a^+ \leq c^+, a'^- \leq c'^-$, and $a'^+ \leq c'^+$ Thus $a \leq b$ and $b \leq c$ implies $a \leq c$. that is, the relation " $\leq$ " is transitive. Hence, the relation " $\leq$ " in an intuitionistic bipolar fuzzy set is a partial order relation.

□

Lemma 3.2 *Let a, b, c be an intuitionistic bipolar fuzzy values where $a = (-a^-, a^+, -a'^-, a'^+)$, $b = (-b^-, b^+, -b'^-, b'^+)$ and $c = (-c^-, c^+, -c'^-, c'^+)$ then:*

1. *$0_{IB} \leq a \leq I_{Ib}$ for any a.*

2. *If $a \leq b$ then $a + c \leq b + c$ and $a \cdot c \leq b \cdot c$.*

3. *$a \leq a + b$ and $b \leq a + b$, then $a + b$ is the leas upper bound of a and b. In other words, if there is an element C satisfying $a \leq c$ and $b \leq c$ then $a + b \leq c$.*

4. *$a \cdot b \leq a$ and $a \cdot b \leq b$, that is, $a \cdot b$ is a lower bound of a and b.*

5. *$a \cdot b \cdot c \leq a \cdot b$.*

Proof:

1. Since $a = (-a^-, a^+, -a'^-, a'^+)$, then for $0 \leq -a^-, a^+, -a'^-, a'^+ \leq 1$. Hence $0_{IB} \leq a \leq I_{Ib}$.

2. Let $a \leq b$ then $a^- \leq b^-, a^+ \leq b^+, a'^- \leq b'^-$ and $a'^+ \leq b'^+$. Therefor, $\max\{a^-, c^-\} \leq \max\{b^-, c^-\}$, $\max\{a^+, c^+\} \leq \max\{b^+, c^+\}, \max\{a'^-, c'^-\} \leq \max\{b'^-, c'^-\}$ and max $\{a'^+, c'^+\} \leq \max\{b'^+, c'^+\}$. Thus $a + b \leq b + c$.
   Also, $\min\{a^-, c^-\} \leq \min\{b^-, c^-\}, \max\{a^+, c^+\} \leq \max\{b^+, c^+\}, \max\{a'^-, c'^-\} \leq \max$ $\{b'^-, c'^-\}$ and $\min\{a'^+, c'^+\} \leq \min\{b'^+, c'^+\}$. Hence $a \cdot c \leq b \cdot c$.

3. We know that $a^- \leq \max\{a^-, b^-\}, a^+ \leq \max\{a^+, b^+\}, a'^- \leq \max\{a'^-, b'^-\}$ and $a'^+ \leq \max\{a'^+, b'^+\}$. So $a \leq a + b$. Similarly $b \leq a + b$. Thus $a + b$ is the upper bound of a and b.
   If possible let $c \neq a + b$ be the least upper bound of a and b, then $a \leq c$ and $b \leq c$, that is, $a^- \leq C^-, a^+ \leq C^+, a'^- \leq C'^-, a'^+ \leq C'^+$ and $b^- \leq C^-, b^+ \leq C^+, b'^- \leq C'^-, b'^+ \leq C'^+$, that is, $\max\{a^-, b^-\} \leq c^-, \max\{a^+, b^+\} \leq c^+, \max\{a'^-, b'^-\} \leq c'^-$ and $\max\{a'^+, b'^+\} \leq c'^+$. Thus
   $$a + b \leq c \qquad (1)$$
   Also, since $a + b$ is the upper bound of a, b and c is the least upper bound, so
   $$c \leq a + b \qquad (2)$$
   by (1) and (2), we can write as $a + b = c$. That is $a + b$ is the least upper bound of a and b.

4. similarly, we can prove that $a \cdot b$ is the greatest lower bound of a and b.

5. We know that $\min\{a^-, c^-, b^-\} \leq \min\{a^-, b^-\}, \max\{a^+, c^+, b^+\} \leq \max\{a^+, b^+\}$, max $\{a'^-, c'^-, b'^-\} \leq \min\{a'^-, b'^-\}$ and $\min\{a'^+, c'^+, b'^+\} \leq \min\{a'^+, b'^+\}$. Therefor $a \cdot c \cdot b \leq a \cdot b$.

