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Bihyperbolic numbers of the Fibonacci type as tridiagonal matrix determinants

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ABSTRACT: Bihyperbolic numbers are extension of hyperbolic numbers to four dimensions. In this paper, we construct a family of tridiagonal matrices which determinants (continuants) and permanents can represent the sequences of bihyperbolic numbers of the Fibonacci type.

Key Words: Bihyperbolic numbers, Fibonacci type numbers, tridiagonal matrix, continuant.

Contents

1. Introduction

Let $n \geq 0$ be an integer. The nth Fibonacci number F_n is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. There are many numbers given by the second order linear recurrence relations. Recall some of them:

Lucas numbers L_n:
$L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$ with $L_0 = 2,\ L_1 = 1$,

Pell numbers P_n:
$P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0,\ P_1 = 1$,

Jacobsthal numbers J_n:
$J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ with $J_0 = 0,\ J_1 = 1$.

Numbers of the Fibonacci type are used as the coefficients of quaternions, split quaternions, hyperbolic numbers and bihyperbolic numbers. Hyperbolic numbers are two dimensional number system. Hyperbolic imaginary unit, so-called *unipotent*, introduced in 1848 by James Cockle (see [10,11,12,13]), is an element $\mathbf{h} \neq \pm 1$ such that $\mathbf{h}^2 = 1$. Some algebraic properties of hyperbolic numbers were given among others in [25,26].

Bihyperbolic numbers are a generalization of hyperbolic numbers. Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + j_1x_1 + j_2x_2 + j_3x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $j_1,\ j_2,\ j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1j_2 = j_2j_1 = j_3,\ j_1j_3 = j_3j_1 = j_2,\ j_2j_3 = j_3j_2 = j_1. \tag{1.1}$$

From the above rules the multiplication of bihyperbolic numbers can be made analogously to the multiplication of algebraic expressions. The addition and the subtraction of bihyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients. The addition and multiplication on $\mathbb{H}_2$ are commutative and associative. Moreover, $(\mathbb{H}_2, +, \cdot)$ is a commutative ring. For the algebraic properties of bihyperbolic numbers, see [1].

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A special kind of bihyperbolic numbers was introduced in [2] as follows.

Let $n \geq 0$ be an integer. The nth bihyperbolic Fibonacci number BhF_n, the nth bihyperbolic Pell number BhP_n and the nth bihyperbolic Jacobsthal number BhJ_n are defined as

$$BhF_n = F_n + j_1 F_{n+1} + j_2 F_{n+2} + j_3 F_{n+3},$$

$$BhP_n = P_n + j_1 P_{n+1} + j_2 P_{n+2} + j_3 P_{n+3},$$

$$BhJ_n = J_n + j_1 J_{n+1} + j_2 J_{n+2} + j_3 J_{n+3},$$

respectively.

Some combinatorial properties of bihyperbolic numbers of the Fibonacci type one can find in [3]. The bihyperbolic Fibonacci, Pell and Jacobsthal numbers satisfy the following recurrence relations. Let $n \geq 2$ be an integer. Then

$$BhF_n = BhF_{n-1} + BhF_{n-2},$$

with

$$BhF_0 = j_1 + j_2 + 2j_3, \quad BhF_1 = 1 + j_1 + 2j_2 + 3j_3,$$

$$BhP_n = 2BhP_{n-1} + BhP_{n-2},$$

with

$$BhP_0 = j_1 + 2j_2 + 5j_3, \quad BhP_1 = 1 + 2j_1 + 5j_2 + 12j_3$$

and

$$BhJ_n = BhJ_{n-1} + 2BhJ_{n-2}$$

with

$$BhJ_0 = j_1 + j_2 + 3j_3, \quad BhJ_1 = 1 + j_1 + 3j_2 + 5j_3.$$

In the literature one can find some connections between determinants or permanents of tridiagonal matrices and the numbers of the Fibonacci type. Strang in [27] gave, probably the first example of determinant of order n, which is equal to the $(n+1)$th Fibonacci number. Cahill et al. in [5] considered matrices with entries being complex numbers. Many authors derived the real or complex matrices which determinants are related to Fibonacci numbers (see e.g. [4,6,17,30]) or numbers of the Fibonacci type, among others Pell numbers ([31]), Jacobsthal numbers ([18]), Horadam numbers ([8]) and different kinds of their generalizations ([19]). For example, Catarino in [7] considered the bicomplex k-Pell quaternions and presented the nth term of this sequence using the determinant of a tridiagonal matrix whose entries are bicomplex k-Pell quaternions. In this paper, we compute determinants and permanents of some tridiagonal matrices which give bihyperbolic numbers of the Fibonacci type.

