Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.68503

Stability Analysis and Optimal Vaccination Approach for a Covid-19 Model with Memory Effects

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ABSTRACT: In this article, we have utilized fractional differential equations to provide a better understanding of the dynamics of the COVID-19 virus. Firstly, we conducted a local stability analysis and discovered that if the basic reproduction rate (R_0) is less than or equal to 1, the disease-free equilibrium point is locally asymptotically stable. If R_0 is greater than 1, the endemic equilibrium point is locally asymptotically stable. Secondly, we developed a more effective vaccination strategy that employs a fractional order derivative. This approach can help minimize the number of infected and susceptible individuals while maximizing the number of recovered individuals. We used Pontryagin's maximum principle to obtain numerical simulations that demonstrate how the optimal adoption of available control measures can reduce the infected population. We observed that the objective function of our optimal control problem decreases with the effects of vaccination and for varying values of the derivative order. Additionally, we discovered that a control strategy utilizing a non-integer order derivative is more efficient compared to one using an integer order derivative.

Key Words: COVID-19 model, fractional derivatives, stability analysis, optimal control, numerical simulations.

Contents

1. Introduction

The COVID-19 pandemic is indeed a serious global health threat, caused by SARS-COV-2 and resulting in severe respiratory damage. It first emerged in China in 2019 and rapidly spread to multiple countries. Currently, there are many vaccines available [17]. One of the primary goals is to determine how much can be spent on vaccination to increase herd immunity while also reducing the cost of vaccination.

The spread of an epidemic is closely linked to the causality principle of inherited informations. As a result, fractional derivatives are considered as non-Markovian processes and are often better suited for modeling epidemics. Unlike classical integer derivatives, fractional derivatives not only operate locally but also exhibit memory effects, making them more appropriate in situations where the current state of the system depends on its previous states up to the initial state [12]. So, we can overcome the errors that come with the values of parameters. It is possible to take a non-integer order of derivative

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2010 *Mathematics Subject Classification:* 26A33, 49K15, 92D25.
Submitted June 07, 2023. Published December 04, 2025

that best matches the available data [5,13]. In the literature, several fractional derivative operators exist in the literature, with the Caputo operator being the most well-known. Caputo derivatives offer several advantages, including taking into account the classical initial conditions of dynamical systems and having a zero derivative of a constant. These properties are not true for other fractional operators, such as Riemann-Liouville derivatives. In [15], Sidi Ammi *et al.* studied the memory of a SIRS model, using a system of Caputo fractional differential equations, they obtained that the solutions of fractional order are stable and converge rapidly towards the point of equilibrium in comparison with the solution obtained by the derivative of integer order, which confirms that the fractional system is more general. In [14], they illustrated the effectiveness of fractional computation using the SEIR model to model the dynamics of the spread of COVID-19 in Morocco, with and without memory. In [3], a new coronavirus infection system with a fuzzy fractional differential equation defined in the sense of Caputo is developed, and the general properties of RNA in COVID-19 are studied for the governing model.

In this work, we proposed a fractional SEIHR model that considered the rate of infected individuals hospitalized (H), which is a generalization of the SEIR model. We then studied the fractional derivative order effects and the impact of vaccination on the dynamics of the pandemic. Our goal was to limit the infection and reduce the cost of the vaccination strategy. To achieve this, we used the Pontryagin maximum principle in the case of fractional derivatives. We sought the optimality system of Caputo's fractional optimal control problem and then solved it numerically in Matlab. We compared the dynamics of our system with different orders of derivatives and found that the objective function of our optimal control problem decreased under the effects of vaccination and for different values of the order of derivative. Additionally, we found a control strategy for a non-integer order of derivative that was more efficient than the control strategy that used an integer-order derivative.

The following sections are organized as follows. We formulate a fractional COVID-19 model in section 2. In section 3, we give the basic elements of fractional calculus. The qualitative analysis is given in section 4. In section 5, we show the existence of equilibria and their local stability. In section 6, we present an optimal control problem constituting a cost function to be minimized subject to a model described by fractional differential equations. Figures obtained by Matlab are given in section 7. We end with section 8 of the conclusion and future perspectives.

2. Fractional Model

The total population can be classified into four categories based on their status with respect to the disease: $S(t)$ denotes individuals who are susceptible to the disease, $E(t)$ represents infected individuals who have not yet developed any symptoms, $I(t)$ stands for those who have been infected and are currently contagious, $H(t)$ represents individuals who require hospitalization due to the severity of their illness, and $R(t)$ represents those who have recovered from the disease. The transmission of Coronavirus (COVID-19) can be visualized through the graph depicted in Figure 1.