□

4. Intuitionistic bipolar fuzzy matrix

In this section we devlop an intuitionistic bipolar fuzzy matrix.

Proposition 4.1 *An intuitionistic bipolar fuzzy algebra is a mathematical system $(I_{BF}, +, \cdot)$ with two binary operations $+$ and $\cdot$ defined on I_{BF} satisfying the following axioms:*

1. *Idenpotent: $a + a = a, a \cdot a = a$.*

2. *Commutitivity: $a + b = b + a, a \cdot b = b \cdot a$.*

3. *Associativity: $a + (b + c) = (a + b) + c, a \cdot (b \cdot c) = (a \cdot b) \cdot c$.*

4. *Absorption: $a + (a \cdot b) = a, a \cdot (a + b) = a$.*

5. *Distributivity: $a \cdot (b + c) = (a \cdot b) + (a \cdot c), a + (b \cdot c) = (a + b) \cdot (a + c)$.*

6. *Universal bounds: $a + 0_{IB} = a, a + I_{IB} = a, a \cdot 0_{IB} = 0_{IB}$ and $a \cdot I_{IB} = a$, where $a = (-a^-, a^+, -a'^-, a'^+), b = (-b^-, b^+, -b'^-, b'^+)$ and $c = (-c^-, c^+, -c'^-, c'^+) \in I_{BF}$.*

Proof: Axioms 1,2,3,4,6 are already proved. The proofs of absorption and second distributive low are given now.
4. To prove the absorption property, we take the left hand side of first as:
$a + (\cdot b) = (-a^-, a^+, -a'^-, a'^+) + [(-a^-, a^+, -a'^-, a'^+) \cdot (-b^-, b^+, -b'^-, b'^+)]$
$= (-a^-, a^+, -a'^-, a'^+) + [-\min\{a^-, b^-\}, \max\{a^+, b^+\}, -\max\{a'^-, b'^-\}, \min\{a'^+, b'^+\}]$
$(-\max\{a^-, \min\{a^-, b^-\}\}, \max\{a^+, \max\{a^+, b^+\}, -\max\{a'^-, \max\{a'^-, b'^-\}, \max\{a'^+, \min$
$\{a'^+, b'^+\}) = (-a^-, a^+, -a'^-, a'^+) = a.$
Similary, we can prove the second part: $a + (b \cdot c) = (-a^-, a^+, -a'^-, a'^+) + [(-b^-, b^+, -b'^-, b'^+) \cdot (-c^-, c^+, -c'^-, c'^+)] = (-a^-, a^+, -a'^-, a'^+) + [-\min\{b^-, c^-\}, \max\{b^+, c^+\}, -\max\{b'^-, c'^-\},$
$\min\{b'^+, c'^+\}] = (-\max\{a^-, \min\{b^-, c^-\}\}, \max\{a^+, \max\{b^+, c^+\}\}, -\max\{a'^-, \max\{b'^-, c'^-\}\},$
$\max\{a'^+, \min\{b'^+, c'^+\}\}) = (-\min\{\max\{a^-, b^-\}, \max\{a^-, c^-\}\}, \max\{\max\{a^+, b^+\}, \max\{a^+,$
$c^+\}\}, -\max\{\max\{a'^-, b'^-\}, \max\{a'^-, c'^-\}\}, \min\{\max\{a'^+, b'^+\}, \max\{a'^+, c'^+\}\})$
$= (-\max\{a^-, b^-\}, \max\{a^+, b^+\}, -\max\{a'^-, b'^-\}, \max\{a'^+, b'^+\}) \cdot (-\max\{a^-, c^-\}, \max\{a^+, c^+\},$
$-\max\{a'^-, c'^-\}, \max\{a'^+, c'^+\}) = (a + b) \cdot (a + c).$
Therefore $(I_{BF}, +, \cdot)$ is an intuitionistic bipolar fuzzy algebra. □

Definition 4.1 An intuitionistic bipolar fuzzy matrix (IBFM) is the matrix over the intuitionistic bipolar fuzzy algebra.

