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Defective topological spaces: A novel framework for topological spaces with incomplete information

Ramazan Ekmekçi* and Esra Korkmaz

ABSTRACT: Recent years have seen a surge of interest in representing and reasoning about uncertain information through extended versions of classical mathematical techniques. In this paper, we introduce the concept of defective topological spaces, which is a novel theory motivated by the problem of approaching a topological space with incomplete knowledge of its open sets. In order to address this problem, we define the concepts of exactly and possibly open sets. Furthermore, we investigate the categorical structure of defective topological spaces, and extend several concepts from classical topological spaces, including continuity, convergence, and separation axioms to defective topological spaces.

Key Words: Exactly open sets, possibly open sets, defective topological spaces, incomplete information.

Contents

1. Introduction

There has been a lot of interest in recent times regarding how to represent and reason uncertain information using extended versions of classical mathematical techniques. Various mathematical frameworks have been developed to represent and manipulate uncertain information, including Dempster-Shafer theory, rough sets, flou sets, and interval-valued sets. Dempster-Shafer theory is based on the concept of a belief function [11], which represents the degree of belief in a particular proposition. Belief functions can be used to combine and revise multiple pieces of ambiguous evidence to arrive at a more accurate belief about the given proposition.

The concept of flou sets [9], on the other hand, can be used to describe sets defined by a particular property, where there is uncertainty about whether an element has that property or not. For example, the set of all "intelligent" people may be considered a flou set, as it is difficult to determine who is and is not intelligent. Unlike the traditional fuzzy model, which is based on membership degrees, flou sets use relative relationships between point sets, and fuzzy regions are represented as flou sets.

Pawlak's rough set theory [10] contains similar ideas. A rough set is defined by using two approximations: the lower approximation and the upper approximation. The lower approximation consists of all elements classified as belonging to the set, while the upper approximation consists of all elements that may potentially belong to the set. In the presence of imprecision or uncertainty, rough sets can be used to draw meaningful conclusions about the set by analyzing the relationship between these approximations.

* Corresponding author

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Yao [12] introduced the concept of an interval set as a way to represent qualitative knowledge based on uncertain data. In this theory, sets are represented as collections of intervals, and various operations can be performed on these sets, such as union, intersection, and difference. It is clear that interval sets can be seen as a generalization of ordinary sets and a specific version of interval-valued fuzzy sets [13]. Kim et al. [6] presented the concept of an interval-valued topology, as well as interval-valued bases and subbases. They also explored several properties of these structures.

The aim of this study is to introduce the concept of defective topological spaces. This is a brand new theory that explores the idea of topological spaces where there is incomplete information available about open sets. The theory of defective topological spaces is a novel approach to understanding and working with incomplete or imperfect information in the realm of topology. The motivation behind the development of this theory was to provide a way to approximate the behavior of a topological space, even when we do not have complete information about all of the open sets in that space.

Consider a scenario that not all of the open sets in the topological space are known. Rather, only certain subfamily of the power set, say δ, are known as open, while the rest remain unspecified. Additionally, it is known for certain that the elements of δ^* are not included in our topology. Then we have $\tau_\delta \subseteq \tau \subseteq \tau_{\mathcal{P}(X)\setminus\delta^*}$, where τ_δ and $\tau_{\mathcal{P}(X)\setminus\delta^*}$ are topologies with subbase δ and $\mathcal{P}(X)\setminus\delta^*$ respectively. As a result, a lower and upper bound is found for the desired topology. Defective topological spaces provide a suitable framework to deal with such cases. They allow for a more flexible and general approach to studying continuity and convergence, since they allow for the possibility of considering situations in which the open sets are not fully known.

Defective topological spaces arise by considering triples consisting of a set and two topologies on this set, such that the first topology is a subset of the second. Defective topological spaces, by the nature of their structure, enable topological spaces that are connected by the relation of being a subset to be analyzed within the same framework. As a result, they can be applied to various fields of mathematics, including algebraic topology, functional analysis, and measure theory.

The rest of the paper is organized as follows. In Section 2, we introduce some basic concepts and results related to interval-valued and bitopological spaces. In Section 3, we explore the categorical structure of defective topological spaces, including their relationships with categories of topological, bitopological, and interval-valued topological spaces. Additionally, we present an algorithm to approximate spaces in cases where information about open sets is incomplete. Section 4 is devoted to the study of convergence in defective topological spaces. Section 5 discusses the separation axioms in the context of defective topological spaces. Finally, in Section 6, we conclude the paper with a summary of our main results and some directions for future research.

2. Preliminaries

The purpose of this section is to quickly review a few of the key concepts and facts we will need later on. There will be no detailed explanation of topological concepts. Readers can refer to [12,6,5,7,1] in the absence of any definitions.

Interval-valued topological spaces:

Definition 2.1 [12] *Given a non-empty set X, the family*

$$[B^-, B^+] = \{A \subseteq X \mid B^- \subseteq A \subseteq B^+\}$$

is called an interval-valued set, where $B^-, B^+ \subseteq X$ and $B^- \subseteq B^+$.

In this case, $\widetilde{\emptyset} = [\emptyset, \emptyset]$ and $\widetilde{X} = [X, X]$ are called the interval-valued empty and whole sets in X, respectively. Clearly, an interval-valued set is a generalization of a classical set.

Let $IVS(X)$ denote the family of all interval-valued sets of X. For any $A, B \in IVS(X)$, the operations of inclusion, complementation, union and intersection on interval-valued sets are defined as $A \subseteq B \Leftrightarrow (B^- \subseteq A^-$ and $A^+ \subseteq B^+)$, $A^c = [(A^+)^c, (A^-)^c]$, $A \cup B = [A^- \cup B^-, A^+ \cup B^+]$, $A \cap B = [A^- \cap B^-, A^+ \cap B^+]$ respectively.