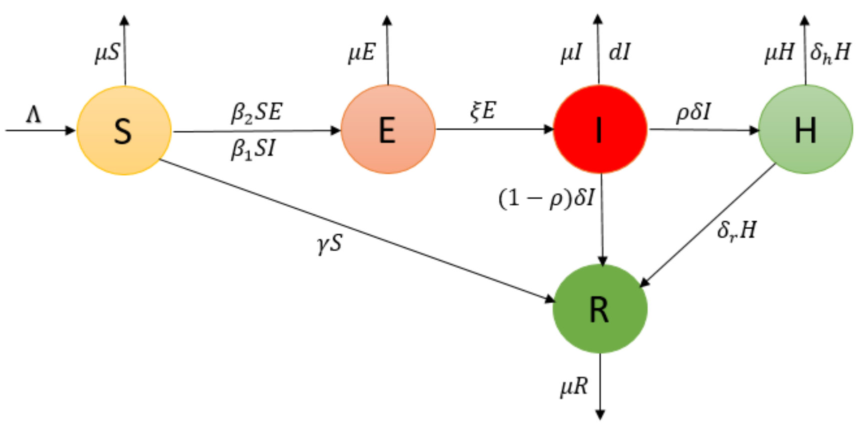

Figure 1: Diagram of COVID-19 dynamics.

Therefore, to describe the dynamics of COVID-19, we consider a fractional order model, which is expressed as follows:

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha S(t) &=& \Lambda^\alpha - \beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)S(t)I(t) - \beta_2^\alpha(1-\gamma^\alpha)S(t)E(t) \\ && -\mu^\alpha S(t) - \gamma^\alpha S(t), \\ {}_0^C\mathcal{D}_t^\alpha E(t) &=& \beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)S(t)I(t) + \beta_2^\alpha(1-\gamma^\alpha)S(t)E(t) \\ && -(\mu^\alpha+\xi^\alpha)E(t), \\ {}_0^C\mathcal{D}_t^\alpha I(t) &=& \xi^\alpha E(t) - (\delta^\alpha + d^\alpha + \mu^\alpha)I(t), \\ {}_0^C\mathcal{D}_t^\alpha H(t) &=& \rho^\alpha\delta^\alpha I(t) - \mu^\alpha H(t) - \delta_h^\alpha H(t) - \delta_r^\alpha H(t), \\ {}_0^C\mathcal{D}_t^\alpha R(t) &=& \delta_r^\alpha H(t) + \gamma^\alpha S(t) + (1-\rho^\alpha)\delta^\alpha I(t) - \mu^\alpha R(t). \end{cases} \tag{2.1}$$

For biological meaning, we consider

$$S(0) \geq 0,\ E(0) \geq 0,\ I(0) \geq 0,\ H(0) \geq 0,\ R(0) \geq 0. \tag{2.2}$$

So that the two members of the equations have the same dimension, we replaced the parameters $*$ by $*^\alpha$. Different parameters intervening in the model are presented in the following table.

Table 1: Values of model parameters.

Name	Description	Value
Λ^α	Recruitment rate of susceptible	0.9
β_1^α	Transmission coefficient due to infected individuals	0.7
δ^α	Isolation rate of infected	0.005
γ^α	Lockdown rate of susceptible	0.02
β_2^α	Transmission coefficient due to exposed individuals	0.2
μ^α	Natural death rate	0.04
ξ^α	Rate at which exposed people become infected	0.8
ρ^α	Rate at which infected become hospitalized	0.1
δ_h^α	Hospitalized death rates due to disease	0.06
δ_r^α	Recovery rate of hospitalized patients	0.9
d^α	Death rate coefficient due to infected	0.03

3. Basic Elements

In the following, we give the basic elements of non-integer derivative and an extension of optimal control theory [1,2].

Definition 3.1 *Let α a real number and $p \in \mathbb{N}$. For $p-1 < \alpha \leq p$ and $g \in C^p$. The forward non-integer derivative in the Caputo sense is given by:*

$${}_a^C\mathcal{D}_s^\alpha g(s) = \frac{1}{\Gamma(p-\alpha)}\int_a^s (s-\tau)^{p-\alpha-1}\frac{d^p}{d\tau^p}g(\tau)d\tau. \tag{3.1}$$

The backward non-integer derivative in the Caputo sense is given by:

$${}_s^C\mathcal{D}_b^\alpha g(s) = \frac{(-1)^p}{\Gamma(p-\alpha)}\int_s^b (\tau-s)^{p-\alpha-1}\frac{d^p}{d\tau^p}g(\tau)d\tau, \tag{3.2}$$

Let the control system consisting by: the cost function to optimize given by:

$$J\left[u(\cdot)\right] = \int_0^{\tau_m} \psi\left(Z(\tau), u(\tau)\right)d\tau \tag{3.3}$$

subject to the fractional system:

$$\begin{cases} {}_0^C\mathcal{D}_\tau^\alpha Z(\tau) = h\left(Z(\tau), u(\tau)\right), \\ Z(0) = Z_0, \end{cases} \tag{3.4}$$

such that $\alpha \in (0,1]$, $Z(\tau)$ is the variable of (3.4), h is a non-linear function, $\tau \in [0, \tau_m]$ and $u(\tau)$ is the control. For finding $u(\tau)$ solution of (3.4)–(3.3), we can based on the extension of Pontryagin maximum principle.

