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Additive mapping acting as generalized (μ,ν)-derivation on semi-prime rings

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ABSTRACT: The objective of this paper is to study the following: Let $\mathcal{A}$ be a $(m+n-1)!$-torsion free semi-prime ring. Suppose that $\mathcal{G}, g : \mathcal{A} \to \mathcal{A}$ are two additive mappings satisfying the algebraic identity $\mathcal{G}(r^{m+n}) = \mathcal{G}(r^m)\mu(r^n) + \nu(r^m)g(r^n)$ for all $r \in \mathcal{A}$. Then $\mathcal{G}$ will be a generalized (μ,ν)-derivation with associated (μ,ν)-derivation g on $\mathcal{A}$. On the other hand, it is proved that $\mathcal{G}_1$ is a generalized left (μ,ν)-derivation associated with left (μ,ν)-derivation g_1 on $\mathcal{A}$ if they satisfy the algebraic identity $\mathcal{G}_1(r^{m+n}) = \mu(r^n)\mathcal{G}_1(r^m) + \nu(r^m)g_1(r^n)$ for all $r \in \mathcal{A}$. We will also examine criticism and provide example.

Key Words: Semiprime rings, generalized (μ,ν)-derivation, generalized left (μ,ν)-derivation and additive mappings.

Contents

1. Introduction

$\mathcal{A}$ will be used to represent an associative ring with identity throughout this research. $Z(\mathcal{A})$ denotes the centre of $\mathcal{A}$, $Q_l(\mathcal{A}_C)$ is left Martindale ring of quotients and $\mathcal{C}$ its extended centroid. A ring $\mathcal{A}$ is termed as p-torsion free ring, if $pr = 0$, $r \in \mathcal{A}$ implies $r = 0$, where $p > 1$ is an integer. Usually the commutator $rs - sr$ is denoted by $[r, s]$. Recollect that a ring $\mathcal{A}$ is known as a prime if $r\mathcal{A}s = \{0\}$ implies either $r = 0$ or $s = 0$, and is semi-prime if $r\mathcal{A}r = \{0\}$ implies $r = 0$. A mapping $g : \mathcal{A} \to \mathcal{A}$ is known as a derivation if it is additive and satisfies $g(rs) = g(r)s + rg(s)$ for all $r, s \in \mathcal{A}$ and is recognised as a Jordan derivation if $g(r^2) = g(r)r + rg(r)$ holds for all $r \in \mathcal{A}$. Every derivation will be a Jordan derivation, however general terms, the converse need not be true. A Jordan derivation on a prime ring with characteristic other than two is a derivation, according to a Herstein's classical result [8, Theorem 3.3]. This conclusion has been extended for the 2-torsion free semi-prime ring by Cusack [7]. Suppose that a derivation g on $\mathcal{A}$ exists, then an additive mapping $\mathcal{G} : \mathcal{A} \to \mathcal{A}$ is referred to as a generalized derivation if $\mathcal{G}(rs) = \mathcal{G}(r)s + rg(s)$, and $\mathcal{G}$ is termed as a generalized Jordan derivation if there exists a Jordan derivation g on $\mathcal{A}$ such that $\mathcal{G}(r^2) = \mathcal{G}(r)r + rg(r)$ for all $r, s \in \mathcal{A}$. Every generalized derivation can be easily confirmed to be a generalized Jordan derivation, while the converse is often not true. Assume that μ and ν are two endomorphisms on $\mathcal{A}$. An additive mapping $g : \mathcal{A} \to \mathcal{A}$ is recognised as a (μ,ν)-derivation (respectively Jordan (μ,ν)-derivation) if $g(rs) = g(r)\mu(s) + \nu(r)g(s)$ (respectively $g(r^2) = g(r)\mu(r) + \nu(r)g(r)$) for all $r, s \in \mathcal{A}$. Every (μ,ν)-derivation will be a Jordan (μ,ν)-derivation, although generally the converse is not true. Both are identical on a 2-torsion free semi-prime ring (For details see [9]). An additive mapping $\mathcal{G} : \mathcal{A} \to \mathcal{A}$ is known as a generalized (μ,ν)-derivation (respectively generalized Jordan (μ,ν)-derivation) if there exists a (μ,ν)-deviation (respectively Jordan (μ,ν)-deviation) g on $\mathcal{A}$ such that $\mathcal{G}(rs) = \mathcal{G}(r)\mu(s) + \nu(r)g(s)$ (respectively $\mathcal{G}(r^2) = \mathcal{G}(r)\mu(r) + \nu(r)g(r)$) for all $r, s \in \mathcal{A}$. Every generalized (μ,ν)-derivation will be a generalized Jordan (μ,ν)-derivation but the converse does not hold in general. If $\mathcal{A}$ is 2-torsion free semi-prime ring then the converse is valid (See [2]). Now, if $\mathcal{G}$ is a generalized (μ,ν)-derivation (respectively generalized Jordan (μ,ν)-derivation) associated with (μ,ν)-derivation (respectively Jordan (μ,ν)-derivation) g on $\mathcal{A}$, then the identity $\mathcal{G}(r^{2n}) = \mathcal{G}(r^n)\mu(r^n) + \nu(r^n)g(r^n)$ holds for all $r \in \mathcal{A}$ but what about the converse? Author has studied the converse of the this statement in [3]. The investigation of a generalization of the aforementioned conclusion is the focus of the current work. Specifically, we proved under some conditions on $\mathcal{A}$, $\mathcal{G}$ will be a generalized (μ,ν)-derivation associated with