If $B \in IVS(X)$ and $b \in B$ then $b_{IVP} = [\{b\}, \{b\}]$ ($b_{IVVP} = [\emptyset, \{b\}]$) is called an interval-valued (vanishing) point in X respectively. We write $b_{IVP} \in B$ if $b \in B^-$ and $b_{IVVP} \in B$ if $b \in B^+$.

Example 2.1 *Let $X = \{1, 2, 3\}$. Then $[\{1\}, X] = \{\{1\}, \{1, 2\}, \{1, 3\}, X\}$ is an interval-valued set where $B^- = \{1\}$ and $B^+ = X$.*

Definition 2.2 [6] *An interval-valued topology $\mathcal{T}$ is a non-empty family of IVSs on a set X satisfying the conditions below:*

$(IVT_1)\widetilde{X} \in \mathcal{T}$ *and* $\widetilde{\emptyset} \in \mathcal{T}$,

$(IVT_2)B_1, B_2 \in \mathcal{T} \Rightarrow B_1 \cap B_2 \in \mathcal{T}$,

$(IVT_3)\{B_j\}_{j\in J} \subseteq \mathcal{T} \Rightarrow \bigcup_{j\in J} B_j \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is referred to as an interval-valued topological space. Additionally, we refer to an element of $\mathcal{T}$ as an interval-valued open set.

Interval-valued topological spaces can be viewed as a generalization of clasical topological spaces. For a interval-valued topological space $(X, \mathcal{T})$, the families $\mathcal{T}^- = \{B^- \subseteq X \mid B \in \mathcal{T}\}$ and $\mathcal{T}^+ = \{B^+ \subseteq X \mid B \in \mathcal{T}\}$ are clasical topologies on X. Thus, $(X, \mathcal{T}^-, \mathcal{T}^+)$ is a bitopological space [5].

Definition 2.3 *An interval-valued base is a non-empty subfamily $\mathcal{B} \subseteq \mathcal{T}$ where every interval-valued open set can be expressed as a union of a subfamily $\mathcal{B}' \subseteq \mathcal{B}$.*

Clearly, $\mathcal{B}^- = \{B^- \mid B \in \mathcal{B}\}$ (resp., $\mathcal{B}^+ = \{B^+ \mid B \in \mathcal{B}\}$) is a base for $\mathcal{T}^-$ (resp., $\mathcal{T}^+$) [6].

Theorem 2.1 [6] *Given a non-empty set X, a family $\mathcal{B} \subseteq IVS(X)$ is an interval-valued base for an interval-valued topology on X if and only if the following conditions hold:*

$(IVB_1)\widetilde{X} = \bigcup \mathcal{B}$,

(IVB_2)*if $B_1, B_2 \in \mathcal{B}$ and $b_{IVP} \in B_1 \cap B_2$ (resp. $b_{IVVP} \in B_1 \cap B_2$) then there is a $B \in \mathcal{B}$ with $b_{IVP} \in B \subseteq B_1 \cap B_2$ (resp. $b_{IVVP} \in B \subseteq B_1 \cap B_2$).*

Example 2.2 *Let $X = \mathbb{N}$. Then the family defined by*

$$\mathcal{T} = \{\widetilde{\emptyset}, [\emptyset, \{1\}], [\{1\}, \{1\}], [\{3\}, \{3\}], [\{3\}, \{1, 3\}], [\{1, 3\}, \{1, 3\}], [\{3, 5\}, \{3, 5, 7\}], [\{1, 3, 5\}, \{1, 3, 5, 7\}], \widetilde{X}\}$$

is an interval-valued topology on X.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two interval-valued topological spaces. A mapping $f : X \to Y$ is called interval-valued continuous if $f^{-1}(B) = [f^{-1}(B^-), f^{-1}(B^+)] \in \mathcal{T}$ for all $B \in \mathcal{S}$. Interval-valued topological spaces together with interval-valued continuous mappings form a category denoted by $\mathbf{IV_{Top}}$ [8].

Now we need to recall bitopological spaces because there exists a significant categorical relationship between specific categories of defective topological spaces and the broader category of bitopological spaces. This relationship is pivotal in bridging our understanding of the complex structures and properties of defective topological spaces. Moreover, it's noteworthy that the separation axioms of bitopological spaces can be readily applied to defective topological spaces as well, as every defective topological space is a bitopological space. This application allows us to harness the rich mathematical machinery of bitopological spaces to gain deeper insights into the separation properties and topological characteristics of defective spaces.

Bitopological spaces: Bitopological spaces are topological spaces equipped with two distinct topologies. In this section, we will discuss bitopological spaces and their fundamental properties, primarily using Kopperman's [7] terminology.

Definition 2.4 *The dual of a bitopological space $\mathcal{X} = (X, \tau, \tau^*)$ is $\mathcal{X}^* = (X, \tau^*, \tau)$. The symmetrization of a bitopological space is $\mathcal{X}^s = (X, \tau^s)$, where τ^s is the supremum of the topologies τ and τ^*.*

Note that the τ-closure and τ^*-closure operators are respectively denoted by cl^{τ} and cl^{τ^*}. A bitopological space $\mathcal{X}$ is dually Q if its dual space $\mathcal{X}^*$ also has property Q. Additionally, if both $\mathcal{X}$ and $\mathcal{X}^*$ have property Q, we refer to space $\mathcal{X}$ as pairwise Q.

Definition 2.5 *A bitopological space $\mathcal{X}$ is*

(1) T_0 if its symmetrization is T_0.

(2) weakly symmetric (ws) if for any $x, y \in X$, $x \notin cl^{\tau}(y)$ implies $y \notin cl^{\tau^*}(x)$,

(3) pseudo-Hausdorff (pH) if for any $x \notin cl^{\tau}(y)$, there exist disjoint sets $G \in \tau$ and $G^* \in \tau^*$ with $x \in G$ and $y \in G^*$,

(4) regular if given any $G \in \tau$ and $x \in G$, there exist $H \in \tau$ and τ^* closed set F^* with $x \in H \subseteq F^* \subseteq G$,

(5) normal if given any τ^* closed set F^* and $G \in \tau$ with $F^* \subseteq G$, there exist $H \in \tau$ and τ^*-closed set K^* such that $F^* \subseteq H \subseteq K^* \subseteq G$.