Theorem 3.1 ([2,6]) *The optimality system corresponding to* (3.4)–(3.3) *is given by*

$$\begin{cases} \frac{\partial \psi}{\partial u}\left(Z(\tau), u(\tau)\right) + Q^T \frac{\partial h}{\partial u}\left(Z(\tau), u(\tau)\right) = 0, \\ {}_0^C\mathcal{D}_\tau^\alpha Z(\tau) = h\left(Z(\tau), u(\tau)\right), \quad Z(0) = Z_0, \\ {}_\tau^C\mathcal{D}_{\tau_m}^\alpha Q(\tau) = \frac{\partial \psi}{\partial Z}\left(Z(\tau), u(\tau)\right) + Q^T \frac{\partial h}{\partial Z}\left(Z(\tau), u(\tau)\right), \quad Q(\tau_m) = 0. \end{cases}$$

4. Qualitative Analysis of the Model

4.1. Well-posedness model

We assume that the functions S, E, I, H, R and their Caputo fractional derivatives are continuous at $t \geq 0$.

Theorem 4.1 *The solution of system* (2.1) *is bounded and non-negative, and the closed set*

$$\Omega = \left\{ (S, E, I, H, R) \in \mathbb{R}_+^5 : S + E + I + H + R \leq N(0) + \frac{\Lambda^\alpha}{\mu^\alpha} \right\}$$

is a positive invariant set of system (2.1).

Proof: The existence of the solution is obtained by applying [8, Theorem 3.1]. By virtue of [8, Remark 3.2], we prove the uniqueness. Next, we show that the solution is non-negative. We proceed as in [10], we consider the following auxiliary system of (2.1):

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha \underline{S}(t) = -\mu^\alpha \underline{S}(t) - \gamma^\alpha \underline{S}(t), \\ {}_0^C\mathcal{D}_t^\alpha \underline{E}(t) = -(\mu^\alpha + \xi^\alpha)\underline{E}(t), \\ {}_0^C\mathcal{D}_t^\alpha \underline{I}(t) = -(\delta^\alpha + d^\alpha + \mu^\alpha)\underline{I}(t), \\ {}_0^C\mathcal{D}_t^\alpha \underline{H}(t) = -\mu^\alpha \underline{H}(t) - \delta_h^\alpha \underline{H}(t) - \delta_r^\alpha \underline{H}(t), \\ {}_0^C\mathcal{D}_t^\alpha \underline{R}(t) = -\mu^\alpha \underline{R}(t), \\ \\ \qquad \underline{S}(0) = \underline{E}(0) = \underline{I}(0) = \underline{H}(0) = \underline{R}(0) = 0. \end{cases} \tag{4.1}$$

It is obvious that $(\underline{S}, \underline{E}, \underline{I}, \underline{H}, \underline{R})^T = (0,0,0,0,0)^T$ is a lower solution of model (2.1). Then, according to [10, Lemma 1], we get $S \geq 0$, $E \geq 0$, $I \geq 0$, $H \geq 0$ and $R \geq 0$.
It remains to prove the boundeness of the solution. Let $N(t) = S(t) + E(t) + I(t) + H(t) + R(t)$. By adding the equations of system (2.1), one can deduce

$$\begin{aligned} {}_0^C D_t^\alpha N(t) &= \Lambda^\alpha - \mu^\alpha N(t) - d^\alpha I(t) - \delta_h^\alpha H(t) \\ &\leq \Lambda^\alpha - \mu^\alpha N(t). \end{aligned}$$

By applying the fractional order comparison theorem, one has

$$N(t) \leq N(0) E_\alpha(-\mu^\alpha t^\alpha) + \frac{\Lambda^\alpha}{\mu^\alpha}(1 - E_\alpha(-\mu^\alpha t^\alpha)).$$

Because $0 \leq E_\alpha(-\mu^\alpha t^\alpha) \leq 1$, we have $N(t) \leq N(0) + \dfrac{\Lambda^\alpha}{\mu^\alpha}$. This completes the proof. $\square$

4.2. Basic Reproduction Number

The system (2.1) has always a disease-free equilibrium point of the form $E_f = (S_f, 0, 0, 0, R_f)$, where $S_f = \frac{\Lambda^\alpha}{\gamma^\alpha+\mu^\alpha}$ and $R_f = \frac{\gamma^\alpha\Lambda^\alpha}{\mu^\alpha(\gamma^\alpha+\mu^\alpha)}$. Using the next generation matrix approach outlined in [18] to our model (2.1), the basic reproduction number can be computed by considering the generation matrices F and V given hereafter. Namely, the Jacobian matrices associated to the rate of appearance of new infections and the flux in and out of the corresponding compartments, respectively,

$$F = \begin{bmatrix} \frac{\beta_2^\alpha\Lambda^\alpha(1-\gamma^\alpha)}{\mu^\alpha+\gamma^\alpha} & \frac{\beta_1^\alpha\Lambda^\alpha(1-\delta^\alpha)(1-\gamma^\alpha)}{\mu^\alpha+\gamma^\alpha} \\ 0 & 0 \end{bmatrix} \quad ; \quad V = \begin{bmatrix} -(\xi^\alpha+\mu^\alpha) & 0 \\ \xi^\alpha & -(\delta^\alpha+\mu^\alpha+d^\alpha) \end{bmatrix}.$$

The basic reproduction number is then

$$R_0 = -F\cdot V^{-1} = \frac{\Lambda^\alpha\beta_2^\alpha(1-\gamma^\alpha)}{(\mu^\alpha+\xi^\alpha)(\mu^\alpha+\gamma^\alpha)} + \frac{\Lambda^\alpha\beta_1^\alpha\xi^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)}{(\delta^\alpha+\mu^\alpha+d^\alpha)(\mu^\alpha+\xi^\alpha)(\mu^\alpha+\gamma^\alpha)}. \tag{4.2}$$

5. Existence of Equilibria and Local Stability

In this section, we firstly discuss the existence of equilibria for model (2.1).