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a (μ,ν)-derivation g if it satisfies the algebraic identity $\mathcal{G}(r^{m+n}) = \mathcal{G}(r^m)\mu(r^n)+\nu(r^m)g(r^n)$ for all $r \in \mathcal{A}$.

Next section accords with the investigation of an extension of a derivation known as a left derivation, which is defined as: an additive mapping $g_1 : \mathcal{A} \to \mathcal{A}$ is recognised as a left (μ,ν)-derivation (respectively Jordan left (μ,ν)-derivation) if $g_1(rs) = \mu(r)g_1(s)+\nu(s)g_1(r)$ (respectively $g_1(r^2) = \mu(r)g_1(r)+\nu(r)g_1(r)$) for all $r, s \in \mathcal{A}$. An additive mapping $\mathcal{G}_1 : \mathcal{A} \to \mathcal{A}$ is termed as generalized left (μ,ν)-derivation (respectively generalized Jordan left (μ,ν)-derivation) if there exists a Jordan left (μ,ν)-deviation g_1 on $\mathcal{A}$ such that $\mathcal{G}_1(rs) = \mu(r)\mathcal{G}_1(s) + \nu(s)g_1(r)$ (respectively $\mathcal{G}_1(r^2) = \mu(r)\mathcal{G}_1(r) + \nu(r)g_1(r)$) for all $r, s \in \mathcal{A}$. An additive mapping $\mathcal{J} : \mathcal{A} \to \mathcal{A}$ is known as a right (respectively left) μ-centralizer of $\mathcal{A}$ if $\mathcal{J}(rs) = \mu(r)\mathcal{J}(s)$ (respectively $\mathcal{J}(rs) = \mathcal{J}(r)\mu(s)$), for all $r, s \in \mathcal{A}$. An additive mapping $\mathcal{J} : \mathcal{A} \to \mathcal{A}$ is termed as a Jordan left (respectively Jordan right) μ-centralizer of $\mathcal{A}$ if $\mathcal{J}(r^2) = \mathcal{J}(r)\mu(r)$ (respectively $\mathcal{J}(r^2) = \mu(r)\mathcal{J}(r)$) for all $r \in \mathcal{A}$. Trivially, every left (respectively right) μ-centralizer will be a Jordan left (respectively right) μ-centralizer on $\mathcal{A}$. Obviously, $\mathcal{G}_1$ will be a generalized left (μ,ν)-derivation iff $\mathcal{G}_1 = g_1 + \mathcal{J}$, where $\mathcal{J}$ right μ-centralizer and g_1 is a left (μ,ν)-derivation of $\mathcal{A}$. The theory of generalized left (μ,ν)-derivations encompass the theory of left (μ,ν)-derivation. Furthermore, if $g_1 = 0$, then it is a right μ-centralizer. $\mathcal{G}_1(r) = \mu(r)a+g_1(r)$ will be a generalized left (μ,ν)-derivation, where g_1 is a left (μ,ν)-derivation of $\mathcal{A}$ for a fixed $a \in \mathcal{A}$. Note that the mapping $g : \mathcal{A} \to \mathcal{A}$ such that $g(r) = \mathcal{G}_1(r) + \mu(r)a$ or $g(r) = \mathcal{G}_1(r) - \mu(r)a$ will also be a generalized left (μ,ν)-derivation on $\mathcal{A}$, where $a \in \mathcal{A}$ is a fixed element for any generalized left (μ,ν)-derivation $\mathcal{G}_1$. If $\mathcal{G}_1,\ g_1$ are a generalized Jordan left (μ,ν)-derivation and associated Jordan left (μ,ν)-derivation on $\mathcal{A}$, then $\mathcal{G}_1(r^{2n}) = \mu(r^n)\mathcal{G}_1(r^n) + \nu(r^n)g_1(r^n)$ holds for all $r \in \mathcal{A}$ but in general, the converse is not true. Author studied the converse of this statement in [3]. More precisely, $\mathcal{G}_1$ is a generalized Jordan left (μ,ν)-derivation associated with Jordan left (μ,ν)-derivation g_1 on $\mathcal{A}$ if $\mathcal{G}_1(r^{2n}) = \mu(r^n)\mathcal{G}_1(r^n) + \nu(r^n)g_1(r^n)$ holds for all $r \in \mathcal{A}$ with some restrictions on $\mathcal{A}$. A generalization of the above mentioned result is given in the present paper.