It is worth remembering that a property of bitopological spaces is said to be self-dual if it holds true for $\mathcal{X}$, then it also holds true for $\mathcal{X}^*$. It can be readily observed that T_0 and normality are self-dual, whereas ws, pH and regularity are not.

Theorem 2.2 [7] *A bitopological space is regular if it is both normal and ws. Furthermore, every regular bitopological space is pH, and every pH bitopological space is ws.*

Theorem 2.3 [7] *Let Q denote one of the properties zs, pH, or regularity for a bitopological space $\mathcal{X}$. If $\mathcal{X}$ satisfies Q, and τ^{**} is a topology finer than τ^* then (X, τ, τ^{**}) also satisfies Q.*

3. Defective topological spaces

In this section, we introduce the notion of defective topological spaces. The main reason for developing this theory is to provide a method which enables us to study topological structures with incomplete knowledge of open sets easily. Furthermore, we investigate the categorical structure of defective topological spaces, and establish their relationships with other related categories, such as topological spaces, bitopological spaces, and interval-valued topological spaces. This analysis provides insights into the fundamental properties and connections between these different types of spaces. Furthermore, we present an algorithm designed specifically to handle incomplete information about open sets. This will enable us to approximate spaces effectively in such scenarios.

Definition 3.1 *Let X be a non empty set. A couple (τ, ρ) is called a defective topology on X if τ and ρ are topologies on X with $\tau \subseteq \rho$. In this case the triple (X, τ, ρ) is called a defective topological space, whereby the elements of τ are called exactly open sets, and the elements of ρ are called possibly open sets.*

Definition 3.2 *Let (X, τ, ρ) be a defective topological space. A subset $A \subseteq X$ is called exactly closed (resp., possibly closed) if the subset $X \setminus A$ is exactly open (resp., possibly open).*

Remark 3.1 *It is obvious that a defective topological space can be equivalently defined by τ^c and ρ^c with $\tau^c \subseteq \rho^c$.*

Example 3.1 *Let (X, τ) is a topological space.*

(1) The triple (X, τ, τ) is obviously a defective topological space.

(2) Assume that $A \subseteq X$ is an open subset of X. In this case, (X, τ_A, τ) is a defective topological space.

(3) Let $X = \mathbb{R}$ and let τ and ρ denote the usual and Sorgenfrey topology on $\mathbb{R}$, respectively. Then $(\mathbb{R}, \tau, \rho)$ is a defective topological space.

(4) Let τ and ρ be cofinite and usual topologies on $\mathbb{R}$, respectively. Then, the triple $(\mathbb{R}, \tau, \rho)$ is a defective topological space.

(5) Let $\mathbb{A}^n(\Bbbk)$ be the affine n-space over $\Bbbk$. It is known that the Zariski topology is coarser than the usual metric topology when $\Bbbk = \mathbb{R}, \mathbb{Q}, \mathbb{C}$. Thus, they define a defective topological space on $\mathbb{A}^n(\Bbbk)$.

(6) Let H be a Hilbert space. Consider the C^*-algebra of bounded linear operators on H, denoted by $B(H)$. It is known that the strong topology is weaker than the norm topology [4]. Thus, they form a defective topological space.

Remark 3.2 *Recall that the specialization order is defined on a topological space τ as "$x \leq_\tau y \Leftrightarrow x \in cl^\tau(y)$". It is useful in domain theory, which is a branch of theoretical computer science that deals with the study of partial orders and their application to the semantics of programming languages. Moreover, it is used to define topological duals, including the de Groot dual (τ^G), the Alexandroff dual (τ^A), and the weak dual (τ^W) [7]. The weak dual is a subset of the de Groot dual, which in turn is a subset of the Alexandroff dual. It follows that the triples (X, τ^W, τ^G), (X, τ^W, τ^A), and (X, τ^G, τ^A) are defective topological spaces. Thus, defective topological spaces enable us to investigate the interplay between different topological duals under the same framework. In particular, defective topological spaces allow us to explore how the existence of a topological property in one dual affects the other duals.*

In [6,8], interval valued topology is defined as a non-empty family of interval valued sets and this makes it toilsome to investigate the structure in many directions. However the definition of defective topology is plain and every defective topology induces an interval valued topology as stated in Corollary 3.1.

As stated in the introduction, defective topological spaces provide a model for situations in which there is a lack of information about the open sets of the space. The following discussion considers a scenario where we are given a topological space without knowing all its open sets.

Let X be a set and τ be a topology with incomplete information about open sets. Assume that δ is the family of subsets that are certain to be open, while δ^* is the family of subsets that are certain not to be open. In the following algorithm, we illustrate how to approach the topology τ using the families δ and δ^*.

Algorithm:

(1) Initialize the lower approximation topology as the set of all known open sets δ.

(2) Iterate over all possible finite intersections of the sets δ, and add them to the lower approximation topology if they are not already included.

(3) Iterate over all possible unions of the sets δ, and add them to the lower approximation topology if they are not already included.

(4) Initialize the upper approximation topology as the complement of δ^* with respect to the power set of X.

(5) Iterate over all possible finite intersections of the sets in the initial set in step 4, and add them to the upper approximation topology.

(6) Iterate over possible unions of the sets obtained in step 5, and add them to the upper approximation topology.

We can denote the lower approximation topology obtained in step 3 by τ_δ, and the upper approximation topology obtained in step 6 by $\tau_{\mathcal{P}(X)\setminus\delta^*}$. In this case, $(X, \tau_\delta, \tau_{\mathcal{P}(X)\setminus\delta^*})$ is a defective topological space, where the elements of τ_δ are exactly open sets, and the elements of $\tau_{\mathcal{P}(X)\setminus\delta^*}$ are possibly open sets.