Theorem 5.1 *The fractional $SEIHR$ model (2.1) has at most two equilibrium points:*

1. *a disease free equilibrium $E_f = (S_f, 0, 0, 0, R_f)$, where $S_f = \frac{\Lambda^\alpha}{\gamma^\alpha+\mu^\alpha}$ and $R_f = \frac{\gamma^\alpha\Lambda^\alpha}{\mu^\alpha(\gamma^\alpha+\mu^\alpha)}$.*
2. *an endemic equilibrium point $E_e = (S^*, E^*, I^*, H^*, R^*)$ if $R_0 > 1$, where $S^* = \frac{\Lambda^\alpha-(\xi^\alpha+\mu^\alpha)E^*}{\gamma^\alpha+\mu^\alpha}$, $I^* = \frac{\xi^\alpha E^*}{\delta^\alpha+\mu^\alpha+d^\alpha}$, $H^* = \frac{\rho^\alpha\delta^\alpha\xi^\alpha E^*}{(\mu^\alpha+\delta_h^\alpha+\delta_r^\alpha)(\delta^\alpha+d^\alpha+\mu^\alpha)}$, $R^* = \frac{1}{\mu^\alpha}(\delta_r^\alpha H^* + \gamma^\alpha S^* + (1-\rho^\alpha)\delta^\alpha I^*)$.*

Proof:

1. For $I = 0$, it's clair that E_f is the unique steady state of system (2.1) .
2. By using the following system

$${}_0^C D_t^\alpha S(t) = {}_0^C D_t^\alpha E(t) = {}_0^C D_t^\alpha I(t) = {}_0^C D_t^\alpha H(t) = {}_0^C D_t^\alpha R(t) = 0,$$

we get the equation

$$\frac{\beta_1^\alpha\xi^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)(\Lambda^\alpha-(\xi^\alpha+\mu^\alpha)E^*)}{(\gamma^\alpha+\mu^\alpha)(\delta^\alpha+\mu^\alpha+d^\alpha)} + \frac{\beta_2^\alpha(1-\gamma^\alpha)(\Lambda^\alpha-(\xi^\alpha+\mu^\alpha)E^*)}{\gamma^\alpha+\mu^\alpha} = (\xi^\alpha+\mu^\alpha)$$

for $E^* \neq 0$. The fact that $S^* = \frac{\Lambda^\alpha-(\xi^\alpha+\mu^\alpha)E^*}{\gamma^\alpha+\mu^\alpha} \geq 0$ yields to $E^* \leq \frac{\Lambda^\alpha}{\xi^\alpha+\mu^\alpha}$. Hence, there is no positive equilibrium point if $E^* > \frac{\Lambda^\alpha}{\xi^\alpha+\mu^\alpha}$. Now, we consider the following function h defined on the interval $\left[0, \frac{\Lambda^\alpha}{\xi^\alpha+\mu^\alpha}\right]$ by

$$h(E) = \frac{\beta_1^\alpha\xi^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)(\Lambda^\alpha-(\xi^\alpha+\mu^\alpha)E)}{(\gamma^\alpha+\mu^\alpha)(\delta^\alpha+\mu^\alpha+d^\alpha)} + \frac{\beta_2^\alpha(1-\gamma^\alpha)(\Lambda^\alpha-(\xi^\alpha+\mu^\alpha)E)}{\gamma^\alpha+\mu^\alpha} - (\xi^\alpha+\mu^\alpha).$$

It's not difficult to show that h is strictly monotonic on $\left[0, \frac{\Lambda^\alpha}{\xi^\alpha+\mu^\alpha}\right]$. Since $h(\frac{\Lambda^\alpha}{\xi^\alpha+\mu^\alpha}) = -(\xi^\alpha+\mu^\alpha) < 0$ and $h(0) = (\xi^\alpha+\mu^\alpha)(R_0-1) > 0$ for $R_0 > 1$, hunce, there exists an unique endemic equilibrium E_e with $0 < E_e < \frac{\Lambda^\alpha}{\xi^\alpha+\mu^\alpha}$. This completes the proof. □

Next, we study the local asymptotic stability of disease free equilibrium point E_f and endemic equilibrium point E_e for system (2.1). The Jacobian matrix of system (2.1) at any equilibrium $\bar{E}_q = (\bar{S}, \bar{E}, \bar{I}, \bar{H}, \bar{R})$

is given by

$$J_{\bar{E}_q} = \begin{pmatrix} c_1 & -\beta_2^\alpha(1-\gamma^\alpha)\bar{S} & -\beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)\bar{S} & 0 & 0 \\ c_2 & \beta_2^\alpha(1-\gamma^\alpha)\bar{S}-(\xi^\alpha+\mu^\alpha) & \beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)\bar{S} & 0 & 0 \\ 0 & \xi^\alpha & -(\delta^\alpha+d^\alpha+\mu^\alpha) & 0 & 0 \\ 0 & 0 & \rho^\alpha\delta^\alpha & -(\mu^\alpha+\delta_h^\alpha+\delta_r^\alpha) & 0 \\ \gamma^\alpha & 0 & (1-\rho^\alpha)\delta^\alpha & \delta_r^\alpha & -\mu^\alpha \end{pmatrix}$$