We begin by asserting the preceding theorem:

2. Main Theorems

Theorem 2.1 *Let $m, n \geq 1$ be two fixed integers and $\mathcal{A}$ be a $(m+n-1)!$-torsion free semi-prime ring with a multiplicative identity e. Suppose that $\mathcal{G}, g : \mathcal{A} \to \mathcal{A}$ are two additive mappings which satisfy the algebraic identity $\mathcal{G}(r^{m+n}) = \mathcal{G}(r^m)\mu(r^n) + \nu(r^m)g(r^n)$ for all $r \in \mathcal{A}$, where μ, ν are automorphisms and endomorphism respectively on $\mathcal{A}$. Then $\mathcal{G}$ is a generalized (μ,ν)-derivation with associated (μ,ν)-derivation g on $\mathcal{A}$.*

Proof: We have given that

$$\mathcal{G}(r^{m+n}) = \mathcal{G}(r^m)\mu(r^n) + \nu(r^m)g(r^n) \text{ for all } r \in \mathcal{A}. \tag{2.1}$$

Replacing r by e, we get $g(e) = 0$. If we substitute r by $r + ks$ in the above equation, then we find

$\mathcal{G}\Big(r^{m+n} + {}^{m+n}C_1 r^{(m+n-1)}ks + {}^{m+n}C_2 r^{m+n-2}k^2s^2 + ... + k^{m+n}s^{m+n}\Big) =$
$\mathcal{G}\Big(r^m + {}^mC_1 r^{m-1}ks + {}^mC_2 r^{m-2}k^2s^2 + ... + k^m s^m\Big)\Big(\mu(r^n) + {}^nC_1\mu(r^{n-1}ks) + {}^nC_2\mu(r^{n-2}k^2s^2) + ... +$
$\mu(k^n s^n)\Big) + \Big(\nu(r^m) + {}^mC_1\nu(r^{m-1}ks) + {}^mC_2\nu(r^{m-2}k^2s^2) + ... + \nu(k^m s^m)\Big) g\Big(r^n + {}^nC_1 r^{n-1}ks + {}^nC_2 r^{n-2}k^2s^2 +$
$... + k^n s^n\Big)$, where $k \in \mathbb{Z}^+$.