Remark 4.1

1. The zero intuitionistic bipolar fuzzy matrix 0_n of $n \times n$ is the matrix where all the elements are $0_{IB} = (0, 0, 0, 0)$.
2. The identity intuitionistic bipolar fuzzy matrix I_n of order $n \times n$ is the matrix where all the diagonal entries are $I_{IB} = (-1, 1, -1, 1)$ and all other entries are $0_{IB} = (0, 0, 0, 0)$.
3. The set of all rectangular intuitionistic bipolar fuzzy matrix of order $n \times m$ is denoted by $M_{n\times m}$.
4. The square intuitionistic bipolar fuzzy matrix of order $n \times n$ is denoted by M_n.
5. If $A = (a_{ij})_{n\times m} \in M_{n\times m}$, then $a_{ij} = (-a^-_{ij}, a^+_{ij}, -a'^-_{ij}, a'^+_{ij}) \in I_{BF}$, where $-a^-_{ij}, a^+_{ij}, -a'^-_{ij}$, $a'^+_{ij} \in [0, 1]$ are the degree of positive membership, degree of negative membership and the degree of positive nonmembership, degree of negative nonmembership values of the element a_{ij}, respectively.

Definition 4.2 Let $A =< a_{ij} >, B =< b_{ij} >\in M_{n\times m}$ be two intuitionistic bipolar fuzzy matrices. Therefore $a_{ij}, b_{ij} \in I_{BF}$, then
1. $A+B =< a_{ij}+b_{ij} >_{n\times m}=< -\max\{a^-_{ij}, b^-_{ij}\}, \max\{a^+_{ij}, b^+_{ij}\}, -\max\{a'^-_{ij}, b'^-_{ij}\}, \max\{a'^+_{ij}, b'^+_{ij}\} >_{n\times m}$.
2. $A \cdot B =< a_{ij} \cdot b_{ij} >_{n\times m}=< -\min\{a^-_{ij}, b^-_{ij}\}, \max\{a^+_{ij}, b^+_{ij}\}, -\max\{a'^-_{ij}, b'^-_{ij}\}, \min\{a'^+_{ij}, b'^+_{ij}\} >_{n\times m}$.

Definition 4.3 Let $A =< a_{ij} >\in M_{n\times m}$ and $B =< b_{ij} >\in M_{m\times k}$ be two intuitionistic bipolar fuzzy matrices. Therefore $a_{ij}, b_{ij} \in I_{BF}$, then
$A \bigodot B = (\sum_{r=1}^{m} a_{ir} \cdot b_{rj})_{n\times k} = (-\max_{r=1}^{m}[\min\{a^-_{ir}, b^-_{rj}\}], \max_{r=1}^{m}[\max\{a^+_{ir}, b^+_{rj}\}],$
$-\max_{r=1}^{m}[\max\{a'^-_{ir}, b'^-_{rj}\}], \max_{r=1}^{m}[\min\{a'^+_{ir}, b'^+_{rj}\}])_{n\times k}$.
If $A \bigotimes B = (\prod_{r=1}^{m} a_{ir} + b_{rj})_{n\times k} = (-\min_{r=1}^{m}[\max\{a^-_{ir}, b^-_{rj}\}], \min_{r=1}^{m}[\max\{a^+_{ir}, b^+_{rj}\}],$
$-\min_{r=1}^{m}[\max\{a'^-_{ir}, b'^-_{rj}\}], \min_{r=1}^{m}[\max\{a'^+_{ir}, b'^+_{rj}\}])_{n\times k}$.

Proposition 4.2 *If the intuitionistic bipolar fuzzy matrices A, B, C are conformal for corresponding operations, then*

1. $A + B = B + A, A \cdot B = B \cdot A$.
2. $A + (B + C) = (A + B) + C, A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

3. $A \cdot (B + C) = A \cdot B + A \cdot C, A + (B \cdot C) = (A + B) \cdot (A + C)$.

4. $A + 0 = 0 + A = A, A \cdot 0 = 0 \cdot A = A$, *where* 0 *is the zero matrix with appropriate order.*

5. $A \bigodot B \neq B \bigodot A, A \bigotimes B \neq B \bigotimes A$, *in general.*

6. $A \bigodot (B \bigodot C) = (A \bigodot B) \bigodot C, A \bigotimes (B \bigotimes C) = (A \bigotimes B) \bigotimes C$.

7. $A \bigodot I = I \bigodot A = A, A \bigotimes I = I \bigotimes A = A$, *where* I *is the identity matrix with appropriate order.*

8. $A \bigodot (B + C) \neq (A \bigodot B) + (A \bigodot C)$,
$A \bigotimes (B \cdot C) \neq (A \bigotimes B) \cdot (A \bigotimes C)$.