where $c_1 = -\beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)\bar{I} - \beta_2^\alpha(1-\gamma^\alpha)\bar{E} - (\mu^\alpha+\gamma^\alpha)$ and $c_2 = \beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)\bar{I} + \beta_2^\alpha(1-\gamma^\alpha)\bar{E}$. We recall that a sufficient condition for the local stability of $\bar{E}_q$ is

$$|arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad i = 1,2,3,4 \text{ and } 5 \tag{5.1}$$

where ξ_i are the eigenvalues of $J_{\bar{E}_q}$ (see [11]). We begin by establishing the local stability of E_f.

Theorem 5.2 *The disease-free equilibrium E_f is locally asymptotically stable if and only if $R_0 \leq 1$.*

Proof: The eigenvalues of J_{E_f} : $\lambda_1 = -\mu$, $\lambda_2 = -(\mu+\delta_r+\delta_h)$, $\lambda_3 = -(\mu+\gamma)$, $\lambda_4 = -(\xi+\mu)$ and $\lambda_5 = -(d+\delta+\mu)$ are all negative, then satisfy condition (5.1). □

We are now concerned with the local stability of E_e.

Theorem 5.3 *The endemic equilibrium E_e is locally asymptotically stable, if and only if $R_0 > 1$.*

Proof: At equilibrium E_e, the characteristic equation for the corresponding linearised system of model (2.1) is $\xi^5 + a_1\xi^4 + a_2\xi^3 + a_3\xi^2 + a_4\xi + a_5 = 0$, where, if the coefficients a_1, a_2, a_3, a_4 and a_5 satisfy condition 6 in [19, Lemma 5.1] the positive equilibrium point E_e is locally asymptotically stable. □

6. Fractional Model with Vaccination

The purpose of this part is to limit the spread of the COVID-19 virus described by the model (2.1), denote $u(t)$ the vaccination rate intended for the susceptible class at the instant t. The COVID-19 pandemic transmission with the vaccination strategy can be given in the diagram as in Figure 2. The

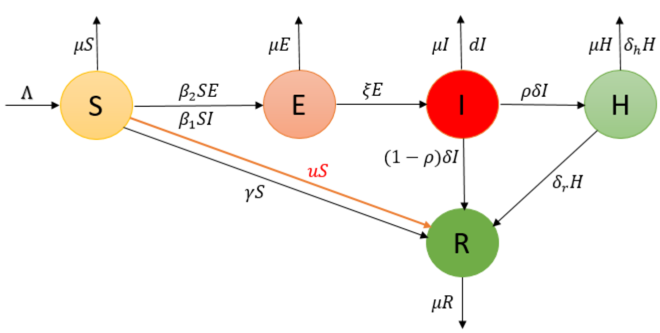

Figure 2: Diagram of COVID-19 dynamics with vaccination strategy.

model associated with the diagram (Figure 2) can be presented as follows

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha S(t) &=& \Lambda^\alpha - \beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)S(t)I(t) - \beta_2^\alpha(1-\gamma^\alpha)S(t)E(t) \\ && -\mu^\alpha S(t) - \gamma^\alpha S(t) - u^\alpha(t)S(t), \\ {}_0^C\mathcal{D}_t^\alpha E(t) &=& \beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)S(t)I(t) + \beta_2^\alpha(1-\gamma^\alpha)S(t)E(t) \\ && -(\mu^\alpha + \xi^\alpha)E(t), \\ {}_0^C\mathcal{D}_t^\alpha I(t) &=& \xi^\alpha E(t) - (\delta^\alpha + d^\alpha + \mu^\alpha)I(t), \\ {}_0^C\mathcal{D}_t^\alpha H(t) &=& \rho^\alpha\delta^\alpha I(t) - \mu^\alpha H(t) - \delta_h^\alpha H(t) - \delta_r^\alpha H(t), \\ {}_0^C\mathcal{D}_t^\alpha R(t) &=& \delta_r^\alpha H(t) + \gamma^\alpha S(t) + u^\alpha(t)S(t) + (1-\rho^\alpha)\delta^\alpha I(t) - \mu^\alpha R(t). \end{cases} \tag{6.1}$$

The system (6.1) can be written as

$${}_0^C\mathcal{D}_t^\alpha Z(t) = h\left(Z(t), u(t)\right), \tag{6.2}$$

where $Z(t) = (S(t), E(t), I(t), H(t), R(t))$. Now, we try to minimize the following objective function:

$$J[u(\cdot)] = \int_0^{t_m} A_1 I(t) - A_2 R(t) + \frac{r}{2}u^2(t)dt \longrightarrow \min, \tag{6.3}$$

where A_i, $i = 1, 2$ are the positive weights, and r is a weighting coefficient. We can write (6.3) as follows:

$$J[u(\cdot)] = \int_0^{t_m} \psi\left(Z(t), u(t)\right)dt \tag{6.4}$$

with

$$\psi\left(Z(t), u(t)\right) = A_1 I(t) - A_2 R(t) + \frac{r}{2}u^2(t).$$

From Theorem 3.1, we have the next optimality system constituted by:
the state system,