The following example serves as an application for the algorithm:

Example 3.2 *Let $X = \{a, b, c\}$ be a set with a topology τ with incomplete information about open sets. Assume that $\delta = \{\{a, b\}, \{a, c\}\}$ and $\delta^* = \{\{c\}, \{b, c\}\}$. Then we obtain the lower approximation topology*

$$\tau_\delta = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$$

by applying steps (1), (2) *and* (3). *Also we have*

$$\mathcal{P}(X) \setminus \delta^* = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$$

by step (4) *and hence we get the upper approximation topology*

$$\tau_{\mathcal{P}(X)\setminus\delta^*} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$$

by applying steps (5) *and* (6). *Thus,* $(X, \tau_\delta, \tau_{\mathcal{P}(X)\setminus\delta^*})$ *is a defective topological space. Note that* $\tau_\delta \subseteq \tau \subseteq \tau_{\mathcal{P}(X)\setminus\delta^*}$.

The following example shows the limitations of the proposed algorithm:

Example 3.3 *Let* $X = \{a, b, c, d\}$, $\delta = \{\{a, b\}, \{a, c\}\}$ *and* $\delta^* = \{\{d\}, \{c\}\}$. *Then, the exactly open sets are as follows:*

$$\tau_\delta = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}.$$

On the other hand, $\tau_{\mathcal{P}(X)\setminus\delta^*} = \mathcal{P}(X)$ *since the sets in* δ^* *can be obtained by intersection of the elements of* $\mathcal{P}(X) \setminus \delta^*$,

$$\{d\} = \{a, d\} \cap \{b, d\} \text{ and } \{c\} = \{a, c\} \cap \{b, c\}.$$

The question of how to find a better approach for τ remains open. Furthermore, the quality of the approximation depends on the size of δ and δ^*. Having a larger number of elements in δ and δ^* can improve the accuracy of approximating the topology τ. Furthermore, having additional information about the topological properties, such as being compact or Hausdorff, can enhance the precision of the approximation.

It is now time to define appropriate notions of continuity between defective topological spaces.

Definition 3.3 *Let* (X, τ_1, ρ_1) *and* (Y, τ_2, ρ_2) *be defective topological spaces. A mapping* $f : X \to Y$ *is called temporarily continuous if* $f^{-1}(A)$ *is exactly open for every exactly open set* A *in* (Y, τ_2, ρ_2) *and* $f : X \to Y$ *is called possibly continuous if* $f^{-1}(A)$ *is possibly open for every possibly open set* A *in* (Y, τ_2, ρ_2). *That is,* f *is called*

(1) temporarily continuous if $f^{-1}(A) \in \tau_1$ *for all* $A \in \tau_2$,

(2) possibly continuous if $f^{-1}(A) \in \rho_1$ *for all* $A \in \rho_2$.

Moreover, f *is called tp-continuous if it is both temporarily and possibly continuous.*

Example 3.4 *For a defective topological space* (X, τ, ρ) *the identity function* $I_X : X \to X$ *is both temporarily and possibly continuous.*

Proposition 3.1 *Let* (X, τ_1, ρ_1), (Y, τ_2, ρ_2) *be defective topological spaces and* $f : X \to Y$ *be a function.*

(1) f *is temporarily continuous iff* $f^{-1}(A) \in \tau_1^c$ *for all* $A \in \tau_2^c$,

(2) f *is possibly continuous iff* $f^{-1}(A) \in \rho_1^c$ *for all* $A \in \rho_2^c$.

Proof: The proofs are straightforward. □

It is evident that temporarily continuity and possibly continuity are preserved by the composition of functions. Consequently, three categories can be derived: **tDtop** for defective topological spaces and temporarily continuous functions between them, **pDtop** for defective topological spaces and possibly continuous functions between them, and **tpDtop** for defective topological spaces and tp-continuous functions between them.

Proposition 3.2 *The functor* $\mathfrak{F} : \mathbf{Top} \to \mathbf{tpDtop}$ *defined by*

$$\mathfrak{F}(f : (X, \tau) \to (Y, \sigma)) = (f : (X, \tau, \tau) \to (Y, \sigma, \sigma))$$

is a full embedding.

Proof: Let (X, τ), (Y, σ) be two topological spaces and $f, g : (X, \tau) \to (Y, \sigma)$ be two continuous functions. If $\mathfrak{F}(f) = \mathfrak{F}(g)$ then we have $f = g$. So, $\mathfrak{F}$ is faithful. Also, $\mathfrak{F}((X, \tau)) = \mathfrak{F}((Y, \sigma))$ implies $(X, \tau, \tau) = (Y, \sigma, \sigma)$ that is, $(X, \tau) = (Y, \sigma)$. That means $\mathfrak{F}$ is injective on objects. Therefore $\mathfrak{F} : \mathbf{Top} \to \mathbf{tpDtop}$ is an embedding. If $f : \mathfrak{F}((X, \tau)) = (X, \tau, \tau) \to \mathfrak{F}((Y, \sigma)) = (Y, \sigma, \sigma)$ is a tp-continuous function then $f : (X, \tau) \to (Y, \sigma)$ is a continuous function, i.e., $\mathfrak{F}(f) = f$. Thus, $\mathfrak{F}$ is full. □

Proposition 3.3 *The functor* $\mathfrak{L} : \mathbf{tpDtop} \to \mathbf{Top}$ *defined by*

$$\mathfrak{L}(f : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)) = (f : (X, \tau_1 \vee \rho_1) \to (Y, \tau_2 \vee \rho_2)) = (f : (X, \rho_1) \to (Y, \rho_2))$$

is a faithful functor.

