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Modeling American Put Options for Oil Producers in the Covid-19 Era

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ABSTRACT: In this paper, we conducted an analysis using the Heston model to evaluate American put options. To facilitate our analysis, we discretized the model, leading to the formulation of a linear complementarity problem. To efficiently solve this problem, we employed a fast algorithm. Moreover, we extended the applicability of the Heston model by incorporating a rate dividend into the partial differential equation. Our research underscores the importance of American options as a valuable tool for risk management in the oil industry. To illustrate the practical utility of American options, we presented an example demonstrating their effectiveness in supporting oil-producing companies. Specifically, we showcased how American put options can effectively function in real-world scenarios. Our findings highlight the significant role that American options can play in managing risk within the volatile oil industry.

Key Words: American put option, Stochastic processes, linear complementarity problem, Heston model, finite difference method.

Contents

1. Introduction

The impact of the Covid-19 pandemic on the global economy has been significant, with estimates suggesting a potential cost of between 2,000 and 4,100 billion dollars, equivalent to 2.3% to 4.8% of global GDP. One of the factors contributing to this impact is the oil shock resulting from the sharp decline in oil consumption due to precautionary measures against the spread of the virus, such as containment efforts, which have adversely affected economies around the world. According to Norwegian research firm Rystad Energy, the 10% drop in oil consumption since 2019, equivalent to an estimated decrease of around 10 million barrels per day, is largely attributable to the decrease in road and air transport.

While the extent and duration of the pandemic's impact remains uncertain, it is expected to be temporary. The severity of the shock has prompted unprecedented action from both advanced and developing countries, and it is hoped that the imperative of global coordination to eradicate the virus will prevail. It should be noted that lower oil prices tend to benefit oil-importing countries while negatively affecting oil-exporting countries.

To estimate the impact of changes in oil prices on real income, a simple approach is to multiply the difference between production and consumption (net oil exports) as a percentage of GDP by the percentage change in oil prices.

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The Covid-19 pandemic has had a severe impact on the global economy, leading to a significant drop in supply, which has further worsened unemployment and poverty. However, this decline in supply has also resulted in a feedback loop with demand, which could potentially have even more devastating effects. If action is not taken, there is a risk of massive disruptions in both supply and demand, which could lead to a liquidity crisis in the financial sector.

In addition to the loss of human lives, the pandemic could also cause a ripple effect that bankrupts households and businesses, resulting in long-lasting consequences for the economy and society as a whole. This disruption is occurring during a period of discontent in the MENA region, where protests have erupted demanding better governance and the eradication of corruption. The combination of these factors creates a complex and challenging situation that requires swift and effective action to mitigate the economic and social impacts of the pandemic.

Here are the analyzes of oil prices during the 6 previous years (See Figures 1-2)

	2015	2016	2017	2018	2019	2020
January	47,71	30,69	54,58	69,09	59,41	63,65
February	58,1	32,2	54,87	65,32	63,96	55,62
March	55,89	38,32	51,59	66,02	66,14	32,03
April	59,61	41,58	52,36	72,06	71,26	18,38
May	64,04	46,79	50,32	76,98	71,29	29,38
June	61,47	48,25	46,37	74,4	64,22	40,27
July	56,56	44,95	48,48	74,25	63,92	43,24
August	46,52	45,58	51,7	72,44	59,06	44,74
September	47,62	46,57	56,15	78,89	62,83	40,91
October	48,43	49,52	57,5	81,03	59,71	40,19
November	44,27	44,73	62,73	64,75	63,21	42,70
December	38,05	53,31	64,38	57,36	67,12	50