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha Z = h\left(Z, u\right), \\ Z(0) = Z_0, \end{cases} \tag{6.5}$$

and the co-system with final condition,

$$\begin{cases} {}_t^C\mathcal{D}_{t_m}^\alpha Q(t) = \frac{\partial\psi}{\partial Z} + Q^T\frac{\partial h}{\partial Z}, \\ Q(t_m) = 0, \end{cases} \tag{6.6}$$

and the stationary condition

$$\frac{\partial\psi}{\partial u} + Q^T\frac{\partial h}{\partial u} = 0, \tag{6.7}$$

where $Q(t) = (Q_1(t), Q_2(t), Q_3(t), Q_4(t), Q_5(t))$ and $h = (h_1, h_2, h_3, h_4, h_5)$ with

$$\begin{aligned} h_1 &= \Lambda^\alpha - \beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)S(t)I(t) - \beta_2^\alpha(1-\gamma^\alpha)S(t)E(t) - \mu^\alpha S(t) \\ &\quad - \gamma^\alpha S(t) - u^\alpha(t)S(t), \\ h_2 &= \beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)S(t)I(t) + \beta_2^\alpha(1-\gamma^\alpha)S(t)E(t) - (\mu^\alpha + \xi^\alpha)E(t), \\ h_3 &= \xi^\alpha E(t) - (\delta^\alpha + d^\alpha + \mu^\alpha)I(t), \\ h_4 &= \rho^\alpha\delta^\alpha I(t) - \mu^\alpha H(t) - \delta_h^\alpha H(t) - \delta_r^\alpha H(t), \\ h_5 &= \delta_r^\alpha H(t) + \gamma^\alpha S(t) + u^\alpha(t)S(t) + (1-\rho^\alpha)\delta^\alpha I(t) - \mu^\alpha R(t). \end{aligned}$$

The adjoint system of (6.1) is given by:

$$\begin{cases} {}_t^C\mathcal{D}_{t_m}^\alpha Q_1(t) &=& (\beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)I(t) + \beta_2^\alpha(1-\gamma^\alpha)E(t))(Q_2(t) - Q_1(t)) \\ & + & (\gamma^\alpha + u^\alpha(t))(Q_5(t) - Q_1(t)) - \mu^\alpha Q_1(t), \\ {}_t^C\mathcal{D}_{t_m}^\alpha Q_2(t) &=& \beta_2^\alpha(1-\gamma^\alpha)S(t)(Q_2(t) - Q_1(t)) - (\mu^\alpha + \xi^\alpha)Q_2(t) + \xi^\alpha Q_3(t), \\ {}_t^C\mathcal{D}_{t_m}^\alpha Q_3(t) &=& \beta_1^\alpha(1-\gamma^\alpha)(1-\delta^\alpha)S(t)(Q_2(t) - Q_1(t)) - (\delta^\alpha + d^\alpha + \mu^\alpha)Q_3(t) \\ & + & \rho^\alpha\delta^\alpha Q_4(t) + (1-\rho^\alpha)\delta^\alpha Q_5(t) + A_1, \\ {}_t^C\mathcal{D}_{t_m}^\alpha Q_4(t) &=& -(\mu^\alpha + \delta_h^\alpha + \delta_r^\alpha)Q_4(t) + \delta_r^\alpha Q_5(t), \\ {}_t^C\mathcal{D}_{t_m}^\alpha Q_5(t) &=& -\mu^\alpha Q_5(t) - A_2, \end{cases} \tag{6.8}$$

subject to the transversality conditions

$$\begin{cases} Q_1(t_m) = 0, \\ Q_2(t_m) = 0, \\ Q_3(t_m) = 0, \\ Q_4(t_m) = 0, \\ Q_5(t_m) = 0. \end{cases} \tag{6.9}$$

We have the stationary condition:

$$u(t) = \frac{(Q_1(t) - Q_5(t))S(t)}{r}. \tag{6.10}$$

7. Numerical Simulations

In this section, we study the effects of fractional order α to the dynamic of infection during a given time interval. We can mention two cases: absence and presence of vaccination. Set $S(0) = 2$, $E(0) = 1$, $I(0) = 0$, $H(0) = 1$, $R(0) = 0$. Take the parameter values given in Table (1). Using the forward-backward sweep method in MATLAB® to solve (6.3) subject to (6.1) (see [7]). The state system and the adjoint equations are numerically integrated by using the approximation of (left/right) Caputo fractional derivative based on the explicit finite differences method [4,9,16]. We propose the following algorithm:

Algorithm 1

1. *Set m the number of time steps, τ the step time, tolerance $\delta = 0.001$, and $test = -1$.*
2. *Initiate the control u_{old}, the state $(S_{old}, E_{old}, I_{old}, H_{old}, R_{old})$ and adjoint $((Q_{old})_1, (Q_{old})_2, (Q_{old})_3, (Q_{old})_4, (Q_{old})_5)$.*