Proof: Let (X, τ_1, ρ_1), (Y, τ_2, ρ_2) be two defective topological spaces and $f, g : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)$ be two tp-continuous functions. Since $\tau_1 \subseteq \rho_1$ and $\tau_2 \subseteq \rho_2$ we have $\tau_1 \vee \rho_1 = \rho_1$ and $\tau_2 \vee \rho_2 = \rho_2$. So, the functions $f, g : (X, \tau_1 \vee \rho_1) \to (Y, \tau_2 \vee \rho_2)$ are continuous. If $\mathfrak{L}(f) = \mathfrak{L}(g)$ then we have $f = g$. So, $\mathfrak{L}$ is faithful. □

Proposition 3.4 *The functor* $\mathfrak{G} : \mathbf{tpDtop} \to \mathbf{BiTop}$ *defined by*

$$\mathfrak{G}(f : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)) = (f : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2))$$

is a full embedding.

Proof: Suppose that (X, τ_1, ρ_1), (Y, τ_2, ρ_2) are defective topological spaces and that $f, g : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)$ are tp-continuous functions. If $\mathfrak{G}(f) = \mathfrak{G}(g)$ then we have $f = g$. So, $\mathfrak{G}$ is faithful. Also, $\mathfrak{G}((X, \tau_1, \rho_1)) = \mathfrak{G}((Y, \tau_2, \rho_2))$ implies $(X, \tau_1, \rho_1) = (Y, \tau_2, \rho_2)$ that is, $\mathfrak{G}$ is injective on objects and $\mathfrak{G} : \mathbf{tpDtop} \to \mathbf{BiTop}$ is an embedding. If $g : \mathfrak{G}((X, \tau_1, \rho_1)) = (X, \tau_1, \rho_1) \to \mathfrak{G}((Y, \tau_2, \rho_2)) = (Y, \tau_2, \rho_2)$ is a bicontinuous function then $g : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)$ is a tp-continuous function, i.e., $\mathfrak{G}(g) = g$. So, $\mathfrak{G}$ is full. □

It is possible to define the following functor using the supremum topology.

Proposition 3.5 *The functor* $\mathfrak{H} : \mathbf{BiTop} \to \mathbf{tpDtop}$ *defined by*

$$\mathfrak{H}(f : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)) = (f : (X, \tau_1, \tau_1 \vee \rho_1) \to (Y, \tau_2, \tau_2 \vee \rho_2))$$

is a faithful functor.

Proof: If (X, τ, ρ) is a bitopological space then $\tau \vee \rho$ is a topology on X and $\tau \subseteq (\tau \vee \rho)$, so $(X, \tau, \tau \vee \rho)$ is a defective topological space. Let (X, τ_1, ρ_1), (Y, τ_2, ρ_2) be two bitopological spaces and $f, g : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)$ be two bicontinuous functions. Since $\tau_1 \cup \rho_1$ is a subbase for $\tau_1 \vee \rho_1$ and $\tau_2 \cup \rho_2$ is a subbase for $\tau_2 \vee \rho_2$, the functions $f, g : (X, \tau_1, \tau_1 \vee \rho_1) \to (Y, \tau_2, \tau_2 \vee \rho_2)$ are tp-continuous. Thus, $\mathfrak{H}$ is a functor. Besides, if $\mathfrak{H}(f) = \mathfrak{H}(g)$ then we have $f = g$. So, $\mathfrak{H}$ is faithful. □

Proposition 3.6 *The functor* $\mathfrak{H}^* : \mathbf{BiTop} \to \mathbf{tpDtop}$ *defined by*

$$\mathfrak{H}^*(f : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)) = (f : (X, \rho_1, \tau_1 \vee \rho_1) \to (Y, \rho_2, \tau_2 \vee \rho_2))$$

is a faithful functor.

Proof: It is similar to the proof of Theorem 3.5. □

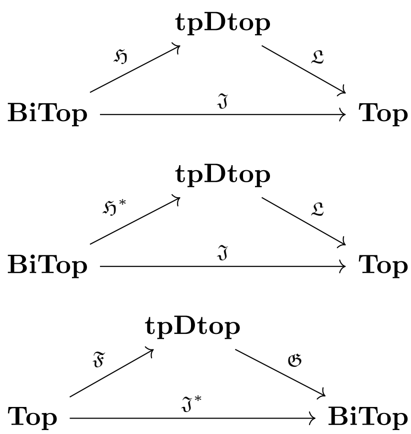
Consider the functors $\mathfrak{J} : \mathbf{BiTop} \to \mathbf{Top}$ and $\mathfrak{J}^* : \mathbf{Top} \to \mathbf{BiTop}$ [3] defined by

$$\mathfrak{J}(f : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)) = (f : (X, \tau_1 \vee \rho_1) \to (Y, \tau_2 \vee \rho_2))$$

and

$$\mathfrak{J}^*(f : (X, \tau_1) \to (Y, \tau_2) = (f : (X, \tau_1, \tau_1) \to (Y, \tau_2, \tau_2)).$$

In this case, there is no question that the diagrams shown below are commutative, i.e., $\mathfrak{J} = \mathfrak{L} \circ \mathfrak{H}$, $\mathfrak{J} = \mathfrak{L} \circ \mathfrak{H}^*$ and $\mathfrak{J}^* = \mathfrak{G} \circ \mathfrak{F}$.



Theorem 3.1 *Let (X, τ, ρ) be a defective topological space and let*

$$\mathcal{B}_{\tau\rho} = \{[int_\tau(A), int_\rho(B)] \ : \ A, B \subseteq X, \ A \subseteq B\}$$

be a family of interval-valued sets, where $int_\tau(A)$ (resp., $int_\rho(B)$) denotes the τ-interior of A (resp., ρ-interior of B). Then $\mathcal{B}_{\tau\rho}$ is base for an IVT on X.