Proof: The proof of $1, 2$, and 4 is simple for the matrices A, B, C of the same order.
(3) this property can be proved by using the distributive property on intuitionistic bipolar fuzzy set.
(5) from the definition, we say that $A \bigodot B$ and $A \bigotimes B$ are possible of the order of A and B is $k \times i$ of $i \times t$ respectively. Thus, if the order of A and B is $k \times i$ of $i \times t$ then $B \bigodot A$ and $B \bigotimes A$ do not exist.

Now if both the matrices are square of the same order, then for both cases the matrices are conformable, but the equality does not hold in general. to verify it, we consider the following example:

Example:

Let $A = \begin{bmatrix} (-0.4, 0.7, -0.5, 0.2) & (-0.5, 0.8, -0.3, 0.2) & (-0.5, 0.6, -0.5, 0.2) \\ (-0.5, 0.6, -0.4, 0.3) & (-0.3, 0.4, -0.2, 0.3) & (-0.3, 0.5, -0.3, 0.2) \\ (-0.4, 0.5, -0.4, 0.3) & (-0.24, 0.36, -0.15, 0.9) & (-0.25, 0.25, -0.15, 0.9) \end{bmatrix}$,

$B = \begin{bmatrix} (-0.6, 0.5, -0.2, 0.4) & (-0.5, 0.6, -0.4, 0.3) & (-0.1, 0.2, -0.6, 0.4) \\ (-0.3, 0.7, -0.8, 0.5) & (-0.1, 0.3, -0.4, 0.2) & (-0.4, 0.8, -0.9, 0.6) \\ (-0.1, 0.2, -0.9, 0.6) & (-0.6, 0.3, -0.1, 0.4) & (-0.3, 0.5, -0.6, 0.9) \end{bmatrix}$

then $A \bigodot B = \begin{bmatrix} (-0.4, 0.8, -0.9, 0.2) & (-0.5, 0.8, -0.5, 0.2) & (-0.4, 0.8, -0.9, 0.2) \\ (-0.5, 0.7, -0.9, 0.3) & (-0.5, 0.6, -0.4, 0.3) & (-0.4, 0.8, -0.9, 0.3) \\ (-0.4, 0.7, -0.9, 0.3) & (-0.4, 0.6, -0.4, 0.3) & (-0.25, 0.8, -0.9, 0.3) \end{bmatrix}$

and $B \bigodot A = \begin{bmatrix} (-0.5, 0.7, -0.6, 0.4) & (-0.5, 0.8, -0.6, 0.3) & (-0.5, 0.6, -0.6, 0.2) \\ (-0.4, 0.8, -0.9, 0.3) & (-0.3, 0.8, -0.9, 0.2) & (-0.3, 0.8, -0.9, 0.2) \\ (-0.5, 0.7, -0.9, 0.3) & (-0.3, 0.8, -0.9, 0.3) & (-0.3, 0.6, -0.9, 0.2) \end{bmatrix}$

Note that $A \bigodot B \neq B \bigodot A$.

Also $A \bigotimes B = \begin{bmatrix} (-0.6, 0.6, -0.5, 0.4) & (-0.5, 0.6, -0.4, 0.2) & (-0.4, 0.6, -0.6, -0.4) \\ (-0.3, 0.5, -0.4, 0.4) & (-0.3, 0.9, -0.3, 0.3) & (-0.3, 0.5, -0.6, 0.4) \\ (-0.25, 0.25, -0.4, 0.4) & (-0.24, 0.3, -0.15, 0.3) & (-0.3, 0.5, -0.6, 0.4) \end{bmatrix}$

and $B \bigotimes A = \begin{bmatrix} (-0.4, 0.5, -0.4, 0.3) & (-0.24, 0.6, -0.3, 0.3) & (-0.25, 0.25, 0.4, 0.3) \\ (-0.4, 0.6, -0.4, 0.3) & (-0.3, 0.4, -0.4, 0.3) & (-0.3, 0.25, -0.4, 0.2) \\ (-0.4, 0.5, -0.4, 0.4) & (-0.3, 0.4, -0.2, 0.4) & (-0.3, 0.5, -0.3, 0.4) \end{bmatrix}$

Hence $A \bigotimes B \neq B \bigotimes A$.