   While *$test < 0$*

3. *Solve the state system (6.1) for (S, E, I, H, R) with initial guess $(S_0, E_0, I_0, H_0, R_0)$, using an explicit finite differences method.*
4. *Solve the co-system (6.8) for $(Q_1, Q_2, Q_3, Q_4, Q_5)$ using the transversality conditions $Q_i(t_m)$ and (S, E, I, H, R).*
5. *Update u by using (6.10).*
6. *Compute the tolerance criterions $\psi_1 = \delta\|S\| - \|S - S_{old}\|$,$\psi_2 = \delta\|E\| - \|E - E_{old}\|$, $\psi_3 = \delta\|I\| - \|I - I_{old}\|$,$\psi_4 = \delta\|H\| - \|H - H_{old}\|$, $\psi_5 = \delta\|R\| - \|R - R_{old}\|$, $\psi_6 = \delta\|Q_1\| - \|Q_1 - (Q_{old})_1\|$, $\psi_7 = \delta\|Q_2\| - \|Q_2 - (Q_{old})_2\|$, $\psi_8 = \delta\|Q_3\| - \|Q_3 - (Q_{old})_3\|$, $\psi_9 = \delta\|Q_4\| - \|Q_4 - (Q_{old})_4\|$, $\psi_{10} = \delta\|Q_5\| - \|Q_5 - (Q_{old})_5\|$, $\psi_{11} = \delta\|u\| - \|u - u_{old}\|$, and calculate $test = min\{\psi_i\}$, $i = 1, \ldots, 11$.*
   Endwhile

In this particular population, the interactions between the compartments of susceptible, infected, and recovered individuals are shown in Figures 3, 4, and 5. These figures depict the numerical results obtained with different values of α, in the absence of control. It is worth noting that when the order of derivative integer $\alpha = 1$, it is considered a special case for our obtained results. From Figure 3, we observe that susceptible individuals become infected over 40 days. The population of susceptible individuals is almost entirely infected within this period. On the other hand, Figure 4 shows that the number of infected individuals increases rapidly in the first five days. Additionally, Figure 5 shows that the number of recovered individuals increases in sync with the number of infected individuals, thanks to the high recovery rate δ_r of Covid-19.

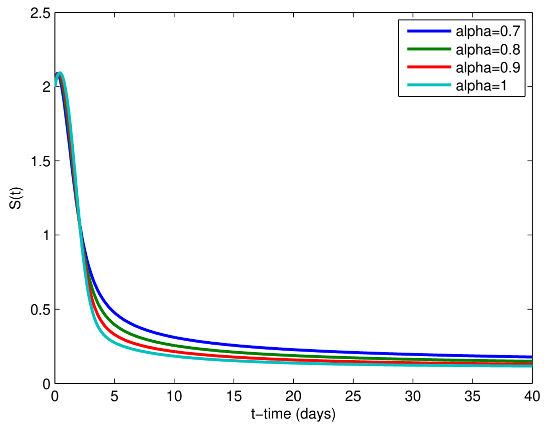

Figure 3: Dynamic of susceptible population without control for different values of α.

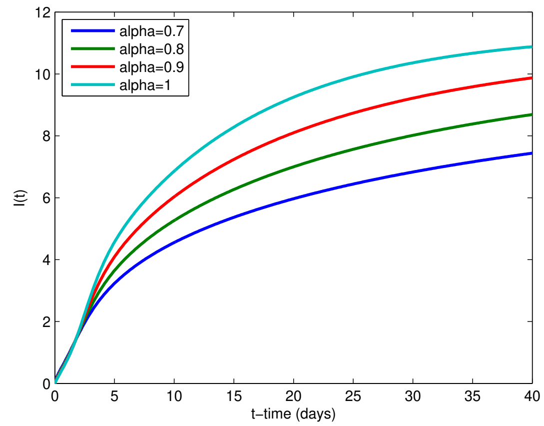

Figure 4: Dynamic of infected population without control.

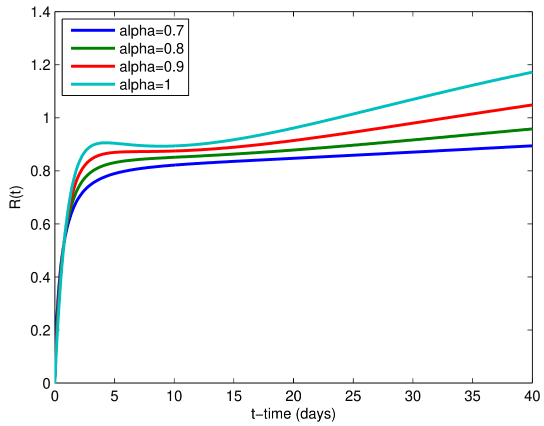

Figure 5: Dynamic of recovered population without control for different values of α.

During a period of 40 days, we analyzed the infection prevalence while implementing a vaccination strategy. Our observations indicate that the number of individuals who have recovered from the infection has increased (see Figure 8). Meanwhile, the number of susceptible and infected individuals have decreased (see Figures 6 and 7) due to the positive effects of the vaccination strategy for susceptible individuals (Figure 9).