Proof: In order to show that $\mathcal{B}_{\tau\rho}$ is a base for an IVT, we must verify that it meets the conditions given in Theorem 2.1. Since $\widetilde{X} = [X, X] \in \mathcal{B}_{\tau\rho}$ then $\widetilde{X} = \bigcup \mathcal{B}_{\tau\rho}$. Now let $C_1 = [int_\tau(A_1), int_\rho(B_1)]$ and $C_2 = [int_\tau(A_2), int_\rho(B_2)]$ be given for $A_1, A_2, B_1, B_2 \subseteq X$ with $A_1 \subseteq B_1$, $A_2 \subseteq B_2$. If $a_{IVP} \in C_1 \cap C_2$ then for $C = [int_\tau(A_1 \cap A_2), int_\rho(B_1 \cap B_2)]$, we have $a_{IVP} \in C \subseteq C_1 \cap C_2$. □

Corollary 3.1 *Each defective topological space (X, τ, ρ) induces an IVTS $(X, \mathcal{T}_{\tau\rho})$ where $\mathcal{T}_{\tau\rho}$ is the interval-valued topology with base $\mathcal{B}_{\tau\rho}$ on X. However, the converse is not valid as shown in the following remark.*

Remark 3.3 *For an interval-valued topological space $(X, \mathcal{T})$, $(X, \mathcal{T}^-, \mathcal{T}^+)$ is not necessarily a defective topological space. If we consider Example 2.2 then we have*

$$\mathcal{T}^- = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, \{3, 5\}, \{1, 3, 5\}, X\}$$

and

$$\mathcal{T}^+ = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, \{3, 5, 7\}, \{1, 3, 5, 7\}, X\}$$

and $\mathcal{T}^- \nsubseteq \mathcal{T}^+$.

Proposition 3.7 *The functor $\mathfrak{K} : \mathbf{tpDtop} \to \mathbf{IV_{Top}}$ defined by*

$$\mathfrak{K}(f : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)) = (f : (X, \mathcal{T}_{\tau_1\rho_1}) \to (Y, \mathcal{T}_{\tau_2\rho_2}))$$

is a full embedding.

Proof: Let (X, τ_1, ρ_1), (Y, τ_2, ρ_2) be two defective topological spaces and $f : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)$ a tp-continuous function. Let $C = [int_{\tau_2}(A), int_{\rho_2}(B)] \in \mathcal{B}_{\tau_2\rho_2}$ an IVS in Y. Since $A \subseteq B$ and $\tau_2 \subseteq \rho_2$ we have $f^{-1}(int_{\tau_2}(A)) \subseteq f^{-1}(int_{\rho_2}(B))$. Also, because of the tp continuity of f we get $f^{-1}(int_{\tau_2}(A)) \in \tau_1$ and $f^{-1}(int_{\rho_2}(B)) \in \rho_1$. So, if we take $D = f^{-1}(int_{\tau_2}(A))$ and $E = f^{-1}(int_{\rho_2}(B))$ then we get $D, E \subseteq X$, $D \subseteq E$ and $f^{-1}(C) = [int_{\tau_1}(D), int_{\rho_1}(E)] \in \mathcal{B}_{\tau_1\rho_1}$. Thus, $f : (X, \mathcal{T}_{\tau_1\rho_1}) \to (Y, \mathcal{T}_{\tau_2\rho_2})$ is interval-valued continuous. That is, $\mathfrak{K} : \mathbf{tpDtop} \to \mathbf{IV_{Top}}$ is a functor.

Now, suppose that (X, τ_1, ρ_1), (Y, τ_2, ρ_2) are defective topological spaces and that $f, g : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)$ are tp-continuous functions. If $\mathfrak{K}(f) = \mathfrak{K}(g)$ then we have $f = g$. So, $\mathfrak{K}$ is faithful. Also, if $(X, \mathcal{T}_{\tau_1\rho_1}) = (Y, \mathcal{T}_{\tau_2\rho_2})$ then $X = Y$ and $\mathcal{T}_{\tau_1\rho_1} = \mathcal{T}_{\tau_2\rho_2}$. Since

$$\begin{aligned} A \in \tau_1 &\Rightarrow [int_{\tau_1}(A), int_{\rho_1}(A)] \in \mathcal{T}_{\tau_1\rho_1} = \mathcal{T}_{\tau_2\rho_2} \Rightarrow [A, int_{\rho_1}(A)] \in \mathcal{T}_{\tau_2\rho_2} \\ &\Rightarrow \exists {B_j}_{j\in J} \subseteq \mathcal{P}(X) \ : \ A = \bigcup_{j\in J} int_{\tau_2}(B_j) \Rightarrow A \in \tau_2. \end{aligned}$$

we have $\tau_1 \subseteq \tau_2$. Similarly it can be shown that $\tau_2 \subseteq \tau_1$ and $\rho_1 = \rho_2$. So, $(X, \mathcal{T}_{\tau_1\rho_1}) = (Y, \mathcal{T}_{\tau_2\rho_2})$ implies $(X, \tau_1, \rho_1) = (Y, \tau_2, \rho_2)$. That is, $\mathfrak{K}$ is injective on objects and so $\mathfrak{K} : \mathbf{tpDtop} \to \mathbf{IV_{Top}}$ is an embedding.

Let $g : \mathfrak{K}((X, \tau_1, \rho_1)) = (X, \mathcal{T}_{\tau_1\rho_1}) \to \mathfrak{G}((Y, \tau_2, \rho_2)) = (Y, \mathcal{T}_{\tau_2\rho_2})$ be an interval-valued continuous function and $A \in \tau_2$. Then we have $[int_{\tau_2}(A), int_{\rho_2}(A)] \in \mathcal{T}_{\tau_2\rho_2}$. So we get $[g^{-1}(int_{\tau_2}(A)), g^{-1}(int_{\rho_2}(A))] = [g^{-1}(A), g^{-1}(int_{\rho_2}(A))] \in \mathcal{T}_{\tau_2\rho_2}$ by the interval-valued continuity of g. Thus we obtain that $g^{-1}(A) \in \tau_1$ by the definition of $\mathcal{T}_{\tau_1\rho_1}$. Similarly it can be shown that $g^{-1}(B) \in \rho_1$ for all $B \in \rho_2$. That is, $g : (X, \tau_1, \rho_1) \to (Y, \tau_2, \rho_2)$ is a tp-continuous function, i.e., $\mathfrak{K}(g) = g$. So, $\mathfrak{K}$ is full. □

4. Convergence in defective topological spaces

In this section, we examine convergence in the context of defective topological spaces. This enables us to gain insight into the behavior of the filters, regardless of whether we know all open sets. Using this perspective, we can develop a more flexible and generalized understanding of convergence, enabling us to analyze a broader range of scenarios. Furthermore, we explore the relationship between continuity and convergence, as well as closure and convergence.