(6) If $A \in M_{m \times k}, B \in M_{k \times r}$ and $C \in M_{r \times n}$ and $B \bigodot C = (d_{ij}) \in M_{k \times n}$. Then ij^{th} entries of $B \bigodot C$ is $d_{ij} = \sum_{t=1}^{r} b_{it} c_{tj}$ such that $B = (b_{ij})$ and $C = (c_{ij})$. Therefore, the ij^{th} entries of $A \bigodot (B \bigodot C)$ is

$$\begin{aligned} \textstyle\sum_{s=1}^{k} a_{is} d_{sj} &= \textstyle\sum_{s=1}^{k} a_{is} \cdot (\sum_{t=1}^{r} b_{st} \cdot c_{tj}) \\ &= \textstyle\sum_{s=1}^{k} \sum_{t=1}^{r} a_{is} \cdot b_{st} \cdot c_{tj} \\ &= \textstyle\sum_{t=1}^{r} \sum_{s=1}^{k} a_{is} \cdot b_{st} \cdot c_{tj} \\ &= \textstyle\sum_{t=1}^{r} (\sum_{s=1}^{k} a_{is} \cdot b_{st}) \cdot c_{tj} \\ &= \textstyle\sum_{t=1}^{r} e_{it} \cdot c_{tj} \end{aligned}$$

where $A \bigodot B = (e_{it}) \in M_{m \times r}$. This is the ij^{th} entries of $(A \bigodot B) \bigodot C$. Therefore $A \bigodot (B \bigodot C) = (A \bigodot B) \bigodot C$. The proof of second part is similar.

(7) For the square matrix $A =< a_{ij} >$ and the identity matrix $I_{IB} = (e_{ij})$ of the same order, suppose n, then the ij^{th} entry of $A \bigodot I = (b_{ij})$, such that $b_{ij} = \sum_{r=1}^{n} a_{ir} e_{rj}$. Therefore $b_{ij} = a_{i1} \cdot e_{1j} + a_{i2} \cdot e_{2j} + \cdots + a_{ij-1} \cdot e_{j-1j} + a_{ij} \cdot e_{jj} + a_{ij+1} \cdot e_{jj+1} + \cdots + a_{in} \cdot e_{nj}$

$= a_{i1}\cdot 0_{IB}+a_{i2}\cdot 0_{IB}+\cdots+a_{ij-1}\cdot 0_{IB}+a_{ij}\cdot 0_{IB}+a_{ij+1}\cdot 0_{IB}+\cdots+a_{in}\cdot 0_{IB}=a_{ij}$. Therefore $A\bigodot I=A$. Similarly, $I\bigodot A=A$. Hence $A\bigodot I=I\bigodot A=A$. Similarly, we can prove the second part.

(8) To verify this part, we consider the following example:

Let $A=\begin{bmatrix}(-0.3,0.6,-0.2,0.4) & (-0.2,0.8,-0.3,0.7)\\(-0.4,0.5,-0.6,0.8) & (-0.1,0.9,-0.5,0.3)\end{bmatrix}$,

$B=\begin{bmatrix}(-0.4,0.6,-0.7,0.5) & (-0.5,0.6,-0.2,0.7)\\(-0.6,0.4,-0.1,0.3) & (-0.7,0.7,-0.4,0.2)\end{bmatrix}$

and $C=\begin{bmatrix}(-0.2,0.7,-0.6,0.1) & (-0.5,0.6,-0.9,0.6)\\(-0.3,0.8,-0.8,0.3) & (-0.4,0.5,-0.3,0.3)\end{bmatrix}$.

Then $B\bigotimes C=\begin{bmatrix}(-0.4,0.7,-0.7,0.5) & (-0.5,0.6,-0.3,0.6)\\(-0.6,0.7,-0.6,0.3) & (-0.6,0.6,-0.4,0.3)\end{bmatrix}$

$A\bigodot(B\bigotimes C)=\begin{bmatrix}(-0.3,0.8,-0.7,0.4) & (-0.3,0.8,-0.4,0.4)\\(-0.4,0.9,-0.7,0.5) & (-0.4,0.9,-0.6,0.6)\end{bmatrix}$.