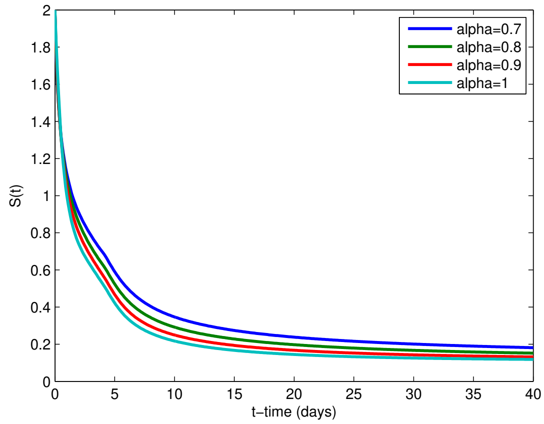

Figure 6: Dynamic of susceptible population with control for different values of α.

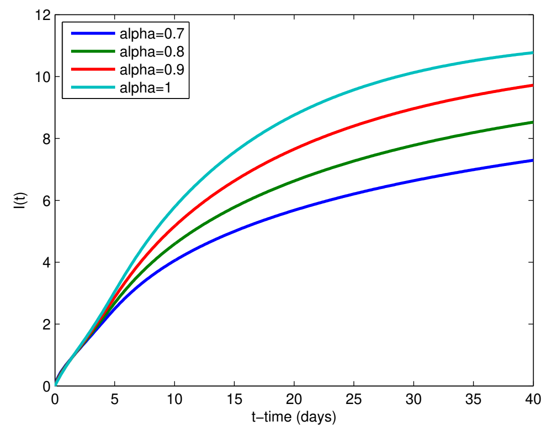

Figure 7: Dynamic of infected population with control for different values of α.

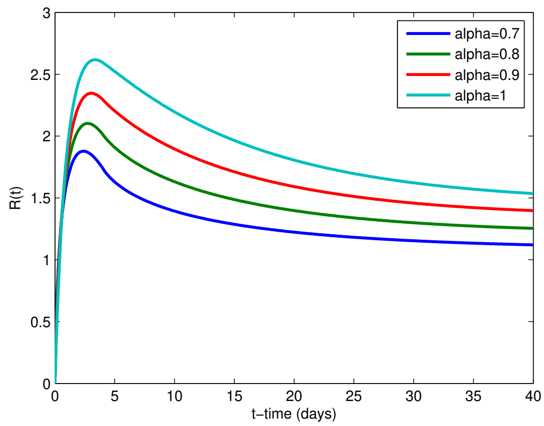

Figure 8: Dynamic of recovered population with control for different values of α.

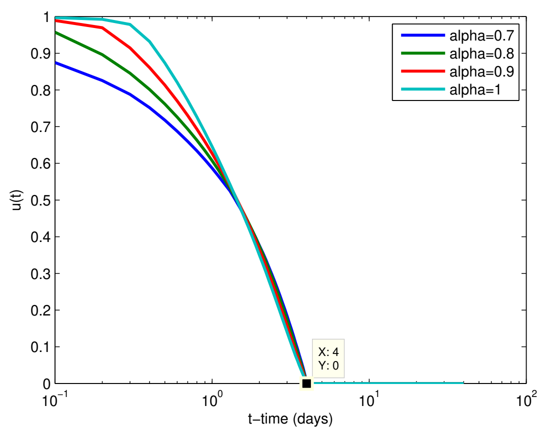

Figure 9: Optimal control for different values of α.

Next, we will examine how the cost function J is affected by the order α in the presence and absence of vaccination. Prior to this, we present the results in Table 2 and Table 3, respectively. In the

Table 2: Values of J without control for different values of α

α	0.7	0.8	0.9	1
J	$5883,51$	$6929,23$	$8004,07$	$9061,46$

Table 3: Values of J with control for different valus of α

α	0.7	0.8	0.9	1
J	$5155,04$	$5984,30$	$6843,29$	$7699,14$

absence of a control strategy, the cost function values are listed in Table 2. We observe that the cost function decreases as the order of derivative α takes smaller values, which is significant from an economic standpoint. Therefore, we can select the order α that best corresponds to the actual data. Table 3 shows that the functional J decreases with both vaccination and non-integer order α. We conclude that J is optimal when $\alpha = 0.7$.

8. Conclusion

In this paper, we utilized the Pontryagin maximum principle to model SARS-COV-2 with a fractional approach. Our objective was to apply a vaccination strategy that could potentially eradicate the COVID-19 virus. We obtained an optimality system described using the Caputo sense derivative to limit the infection while decreasing the cost associated with the vaccination strategy. We compared the dynamics of our system using different values of the derivative order and concluded that the cost function decreases under the effects of vaccination for various values of α, which is crucial in the economic world. We can choose the order α that best corresponds to the real data.

Furthermore, we discovered that the optimal vaccination strategy leads to the smallest value of the cost function J when α is set to 0.7. Therefore, we can find a control strategy for a non-integer order of derivative that is more efficient than the control strategy that employs a derivative of integer order.

As for future research, we plan to use numerical methods such as the piecewise-spectral homotopy analysis method (PSHAM) and the new extended modal series method for resolving the infinite horizon optimal control problem of nonlinear interconnected large-scale dynamic systems. We will also use a recursive shooting method to solve the optimal control problem of linear time-varying systems with state time-delay. Lastly, we aim to use real data from some countries to further validate our findings.

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