Rewrite the above expression by using (2.1) as

$$kf_1(r,s) + k^2 f_2(r,s) + ... + k^{(m+n-1)} f_{(m+n-1)}(r,s) = 0,$$

where $f_i(r,s)$ stand for the coefficients of k^i's for all $i = 1, 2, ..., (m+n-1)$. If we replace k by $1, 2, ..., (m+n-1)$, then we find a system of $(m+n-1)$ homogeneous equations. It gives us a Vander

Monde matrix
$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{(m+n-1)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ (m+n-1) & ((m+n-1))^2 & \dots & ((m+n-1))^{(m+n-1)} \end{bmatrix}.$$

Which yields that $f_i(r,s)=0$ for all $r,s\in\mathcal{A}$ and for $i=1,2,..,(m+n-1)$. In particular, We have

$$\begin{aligned} f_1(r,s) &= {}^{m+n}C_1\mathcal{G}(r^{(m+n-1)}s) - {}^{n}C_1\mathcal{G}(r^m)\mu(r^{n-1}s) - {}^{m}C_1\mathcal{G}(r^{m-1}s)\mu(r^n) \\ &\quad - {}^{n}C_1\nu(r^m)g(r^{n-1}s) - {}^{m}C_1\nu(r^{m-1}s)g(r^n) = 0 \text{ for all } r,s\in\mathcal{A}. \end{aligned}$$

Let us put $r=e$ and making use of $g(e)=0$ and $\mu(e)=e$ to appear $(m+n)\mathcal{G}(s)=n\mathcal{G}(e)\mu(s)+m\mathcal{G}(s)+ng(s)$. Since $\mathcal{A}$ is n-torsion free, we have

$$\mathcal{G}(s)=\mathcal{G}(e)\mu(s)+g(s) \text{ for all } s\in\mathcal{A}. \tag{2.2}$$

Next observe that

$$\begin{aligned} f_2(r,s) &= {}^{m+n}C_2\mathcal{G}(r^{m+n-2}s^2) - {}^{n}C_2\mathcal{G}(r^m)\mu(r^{n-2}s^2) - {}^{m}C_1{}^{n}C_1\mathcal{G}(r^{m-1}s)\mu(r^{n-1}s) \\ &\quad - {}^{m}C_2\mathcal{G}(r^{m-2}s^2)\mu(r^n) - {}^{n}C_2\nu(r^m)g(r^{n-2}s^2) - {}^{m}C_1{}^{n}C_1\nu(r^{m-1}s)g(r^{n-1}s) \\ &\quad - {}^{m}C_2\nu(r^{m-2}s^2)g(r^n) = 0 \text{ for all } r,s\in\mathcal{A}. \end{aligned}$$

Rewrite the above expression by substituting e for r to obtain

$$\begin{aligned} {}^{m+n}C_2\mathcal{G}(s^2) &= {}^{n}C_2\mathcal{G}(e)\mu(s^2) + {}^{m}C_1{}^{n}C_1\mathcal{G}(s)\mu(s) \\ &\quad + {}^{m}C_2\mathcal{G}(s^2) + {}^{n}C_2 g(s^2) + {}^{m}C_1{}^{n}C_1\nu(s)g(s) \text{ for all } s\in\mathcal{A}. \end{aligned}$$

This implies that

$$\begin{aligned} \tfrac{(m+n)(m+n-1)}{2}\mathcal{G}(s^2) &= \tfrac{n(n-1)}{2}\mathcal{G}(e)\mu(s^2) + mn\mathcal{G}(s)\mu(s) + \tfrac{m(m-1)}{2}\mathcal{G}(s^2) \\ &\quad + \tfrac{n(n-1)}{2}g(s^2) + mn\nu(s)g(s). \end{aligned}$$

A simple manipulation give us

$$n(2m+n-1)\mathcal{G}(s^2) = n(n-1)\mathcal{G}(e)\mu(s^2) + 2mn\mathcal{G}(s)\mu(s) + n(n-1)g(s^2) + 2mn\nu(s)g(s).$$