Now $A\bigodot B=\begin{bmatrix}(-0.3,0.8,-0.7,0.4) & (-0.3,0.8,-0.4,0.4)\\(-0.4,0.9,-0.7,0.5) & (-0.4,0.9,-0.6,0.7)\end{bmatrix}$

and $A\bigodot C=\begin{bmatrix}(-0.2,0.8,-0.8,0.3) & (-0.3,0.8,-0.9,0.4)\\(-0.2,0.9,-0.8,0.3) & (-0.4,0.9,-0.9,0.6)\end{bmatrix}$.

Thus $A\bigodot B\bigotimes A\bigodot C=\begin{bmatrix}(-0.3,0.8,-0.8,0.4) & (-0.3,0.8,-0.9,0.4)\\(-0.4,0.9,-0.8,0.5) & (-0.4,0.9,-0.9,0.5)\end{bmatrix}$.

Therefore $A\bigodot(B\bigotimes C)\neq(A\bigodot B)\bigotimes(A\bigodot C)$. Second part can be verified by similar way. □

Definition 4.4 A sequence of an intuitionistic bipolar fuzzy matrices $A_1,A_2,\ldots,A_n,A_{n+1}$, $\ldots$, that is, $\{A_n\}$ is said to be converged to a finite intuitionistic bipolar fuzzy matrix A (if exist) if $\lim_{n\to\infty}A_n=A$.

Definition 4.5 A converence power of an intuitionistic bipolar fuzzy matrix A is a least positive integer k in respect to a binary composition " $\star$ " if:
$A^{k+m}=A^{k+m-1}=A^{k+m-2}=\cdots=A^{k+1}=A^k$ where $m\in N$ and $A^2=A\star A$, $A^3=A\star A\star A=A^2\star A$ and so on. The number k is said to be the index of A and is denoted by $i(A)$.

Definition 4.6 The partial order relation " $\leq$ " over M_n is defined as $A\leq B$ if and only if $a_{ij}\leq b_{ij}$ for all $i,j\in\{1,2,\ldots,n\}$ such that $A=<a_{ij}>, B=<b_{ij}>\in M_n$. That is, $A\leq B$ if and only if $A+B=B$, $A<B$ holds if and only if $A\leq B$ and $A\neq B$.

Definition 4.7 If $A=<a_{ij}>$ is an intuitionistic bipolar fuzzy matrix. The ij^{th} entry of the square matrix A^t is denoted by $a_{ij}^{(t)}$ and

$$a_{ij}^{(t)}=\sum_{1\leq j_1,j_2,\ldots,j_{r-1}\leq n}\{a_{ij_1},a_{ij_2},a_{ij_3},\ldots,a_{ij_{r-1}}\}\qquad(\star)$$

Definition 4.8 An intuitionistic bipolar fuzzy matrix A is said to be nilpotent of order r if $A^r=(0_{IB})$ for some $r\in N$, and A is idempotent if $A^2=A$.

Proposition 4.3 *If A is an intuitionistic bipolar fuzzy matrix of order n and if $k>n$, then $A^k\leq\sum_{i=0}^{n-1}A^i$, such that $A^0=(I_{IB})_n$. As a result $A^{k+1}\leq\sum_{i=1}^{n}A^i$.*

Proof: Suppose that $B=\sum_{i=0}^{n-1}A^i$. Now $a_{ii}^{(k)}\leq i_{IB}=b_{ii}$ since $a_{ii}^{(0)}=i_{IB}$. If $i\neq j$, we consider an arbitrary summand of right hand side of equality $(\star)$ this means that, $a_{ij_1}\cdot a_{j_1j_2}\cdot a_{j_2j_3}\cdot\ldots a_{j_{k-1}j}$ since $i,j_1,j_2,j_3,\ldots,j_{k-1},j\in\{1,2,\ldots,n\}$ and $k+1>n$, then by delete $a_{j_uj_{u+1}}\cdot a_{u+1j_{u+2}}\cdot a_{u+2j_{u+3}}\cdot\ldots\cdot a_{u-1j_u}$ from the summand $a_{ij_1}\cdot a_{j_1j_2}\cdot a_{j_2j_3}\cdots a_{j_{u-1}j}$, we get $a_{ij_1}\cdot a_{j_1j_2}\cdot a_{j_2j_3}\cdots a_{j_{u-1}j}\leq a_{ij_1}\cdot a_{j_1j_2}\cdot a_{j_2j_3}\cdots a_{j_{u-1}j}\cdot a_{j_uj_{u-1}}\cdots\cdot a_{j_{k-1}j}$.
If the number $u+v-v+2$ of the subscripts in the right hand side of the above inequality still more that n, the same deleting method is used. Hence there is a positive integer $z\leq n-1$ such that $a_{ij_1}\cdot a_{j_1j_2}\cdot a_{j_2j_3}\ldots\cdot$