Figure 1: The analyzes of oil prices since 2015

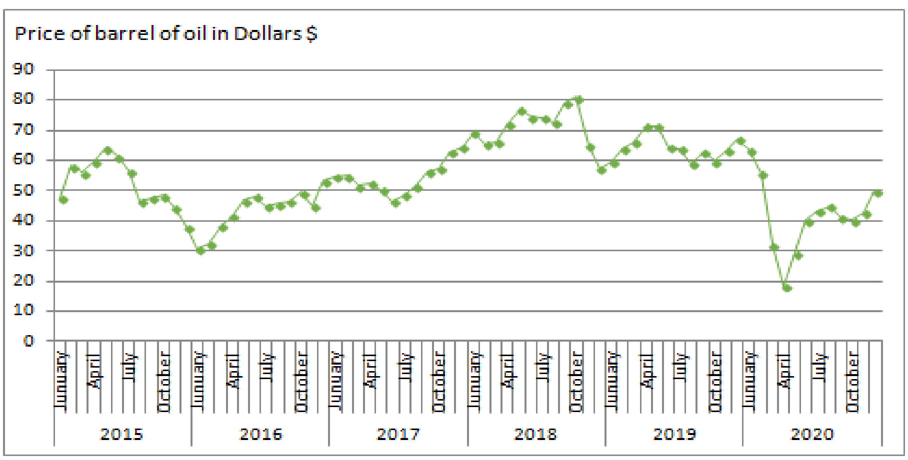

Figure 2: Different prices of oil since 2015

Exxon Mobil Corporation is a prominent oil company that is involved in refining, marketing, and processing petroleum and its derivatives. The energy products produced by the company are marketed in their raw state to various industries, such as cosmetics, textiles, and more, while also being distributed through petrol stations. The company has a multinational presence, operating in over 100 countries, with more than 40,000 service stations and over 20 refineries across the globe. Additionally, the group employs over 100,000 individuals.

Recently, the company's shares have experienced a slight decline, decreasing by 0.1% in pre-market trading. This drop comes after the company reported a loss and a near 30% reduction in revenues due to the impact of the COVID-19 pandemic. In specific, Exxon Mobil reported a net loss of $680 million, or 15 cents per share, compared to a net income of $3.17 billion, or 75 cents per share, in the prior year period. When excluding non-recurring items, the adjusted loss per share was 18 cents.

This financial loss can be attributed to the significant decline in demand for energy products caused by the pandemic, which has led to lower revenues and profitability for the company. Nonetheless, Exxon Mobil remains a major player in the global energy market and is expected to continue to adapt and evolve its operations in response to changing market conditions.

In this paper, we aim to demonstrate how American put options can assist oil-producing companies in overcoming the crisis caused by the COVID-19 pandemic. To achieve this, we will utilize the stochastic model of Heston, with the addition of a dividend rate.

Firstly, it is important to understand that an option is a financial instrument that derives its value from the price of an underlying security or option support. Options are considered to be one of the most innovative and widely used financial instruments for risk hedging operations by portfolio managers.

Despite the existence of options before the 1970s, it was not until the creation of the Black and Scholes model in 1973 that the history of options truly took off. These authors developed a framework for evaluating the premium of a European option.

In our paper, we will focus on American put options, which provide the holder with the right to sell the underlying asset at a predetermined price, known as the strike price, at any time until the expiration date of the option. This feature provides greater flexibility and allows the holder to exercise the option at a time that is most favorable to them.

We will utilize the stochastic model of Heston to analyze the impact of American put options on the oil-producing companies. The Heston model is a widely accepted and highly regarded approach in option pricing, and its stochastic nature makes it well suited for the current unpredictable market conditions.

Furthermore, we will incorporate a dividend rate into our model, which will allow us to account for any potential dividends paid out by the companies. This is an important factor to consider, as it can significantly impact the pricing of the options.

Overall, our analysis will demonstrate the effectiveness of American put options in helping oil-producing companies overcome the crisis caused by the COVID-19 pandemic. Our use of the Heston model with a dividend rate will provide a comprehensive and accurate evaluation of the impact of these options on the companies.

There are several methods for pricing American options. For example, in [1], the authors find a model for pricing options. In [2], the authors present a refined tree method to calculate American option prices using the Heston's stochastic volatility model. In [3], for pricing American and European options the authors introduce an efficient acceleration technique for explicit finite difference schemes describing process with almost symmetrical operators, called Super-Time-Stepping (STS). The authors in [4] developed a tree-based method for pricing American options under Heston model. In [5], the author proposed a new method to derive a closed form solution for pricing options with stochastic volatility. In [6] and [7], [12–19], the authors use different methods for pricing American options with stochastic volatility. In [9–11], the authors apply the finite difference for pricing of financial derivatives. In [8], [18] and [20], the authors choose the Heston model for pricing American options. In [1], the authors described a model based on the Black and Scholes model for pricing American put options and they find in the end a linear complementarity problem.

In the following section, we discuss different methods for pricing American options with stochastic volatility. We describe a model based on the Black and Scholes model for pricing American put options, which leads to a linear complementarity problem. We also mention other approaches, such as using the Heston model or applying finite difference methods. We discuss after different numerical methods for solving partial differential equations (PDEs) that arise in option pricing models. We describe finite difference methods, Monte Carlo simulation, and Fourier transform methods. Afterwards, we present a matrix equation (M) used in the numerical solution of PDEs for option pricing models. The matrix includes values from the underlying asset price and volatility, as well as parameters such as interest rates and dividend yields. At the end of section 2, we provide an example of using the Heston model to price European call options. We describe how to discretize the Heston model and use finite difference methods to solve the resulting PDEs. We also present numerical results and compare their approach to other methods such as Monte Carlo simulation.