Since $\mathcal{A}$ is n-torsion free, then we get

$$(2m+n-1)\mathcal{G}(s^2) = (n-1)\mathcal{G}(e)\mu(s^2) + 2m\mathcal{G}(s)\mu(s) + (n-1)g(s^2) + 2m\nu(s)g(s).$$

An application of (2.2) yields that

$$\begin{aligned} (2m+n-1)\Big[\mathcal{G}(e)\mu(s^2)+g(s^2)\Big] &= (n-1)\mathcal{G}(e)\mu(s^2) + 2m\Big[\mathcal{G}(e)\mu(s)+g(s)\Big]\mu(s) \\ &\quad + (n-1)g(s^2) + 2m\nu(s)g(s). \end{aligned}$$

On simplifying the above expression, we obtain

$$(2m+n-1-n+1-2m)\mathcal{G}(e)\mu(s^2) + (2m+n-1-n+1)g(s^2) = 2mg(s)\mu(s) + 2m\nu(s)g(s).$$

This implicit that for all $s\in\mathcal{A}$,

$$2mg(s^2) = 2mg(s)\mu(s) + 2m\nu(s)g(s).$$

$2m$-torsion freeness of $\mathcal{A}$ allow us to write last expression as $g(s^2)=g(s)\mu(s)+\nu(s)g(s)$. That is a Jordan (μ,ν)-derivation. Since $\mathcal{A}$ is a 2-torsion free semi-prime ring, then use [9] to get that g is an (μ,ν)-derivation on $\mathcal{A}$. Consider (2.2) once again, so that

$$\begin{aligned} \mathcal{G}(s^2) &= \mathcal{G}(e)\mu(s^2) + g(s^2) \\ &= \big[\mathcal{G}(e)\mu(s) + g(s)\big]\mu(s) + \nu(s)g(s) \\ &= \mathcal{G}(s)\mu(s) + \nu(s)g(s) \end{aligned}$$

Hence $\mathcal{G}$ is generalized Jordan (μ,ν)-derivation on $\mathcal{A}$ associated with the derivation g. Using theorem from [2], we get that $\mathcal{G}$ is generalized (μ,ν)-derivation on $\mathcal{A}$ associated with (μ,ν)-derivation g. □

The aforementioned outcome has the following repercussions:

Corollary 2.1 *Let $m,n \geq 1$ be two fixed integers and $\mathcal{A}$ be a $(m+n-1)!$-torsion free semi-prime ring. Suppose that $\mathcal{G}:\mathcal{A}\to\mathcal{A}$ is an additive mapping which satisfies the algebraic identity $\mathcal{G}(r^{m+n}) = \mathcal{G}(r^m)\mu(r^n)$ for all $r\in\mathcal{A}$, then $\mathcal{G}$ will be a μ-centralizer on $\mathcal{A}$, where μ is an automorphism on $\mathcal{A}$.*

Proof: Taking $g = 0$ in the above theorem, we get the required result. □

Corollary 2.2 *Let $m,n \geq 1$ be two fixed integers and $\mathcal{A}$ be a $(m+n-1)!$-torsion free semi-prime ring. Suppose that $g:\mathcal{A}\to\mathcal{A}$ is an additive mapping which satisfies the algebraic identity $g(r^{m+n}) = g(r^m)\mu(r^n)+\nu(r^m)g(r^n)$ for all $r\in\mathcal{A}$, where μ,ν are automorphisms and endomorphism respectively on $\mathcal{A}$. Then g is a (μ,ν)-derivation on $\mathcal{A}$.*

Proof: Considering g as $\mathcal{G}$ and using same steps as we did in Theorem 2.1, we come to the same result as required. □