2. Some matrix representations of bihyperbolic numbers of the Fibonacci type

For an integer $n \geq 1$ let

$$\mathbf{M}_n = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{bmatrix}_{n \times n} \tag{2.1}$$

be any tridiagonal matrix of order n with entries being bihyperbolic numbers. Recall that tridiagonal matrix is called a continuant matrix, and its determinant a continuant, see [23]. The continuant of the matrix $\mathbf{M}_n$ satisfies the recurrence relations

$$\begin{aligned} &\det \mathbf{M}_1 = a_1, \\ &\det \mathbf{M}_2 = a_1 a_2 - c_1 b_1, \\ &\det \mathbf{M}_n = a_n \det \mathbf{M}_{n-1} - c_{n-1} b_{n-1} \det \mathbf{M}_{n-2}, \quad \text{for } \ n \geq 3. \end{aligned}$$

In the next part of this paper zero elements in tridiagonal matrix will be omitted.

In [19], the authors obtained determinants and permanents of some tridiagonal matrices which give the terms of Fibonacci sequence, Pell sequence and Jacobsthal sequence. Let

$$B_n = \begin{bmatrix} c_1 & -c_2 & & & \\ 1 & c_1 & & & \\ & & \ddots & & \\ & & & c_1 & -c_2 \\ & & & 1 & c_1 \end{bmatrix}_{n\times n}.$$

The following results were proved.

If $c_1 = 1$, $c_2 = 1$ then $\det B_n = F_{n+1}$.

If $c_1 = 2$, $c_2 = 1$ then $\det B_n = P_{n+1}$.

If $c_1 = 1$, $c_2 = 2$ then $\det B_n = J_{n+1}$.

By applying some modifications to the elements of the matrix B_n, we can obtain bihyperbolic numbers as determinants of the resulting matrices.

Theorem 2.1 *Let $n \geq 1$ be an integer and*

$$\mathbf{B}_n = \begin{bmatrix} C_2 & -c_2 & & & \\ C_1 & c_1 & & & \\ & & \ddots & & \\ & & & c_1 & -c_2 \\ & & & 1 & c_1 \end{bmatrix}_{n\times n}.$$

If $C_2 = BhF_2$, $C_1 = BhF_1$, $c_1 = 1$, $c_2 = 1$ then $\det \mathbf{B}_n = BhF_{n+1}$.

If $C_2 = BhP_2$, $C_1 = BhP_1$, $c_1 = 2$, $c_2 = 1$ then $\det \mathbf{B}_n = BhP_{n+1}$.

If $C_2 = BhJ_2$, $C_1 = BhJ_1$, $c_1 = 1$, $c_2 = 2$ then $\det \mathbf{B}_n = BhJ_{n+1}$.

In the following theorems, we will present bihyperbolic Fibonacci type numbers as the determinants of some tridiagonal matrices.

Theorem 2.2 *Let $n \geq 1$ be an integer and*

$$\mathbf{A}_n = \begin{bmatrix} j_1 + j_2 + 2j_3 & j_2 & & & & \\ -1 - j_1 - j_2 & 1 & 1 & & & \\ & -1 & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & -1 & 1 \end{bmatrix}_{n\times n}.$$

Then $BhF_{n-1} = \det \mathbf{A}_n$.

Proof: (by induction on n)

If $n = 1$ then $\det \mathbf{A}_1 = \det [j_1 + j_2 + 2j_3] = j_1 + j_2 + 2j_3 = BhF_0$.

Let $n = 2$. Then by (1.1) we get

$$\begin{aligned} \det \mathbf{A}_2 = \det \begin{bmatrix} j_1 + j_2 + 2j_3 & j_2 \\ -1 - j_1 - j_2 & 1 \end{bmatrix} \\ &= j_1 + j_2 + 2j_3 - (-1 - j_1 - j_2) \cdot j_2 \\ &= j_1 + j_2 + 2j_3 + j_2 + j_3 + 1 \\ &= 1 + j_1 + 2j_2 + 3j_3 = BhF_1. \end{aligned}$$

Now assume that for any $n \geq 1$ $BhF_{n-1} = \det \mathbf{A}_n$ and $BhF_n = \det \mathbf{A}_{n+1}$. We shall show that $BhF_{n+1} = \det \mathbf{A}_{n+2}$. By induction hypothesis we have

$$\begin{aligned}
\det \mathbf{A}_{n+2} &= \det \begin{bmatrix} j_1+j_2+2j_3 & j_2 & & & & \\ -1-j_1-j_2 & 1 & 1 & & & \\ & -1 & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & -1 & 1 \end{bmatrix}_{(n+2)\times(n+2)} \\
&= 1 \cdot \det \mathbf{A}_{n+1} - 1 \cdot (-1) \cdot \det \mathbf{A}_n \\
&= BhF_n + BhF_{n-1} = BhF_{n+1},
\end{aligned}$$

which ends the proof. □

In the same way one can prove the next theorems.