Let (X, τ, ρ) be a defective topological space and $x \in X$. We denote $\tau-$neighborhoods of x by $\mathcal{N}_\tau(x)$ and $\rho-$neighborhoods of x by $\mathcal{N}_\rho(x)$. That is,

$$\mathcal{N}_\tau(x) = \{N \subseteq X \mid \exists G \in \tau \ : \ x \in G \subseteq N\}$$

$$\mathcal{N}_\rho(x) = \{N \subseteq X \mid \exists H \in \rho \ : \ x \in H \subseteq N\}.$$

Since $\tau \subseteq \rho$ we get $\mathcal{N}_\tau(x) \subseteq \mathcal{N}_\rho(x)$.

Definition 4.1 *Let (X, τ_1, ρ_1) and (Y, τ_2, ρ_2) be defective topological spaces, $x \in X$. A function $f : X \to Y$ is called*

(1) temporarily continuous at x if $f^{-1}(N) \in \mathcal{N}_{\tau_1}(x)$ for all $N \in \mathcal{N}_{\tau_2}(f(x))$,

(2) possibly continuous at x if $f^{-1}(N) \in \mathcal{N}_{\rho_1}(x)$ for all $N \in \mathcal{N}_{\rho_2}(f(x))$.

Remark 4.1 *It is obvious that $f : X \to Y$ is temporarily (resp., possibly) continuous if and only if it is temporarily (resp., possibly) continuous at every point $x \in X$ respectively.*

Definition 4.2 *A filter $\mathcal{F}$ on (X, τ, ρ) is said to be*

(1) temporarily converges to x if $\mathcal{N}_\tau(x) \subseteq \mathcal{F}$ (denoted by $\mathcal{F} \xrightarrow{t} x$),

(2) exactly converges to x if $\mathcal{N}_\rho(x) \subseteq \mathcal{F}$ (denoted by $\mathcal{F} \xrightarrow{e} x$).

Remark 4.2 *It is clear that $\mathcal{F} \xrightarrow{e} x$ implies $\mathcal{F} \xrightarrow{t} x$ since $\mathcal{N}_\tau(x) \subseteq \mathcal{N}_\rho(x)$.*

Proposition 4.1 *Let (X, τ_1, ρ_1) and (Y, τ_2, ρ_2) be defective topological spaces, $f : X \to Y$ a function and A a subset of X. Then we have:*

(1) $x \in cl^{\tau}(A)$ if and only if there exists a filter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{t} x$.

(2) $x \in cl^{\rho}(A)$ if and only if there exists a filter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{e} x$.

(3) f is temporarily continuous at $x \Leftrightarrow$ for every filter $\mathcal{F}$ on X with $\mathcal{F} \xrightarrow{t} x$, $f(\mathcal{F}) \xrightarrow{t} f(x)$.

(4) f is possibly continuous at $x \Leftrightarrow$ for every filter $\mathcal{F}$ on X with $\mathcal{F} \xrightarrow{e} x$, $f(\mathcal{F}) \xrightarrow{e} f(x)$.

Proof: It is similar to the ordinary topological case. □

Remark 4.3 *Since $cl^{\tau}(A)$ is the intersection of the all exactly closed sets contains A and $cl^{\rho}(A)$ is the intersection of the all possibly closed sets contains A, we refer to $cl^{\tau}(A)$ (resp., $cl^{\rho}(A)$) as the exactly (resp., possibly) closure of A respectively.*

5. Separation axioms in defective topological spaces

Separation axioms in defective topological spaces have different properties than their standard counterparts in ordinary topological spaces. It is well known that a bitopological space (X, τ, τ^*) has a dual defined by (X, τ^*, τ). When it comes to defective topological spaces, there is no way of talking about a dual unless the two topologies are equal. This makes the study of separation axioms in these spaces particularly intriguing.

As you may recall, in Section 3, we have presented a procedure to obtain an approximation of a topology when we don't know all of its open sets. Additionally, we have discussed how additional information can help us obtain a more accurate approximation. This led us to investigate separation axioms in defective topological spaces. In this section, we will explore how separation axioms can be applied in defective topological spaces, and what implications this has for the study of these spaces, given the absence of a dual.

The separation axioms stated in Definition 2.5 can be applied to defective topological spaces as well, since every defective topological space is a bitopological space. However, since defective topological spaces lack a dual, it is not feasible to discuss the concept of dual Q. Thus, subsequent definitions must be established without regard to duality.

Definition 5.1 *A defective topological space is called*

(1) defectively weakly symmetric (d-ws) if for any $x, y \in X$, $x \notin cl^{\rho}(y)$ implies $y \notin cl^{\tau}(x)$,

(2) defectively pseudo-Hausdorff (d-pH) if for any $x \notin cl^{\rho}(y)$, there exist disjoint sets $G \in \rho$ and $H \in \tau$ with $x \in G$ and $y \in H$,

(3) defectively regular (d-regular) if given any $G \in \rho$ and $x \in G$, there exist $H \in \rho$ and τ-closed set F with $x \in H \subseteq F \subseteq G$,

(4) defectively normal (d-normal) if given any τ-closed set F and $G \in \rho$ with $F \subseteq G$, there exist $H \in \rho$ and τ-closed set K such that $F \subseteq H \subseteq K \subseteq G$.

The following theorem can be proved by a routine argument.