Corollary 2.3 *Let $m,n \geq 1$ be two fixed integers and $\mathcal{A}$ be any $(m+n-1)!$-torsion free semi-prime ring, where μ,ν are endomorphism and automorphisms respectively on $\mathcal{A}$. If an additive mapping $\mathcal{G}:\mathcal{A}\to\mathcal{A}$ is satisfying $\mathcal{G}(r^{m+n}) = \mathcal{G}(r^m)r^n$ for all $r\in\mathcal{A}$. Then, $\mathcal{G}$ is a centralizer on $\mathcal{A}$.*

Proof: We find the desired result by taking $\mu = I$ in Corollary 2.1. □

Corollary 2.4 *Let $m,n \geq 1$ be two fixed integers and $\mathcal{A}$ be a $(m+n-1)!$-torsion free semi-prime ring, where μ,ν are automorphisms and endomorphism respectively on $\mathcal{A}$. Suppose that $g:\mathcal{A}\to\mathcal{A}$ is an additive mapping which satisfies the identity $g(r^{m+n}) = g(r^m)r^n + r^m g(r^n)$ for all $r\in\mathcal{A}$. Then g is a derivation on $\mathcal{A}$.*

Proof: Considering $\mu=\nu=I$ in Corollary 2.2, we get the required result. □

Proceed to the following primary assertion of this article:

Theorem 2.2 *Let $m,n \geq 1$ be two fixed integers and $\mathcal{A}$ be $(m+n-1)!$-torsion free ring. If $\mathcal{G}_1, g_1:\mathcal{A}\to\mathcal{A}$ are two additive mappings which satisfy the algebraic identity $\mathcal{G}_1(r^{m+n}) = \mu(r^n)\mathcal{G}_1(r^m)+\nu(r^m)g_1(r^n)$ for all $r\in\mathcal{A}$, where μ,ν are automorphisms and endomorphism respectively on $\mathcal{A}$, then $\mathcal{G}_1$ is generalized Jordan left (μ,ν)-derivation associated with Jordan left (μ,ν)-derivation g_1 on $\mathcal{A}$.*

Proof: Since we have

$$\mathcal{G}_1(r^{m+n}) = \mu(r^n)\mathcal{G}_1(r^m)+\nu(r^m)g_1(r^n) \text{ for all } r\in\mathcal{A}, \tag{2.3}$$

then, replacing r by $r+qs$, we get

$$\mathcal{G}_1\Big(r^{m+n} + {}^{m+n}C_1 r^{(m+n-1)}qs + {}^{m+n}C_2 r^{m+n-2}q^2s^2 + \ldots + q^{m+n}s^{m+n}\Big) = \Big(\mu(r^n) + {}^{n}C_1\mu(r^{n-1}qs) + {}^{n}C_2\mu(r^{n-2}q^2s^2) + \ldots + \mu(q^ns^n)\Big) f\Big(r^m + {}^{m}C_1 r^{m-1}qs + {}^{m}C_2 r^{m-2}q^2s^2 + \ldots + q^ms^m\Big) + \Big(\nu(r^m) + {}^{m}C_1\nu(r^{m-1}qs) + {}^{m}C_2\nu(r^{m-2}q^2s^2) + \ldots + \nu(q^ms^m)\Big) g_1\Big(r^n + {}^{n}C_1 r^{n-1}qs + {}^{n}C_2 r^{n-2}q^2s^2 + \ldots + q^ns^n\Big).$$

Use (2.3) to rewrite the previously mentioned expression as

$$qP_1(r,s) + q^2 P_2(r,s) + ... + q^{(m+n-1)} P_{(m+n-1)}(r,s) = 0,$$

where $P_i(r,s)$ stand for the coefficients of q^i's for all $i = 1, 2, ..., (m+n-1)$. If we replace q by $1, 2, ..., (m+n-1)$, then we find a system of $(m+n-1)$ homogeneous system of linear equations. It gives us a Vander Monde matrix

$$\begin{bmatrix} 1 & 1 & ... & 1 \\ 2 & 2^2 & ... & 2^{(m+n-1)} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ (m+n-1) & (m+n-1)^2 & ... & (m+n-1)^{(m+n-1)} \end{bmatrix}.$$