Theorem 2.3 *Let $n \geq 1$ be an integer and*

$$\mathbf{F}_n = \begin{bmatrix} j_1+j_2+2j_3 & j_1 & & & & \\ -j_1-j_2-j_3 & 1 & c_1 & & & \\ & -c_1 & 1 & c_1 & & \\ & & & \ddots & & \\ & & & & 1 & c_1 \\ & & & & -c_1 & 1 \end{bmatrix}_{n\times n}.$$

If $c_1 = j_1$ or $c_1 = j_2$ or $c_1 = j_3$, then $BhF_{n-1} = \det \mathbf{F}_n$.

Theorem 2.4 *Let $n \geq 1$ be an integer and*

$$\mathbf{P}_n = \begin{bmatrix} j_1+2j_2+5j_3 & -3+j_1+3j_2+3j_3 & & & & \\ -1-j_1-j_2 & 1 & 1 & & & \\ & -1 & 2 & 1 & & \\ & & & \ddots & & \\ & & & & 2 & 1 \\ & & & & -1 & 2 \end{bmatrix}_{n\times n}.$$

Then $BhP_{n-1} = \det \mathbf{P}_n$.

Theorem 2.5 *Let $n \geq 1$ be an integer and*

$$\mathbf{J}_n = \begin{bmatrix} j_1+j_2+3j_3 & -j_1-4j_3 & & & & \\ 1 & j_1 & 1 & & & \\ & -2 & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & -2 & 1 \end{bmatrix}_{n\times n}.$$

Then $BhJ_{n-1} = \det \mathbf{J}_n$.

Let S_n be the symmetric group which consists of all permutations of $\{1, 2, 3, \ldots, n\}$ and σ be an element of this group where $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Then, the permanent of a square matrix $A = [a_{ij}]$ of order n is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ in the symmetric group S_n, see [22].

The permanent of a matrix is analogous to the determinant of the matrix, where all of the signs used in the Laplace's expansion of minors are positive. In particular, for matrix $\mathbf{M}_n$ defined by (2.1) we have (see [21])

$$\begin{aligned}
&\operatorname{per}\mathbf{M}_1 = a_1,\\
&\operatorname{per}\mathbf{M}_2 = a_1a_2 + c_1b_1,\\
&\operatorname{per}\mathbf{M}_n = a_n \operatorname{per}\mathbf{M}_{n-1} + c_{n-1}b_{n-1}\operatorname{per}\mathbf{M}_{n-2}, \quad \text{for} \quad n \geq 3.
\end{aligned}$$

Now, we can give a connection between permanent of the matrix and bihyperbolic Fibonacci number.

Theorem 2.6 *Let $n \geq 1$ be an integer. Then*

$$BhF_{n-1} = \operatorname{per}\begin{bmatrix} j_1+j_2+2j_3 & j_2 & & & & \\ 1+j_1+j_2 & 1 & 1 & & & \\ & 1 & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & 1 & 1 \end{bmatrix}_{n\times n}.$$

Note that the matrix in Theorem 2.6 was obtained from $\mathbf{A}_n$ (Theorem 2.2) by changing the elements lying in the sub-diagonal directly below the principal diagonal to opposite numbers. Recall that the Hadamard product of two matrices is the matrix such that each entry is the product of the corresponding entries of the input matrices. In the literature it is also used the term Schur product instead of Hadamard product, see [14].

Let K_n be any tridiagonal matrix of order n, O_n be a matrix of order n defined by

$$O_n = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & -1 & 1 \end{bmatrix} \tag{2.2}$$

and $O_n \circ K_n$ denotes the Hadamard product of O_n and K_n. Then $\operatorname{per}(O_n \circ K_n) = \det K_n$, see [15].

Remark 2.1 Let $n \geq 1$ be an integer, the matrices $\mathbf{F}$, $\mathbf{P}$, $\mathbf{J}$ are defined in Theorems 2.3-2.5 and the matrix O_n is defined by (2.2). Then

$$\begin{aligned}
BhF_{n-1} &= \operatorname{per}(O_n \circ \mathbf{F}_n),\\
BhP_{n-1} &= \operatorname{per}(O_n \circ \mathbf{P}_n),\\
BhJ_{n-1} &= \operatorname{per}(O_n \circ \mathbf{J}_n).
\end{aligned}$$

3. Concluding remarks

One of the generalization of the Fibonacci type numbers are Fibonacci type polynomials. In [24], the authors presented closed formulas for the Fibonacci type polynomials in terms of tridiagonal determinants. The determinantal and permanental representations of Fibonacci type numbers and polynomials can be found also in [20]. The bihypernomials as generalizations of bihyperbolic numbers of the Fibonacci type were introduced quite recently, see [28,29].

The topic of this article may provide motivation for future research in matrix theory. In [9,16], the Fibonacci numbers, generalized Fibonacci numbers and generalized Lucas numbers were presented as determinants of some pentadiagonal matrices. The future work may concern three- or five-diagonal matrices with determinants or permanents being the bihyperbolic Fibonacci type numbers or Fibonacci type bihypernomials.

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