Theorem 5.1 *A defective topological space is d-normal if it is both d-regular and d-ws. Moreover, d-regular defective topological spaces are d-pH, and d-pH defective topological spaces are d-ws.*

Example 5.1 *Consider the defective topological space given in Example 3.1 (4). It is obviously T_0. Due to the fact that singletons are closed to both topologies, it is also ws and d-ws. However, (X, τ, τ^*) is neither pH nor d-ph. This is because, for any open set $G \in \tau$ and any open interval $(a, b) \in \rho$, the intersection of G and (a, b) is not empty. Accordingly, it does not satisfy any of the properties of being normal, d-normal, regular, or d-regular by Definitions 2.5 and 5.1.*

Based on the fact that the symmetrization of a defective topological space is ρ, we have the following proposition.

Proposition 5.1 *A defective topological space* (X, τ, ρ) *is* T_0 *if and only if* ρ *is* T_0.

In a bitopological space (X, τ, τ^*), there is no connection between τ being T_0 and τ^* being T_0. On the other hand, in a defective topological space (X, τ, ρ), if τ is T_0, then ρ is T_0 as well. However, it is essential to note that the converse of this statement is not always valid, as shown in the example below.

Example 5.2 *Consider the defective topological space* (X, τ, ρ) *where* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, *and* ρ *is the discrete topology on* X. *Then,* (X, τ, ρ) *is obviously weakly symmetric. However,* τ *is not* T_0, *whereas* ρ *is.*

Proposition 5.2 *Given a d-ws defective topological space* (X, τ, ρ), τ *is* T_0 *if and only if* ρ *is* T_0.

Proof: Let ρ be T_0 and $x \neq y$. Then, there exist $G \in \rho$ with $x \in G$ and $y \notin G$ (or, $y \in G$ and $x \notin G$). Since the symmetrization of the defective topological space is ρ, $\tau \cup \rho$ can be considered as a subbase for the topology ρ. Thus, we have $U \in \tau$ and $V \in \rho$ with $x \in U \cap V \subseteq G$. Moreover, $y \notin U \cap V$. Now, if $y \notin U$, we are done. However, if $y \notin V$, then $x \notin cl^{\rho}(y)$ and d-ws implies $y \notin cl^{\tau}(x)$. Hence, by definition of τ-closure, we can find an $T \in \tau$ with $y \in T$ and $x \notin T$. □

Recall that a topological space is ws if $x \notin cl^{\tau}(y)$ implies $y \notin cl^{\tau}(x)$. The terms "weak regularity" or "R_0" generally refer to weak symmetry in topological spaces. Moreover, the axiom pH in a topological space is obtained by separating out the T_0 portion of a Hausdorff space as is done for higher separation axioms [2] (for example, T_3 =regular + T_0). A topological space (X, τ) is pH if given any $x, y \in X$ with $x \notin cl^{\tau}(y)$, there exist $G, H \in \tau$ with $x \in G$, $y \in H$ and $G \cap H = \emptyset$.

Proposition 5.3 *If* (X, τ, ρ) *is d-ws, and* $\tau \subseteq \tau' \subseteq \rho$ *then* τ' *is ws.*

Proof: Let (X, τ, ρ) be d-ws and $x \notin cl^{\tau'}(y)$. By the fact that $cl^{\rho}(z) \subseteq cl^{\tau'}(z) \subseteq cl^{\tau}(z)$ for every $z \in X$, we have $x \notin cl^{\rho}(y)$. We now have $y \notin cl^{\tau}(x)$ by d-ws. Thus, $y \notin cl^{\rho}(x)$, as well. □

Theorem 5.2 *A defective topological space* (X, τ, ρ) *is ws (resp., pH, regular) if* τ *is ws (resp., pH, regular).*

Proof: It is an easy consequence of Theorem 2.3 and the fact that $\tau \subseteq \rho$ □

Corollary 5.1 *A defective topological space* (X, τ, ρ) *is ws if it is d-ws.*

Proof: The assertion is a consequence of Proposion 5.3 and Theorem 5.2. □

Proposition 5.4 *A defective topological space is normal if and only if d-normal.*

Proof: It can be done in a straightforward manner. □

Proposition 5.5 *(1) A defective topological space is pH if and only if it satisfies the following condition: For every filter* $\mathcal{F}$ *on* X, *if* $\mathcal{F}$ *temporarily converges to* x *and exactly converges to* y *then* $x \in cl^{\tau}(y)$.

(2) If a defective topological space is pH and d-ws then it is d-pH.

Proof: They can be proven in a similar manner to that of [7, Lemma 2.5]. □

6. Conclusion

In this paper, we have introduced the concept of defective topological spaces, which provides a framework for dealing with incomplete information about open sets in a topological space. We have presented an algorithm to approximate spaces in cases where information about open sets is incomplete. We have also explored the categorical structure of defective topological spaces and their relationships with categories of topological, bitopological, and interval-valued spaces. Moreover, we studied continuity, convergence, and separation axioms in the context of defective topological spaces.

Defective topological spaces are useful for modeling uncertainty about open sets in a topological space. Moreover, they offer a more flexible and general approach to studying continuity and convergence, as it allows for the possibility of considering situations where open sets are not fully known.

The theory of defective topological spaces offers a promising new direction for studying topological concepts. Moreover, it represents a significant advance in our understanding of topological spaces, and may lead to important developments in various fields of mathematics, including algebraic topology, functional analysis, and measure theory.

There are several directions for future research in this area. One possible direction is to study other bitopological concepts, such as compactness, stability, and local compactness, in the context of defective topological spaces. Another interesting direction is to investigate the relation between defective topological spaces and lattice theoretical structures such as biframes. These areas of research may provide further insight into the behavior of defective topological spaces and their applications in different areas of mathematics.

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Ramazan Ekmekçi,
Department of Mathematics,
Kırklareli University,
Turkey.
E-mail address: rekmekci@klu.edu.tr

and

Esra Korkmaz,
Department of Computer Technologies,
Düzce University,
Turkey.
E-mail address: esrakorkmaz@duzce.edu.tr