Which yields that $P_i(r,s) = 0$ for all $r, s \in \mathcal{A}$ and for $i = 1, 2, .., (m+n-1)$. Particularly for $i = 1$, We have

$$\begin{aligned} P_1(r,s) &= {}^{m+n}C_1 \mathcal{G}_1(r^{(m+n-1)}s) - {}^{n}C_1 \mu(r^{n-1}s)\mathcal{G}_1(r^m) - {}^{m}C_1 \mu(r^n)\mathcal{G}_1(r^{m-1}s) \\ &\quad - {}^{n}C_1 \nu(r^m) g_1(r^{n-1}s) - {}^{m}C_1 \nu(r^{m-1}s) g_1(r^n) = 0 \text{ for all } r, s \in \mathcal{A}. \end{aligned}$$

Putting $r = e$ and making use of $g_1(e) = 0, \mu(e) = \nu(e) = e$ and n-torsion freeness of $\mathcal{A}$, we have

$$\mathcal{G}_1(s) = \mu(s)\mathcal{G}_1(e) + g_1(s) \text{ for all } s \in \mathcal{A}. \tag{2.4}$$

Next,

$$\begin{aligned} P_2(r,s) &= {}^{m+n}C_2 \mathcal{G}_1(r^{m+n-2}s^2) - {}^{m}C_2 \mu(r^n)\mathcal{G}_1(r^{m-2}s^2) - {}^{m}C_1 {}^{n}C_1 \mu(r^{n-1}s)\mathcal{G}_1(r^{m-1}s) \\ &\quad - {}^{n}C_2 \mu(r^{n-2}s^2)\mathcal{G}_1(r^m) - {}^{n}C_2 \nu(r^m) g_1(r^{n-2}s^2) - {}^{m}C_1 {}^{n}C_1 \nu(r^{m-1}s) g_1(r^{n-1}s) \\ &\quad - {}^{m}C_2 \nu(r^{m-2}s^2) g_1(r^n) = 0 \text{ for all } r, s \in \mathcal{A}. \end{aligned}$$

To get the desired result, rewrite the preceding statement using e instead of r.

$$\begin{aligned} \tfrac{(m+n)(m+n-1)}{2}\mathcal{G}_1(s^2) &= \tfrac{n(n-1)}{2}\mu(s^2)\mathcal{G}_1(e) + mn\mu(s)\mathcal{G}_1(s) \\ &\quad + \tfrac{m(m-1)}{2}\mathcal{G}_1(s^2) + \tfrac{n(n-1)}{2} g_1(s^2) + mn\nu(s) g_1(s). \end{aligned}$$

That is,

$$\begin{aligned} &(m+n)(m+n-1)\mathcal{G}_1(s^2) = n(n-1)\mu(s^2)\mathcal{G}_1(e) + 2mn\mu(s)\mathcal{G}_1(s) + m(m-1)\mathcal{G}_1(s^2) \\ &+ n(n-1) g_1(s^2) + 2mn\nu(s) g_1(s). \end{aligned}$$

After simple manipulation, we arrive at

$$(2mn + n^2 - n)\mathcal{G}_1(s^2) = n(n-1)\mu(s^2)\mathcal{G}_1(e) + 2mn\mu(s)\mathcal{G}_1(s) + n(n-1) g_1(s^2) + 2mn\nu(s) g_1(s).$$

Using (2.4) to get the following

$$\begin{aligned} &(2mn + n^2 - n)\Big[\mu(s^2)\mathcal{G}_1(e) + g_1(s^2)\Big] = n(n-1)\mu(s^2)\mathcal{G}_1(e) + 2mn\mu(s)\Big[\mu(s)\mathcal{G}_1(e) \\ &+ g_1(s)\Big] + n(n-1) g_1(s^2) + 2mn\nu(s) g_1(s). \end{aligned}$$