$a_{j_{k-1}j} \leq a_{iw_1} \cdot a_{w_1w_2} \cdot a_{w_2w_3} \cdot \ldots a_{w_{z-1}j}$. Therefore by definition of A^k we have $a_{ij}^{(k)} \leq \sum_{i=1}^{n-1} a^i t = b_{st}$ this means that $A^k \leq \sum_{i=1}^{n-1} A^i \leq \sum_{i=0}^{n} A^i$. □

Definition 4.9 If A, B, C are intuitionistic bipolar matrices of order n. Then:

1. A is called transitive if $A^2 \leq A$.
2. B is called transitive closure of A if B is transitive, $A \leq B$ and $B \leq C$ for any transitive matrix C, satisfying $A \leq C$.

Notation 4.1 *The transitive closure of A is denoted by $t(A)$.*

Theorem 4.2 *Let A be an intuitionistic bipolar fuzzy matrix of order n. Then the transitive closure of A is given by $t(A) = \sum_{m=1}^{n} A^m$.*

Proof: Suppose that $B = \sum_{m=1}^{n} A^m$, since $A \leq B$ of the set of intuitionistic bipolar matrices is idempotent under addition, we have $B^2 = \sum_{m=2}^{2n} A^m \leq \sum_{m=1}^{2n} A^m$ or $B^2 \leq B + \sum_{m=n+1}^{2n} A^m$ by propositin 4.3 $A^m \leq \sum_{i=1}^{n} A^i = B$ as $m > n$. Hence $B^2 \leq B$.

If there is a matrix C such that $A \leq C$ and $C^2 \leq C$, then $A^2 \leq AC \leq C^2 \leq C$ and by induction we have $A^m \leq C^m \leq C, \forall m \in Z^+$. Hence $B \leq C$ and we get $B = t(A) = \sum_{m=1}^{n} A^m$ by definition of transitive closure. □

Example 4.1 Let $A = \begin{bmatrix} (-0.4, 0.6, -0.3, 0.2) & (-0.3, 0.5, -0.4, 0.7) \\ (-0.1, 0.7, -0.8, 0.3) & (-0.2, 0.4, -0.5, 0.6) \end{bmatrix}$ then

$$A^2 = \begin{bmatrix} (-0.4, 0.6, -0.3, 0.2) & (-0.3, 0.5, -0.4, 0.7) \\ (-0.1, 0.7, -0.8, 0.3) & (-0.2, 0.4, -0.5, 0.6) \end{bmatrix} \bigodot \begin{bmatrix} (-0.4, 0.6, -0.3, 0.2) & (-0.3, 0.5, -0.4, 0.7) \\ (-0.1, 0.7, -0.8, 0.3) & (-0.2, 0.4, -0.5, 0.6) \end{bmatrix}$$

$$= \begin{bmatrix} (-0.4, 0.7, -0.8, 0.3) & (-0.3, 0.6, -0.5, 0.6) \\ (-0.1, 0.7, -0.8, 0.3) & (-0.2, 0.7, -0.8, 0.6) \end{bmatrix}.$$

$$t(A) = A + A^2 = \begin{bmatrix} (-0.4, 0.7, -0.8, 0.3) & (-0.3, 0.6, -0.5, 0.7) \\ (-0.1, 0.7, -0.8, 0.3) & (-0.2, 0.7, -0.8, 0.6) \end{bmatrix}.$$

Conclusion

In this article, we defined intuitionistic bipolar fuzzy set, intuitionistic bipolar fuzzy relation, intuitionistic bipolar fuzzy matrix based on an intuitionistic bipolar fuzzy algebras. Also, some results on an intuitionistic bipolar fuzzy matrix are investigated. On an intuitionistic bipolar fuzzy matrix more results can be done such as rank, determinate, invertible matrix, etc. We will work on these topics in our future study.

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