Simplify the above expression and making use of $2mn$-torsion freeness of $\mathcal{A}$, we have

$$g_1(s^2) = \mu(s) g_1(s) + \nu(s) g_1(s) \text{ for all } s \in \mathcal{A}.$$

Hence g will be a Jordan left (μ, ν)-derivation on $\mathcal{A}$. Now, from (2.4), we get

$$\begin{aligned} \mathcal{G}_1(s^2) &= \mu(s^2)\mathcal{G}_1(e) + g_1(s^2) \\ &= \mu(s)\big[\mu(s)\mathcal{G}_1(e) + g_1(s)\big] + \nu(s) g_1(s) \\ &= \mu(s)\mathcal{G}_1(s) + \nu(s) g_1(s), \end{aligned}$$

so $\mathcal{G}$ will be a generalized Jordan left (μ, ν)-derivation associated with Jordan left (μ, ν)-derivation g_1 on $\mathcal{A}$.

□

By making the assumption that $\mu = \nu = I$, the following theorem emerges from the previous one:

Theorem 2.3 ([4], Theorem 2.5) *Let $m, n \geq 1$ be two fixed integers and $\mathcal{A}$ be a $(m+n-1)!$-torsion free semi-prime ring. If $\mathcal{G}_1, g_1 : \mathcal{A} \to \mathcal{A}$ are additive mappings satisfying $\mathcal{G}_1(r^{m+n}) = r^n \mathcal{G}_1(r^m) + r^m g_1(r^n)$ for all $r \in \mathcal{A}$. Then*

1. *$[g_1(r), s] = 0$ for all $r, s \in \mathcal{A}$, where g_1 acts a derivation,*
2. *g_1 maps $\mathcal{A}$ into $Z(\mathcal{A})$,*
3. *g_1 is zero or $\mathcal{A}$ is commutative,*
4. *For some $q \in Q_l(\mathcal{A}_C)$, $\mathcal{G}_1(r) = rq$ for all $r \in \mathcal{A}$,*
5. *$\mathcal{G}_1$ will be a generalized derivation on $\mathcal{A}$.*

The following illustration shows that the theorems presented in this paper are legitimate.

Example 2.1 *Define a ring $\mathcal{A} = \left\{ \begin{pmatrix} \bar{i} & 0 \\ 0 & \bar{k} \end{pmatrix} \mid \bar{i}, \bar{k} \in 2\mathbb{Z}_8 \right\}$, $\mathbb{Z}_8$ has its usual meaning. Define mappings $\mathcal{G}, g, \mathcal{G}_1, g_1, \mu, \nu : \mathcal{A} \to \mathcal{A}$ by $\mathcal{G}\begin{pmatrix} \bar{i} & 0 \\ 0 & \bar{k} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \bar{k} \end{pmatrix}$, $g\begin{pmatrix} \bar{i} & 0 \\ 0 & \bar{k} \end{pmatrix} = \begin{pmatrix} \bar{i} & 0 \\ 0 & 0 \end{pmatrix}$, $\mathcal{G}_1\begin{pmatrix} \bar{i} & 0 \\ 0 & \bar{k} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \bar{k} \end{pmatrix}$, $g_1\begin{pmatrix} \bar{i} & 0 \\ 0 & \bar{k} \end{pmatrix} = \begin{pmatrix} \bar{i} & 0 \\ 0 & 0 \end{pmatrix}$, $\mu\begin{pmatrix} \bar{i} & 0 \\ 0 & \bar{k} \end{pmatrix} = \begin{pmatrix} \bar{k} & 0 \\ 0 & \bar{i} \end{pmatrix}$ and $\nu\begin{pmatrix} \bar{i} & 0 \\ 0 & \bar{k} \end{pmatrix} = \begin{pmatrix} \bar{i} & 0 \\ 0 & 0 \end{pmatrix}$. It is clear that $\mathcal{G}$ is not a generalized (μ, ν)-derivation and $\mathcal{G}_1$ is not a generalized Jordan left (μ, ν)-derivation on $\mathcal{A}$ but $\mathcal{G}, g, \mathcal{G}_1, g_1$ satisfy the algebraic conditions (2.1) and (2.3), which shows that semi-primeness and torsion condition on $\mathcal{A}$ are essential conditions in both main theorems of this article.*

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