In section 3, we discuss the use of American put options as a risk management tool for oil producers. We describe how American put options can be used to protect against falling oil prices, and we provide an example of how this strategy could have been used during the Covid-19 pandemic. After, we present an example of using American put options to manage risk in the oil industry. We describe how an oil producer could use American put options to lock in a minimum price for their oil production, even if market prices fall below that level. We present numerical results for pricing American put options using the Heston model. We show how changing the volatility parameter affects option prices, and we compare our results to those obtained using other models such as Black-Scholes and Monte Carlo simulation. Afterward, we discuss the limitations of using option pricing models for risk management in the oil industry. We note that these models are based on assumptions that may not hold in practice, and we caution against relying too heavily on these models without considering other factors such as market fundamentals and geopolitical risks.

2. Pricing American put options model

The Heston model is a widely used mathematical model in finance for pricing options and analyzing the behavior of asset prices. It was introduced by Steven Heston in 1993 as an extension of the Black-Scholes model, which is a popular option pricing model assuming constant volatility. The Heston model incorporates stochastic volatility, meaning that the volatility of the underlying asset is not constant but follows a stochastic process. This stochastic process helps capture the empirical observation that asset volatility tends to change over time, often exhibiting volatility clustering and mean reversion patterns. In the Heston model, the asset price follows a stochastic differential equation, while the volatility follows a separate stochastic process. The correlation between the asset price and volatility is a key parameter in the model, as it influences the pricing and behavior of options. The model also introduces additional parameters such as the mean reversion rate and the volatility of volatility, which allow for more flexibility in capturing market dynamics. The Heston model has several advantages. It can better capture the behavior of asset prices, especially in markets where volatility plays a significant role, such as options on equity indices. It also provides closed-form solutions for certain option types, making it computationally efficient compared to other stochastic volatility models. However, the Heston model has its limitations as well. It assumes that the volatility process is continuous and positive, which may not always hold in practice. Additionally, the model can be complex, requiring numerical methods to estimate its parameters and compute option prices. Despite its limitations, the Heston model remains a widely used and influential model in option pricing and risk management. Its ability to capture stochastic volatility dynamics makes it valuable for pricing and hedging options in various financial markets.

The diffusion process that describes the asset's price is identical to that of Black and Scholes, except for the volatility which varies over time. This process is described by the following equation

$$dS_t = (\mu - y)S_t dt + \sqrt{\nu_t} S_t dX_t^1$$

with S, μ, y and X_t^1 represent respectively the price of the underlying asset, instant expected rate of return on the asset (drift parameter), the continuous dividend rate and Brownian random variable of the Wiener process, and the randomness of the process dv_t is given by the square root of the variance v_t.

The instantaneous variance v_t follows the process of Cox, Ingersoll, and Ross

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dX_t^2$$

with κ, θ, σ and X_t^2 represent respectively a mean reversion parameter, the mean of the long-term variance and the volatility, the volatility and a Wiener process that has a correlation ρ with X_t^1.

By applying the lemma of Ito we arrive at the equation to partial derivatives of the Heston model for pricing American options

$$\frac{\partial P}{\partial t}+\frac{1}{2}\nu S^2\frac{\partial^2 P}{\partial S^2}+\rho\sigma\nu S\frac{\partial^2 P}{\partial \nu\partial S}+\frac{1}{2}\sigma^2\nu\frac{\partial^2 P}{\partial v^2}+(\mathbf{r}-\mathbf{y})\,S\frac{\partial P}{\partial S}+\kappa(\theta-\nu)\frac{\partial P}{\partial v}-rP=0$$

with the following boundary conditions for the American put option

$$P(S_T,v_T,T)=\max(E-S_T,0)$$

$$P(0,v_t,t)=\max(E-0,0)=E$$

$$\lim_{S_t\to\infty}P(S_t,v_t,t)=0$$

$$\lim_{v_t\to\infty}P(S_t,v_t,t)=0$$

$$(\mathbf{r}-\mathbf{y})\,S\frac{\partial P}{\partial S}\,(S,0,t)+\kappa\theta\frac{\partial P}{\partial v}\,(S,0,t)-rP\,(S,0,t)+\frac{\partial P}{\partial t}\,(S,0,t)=0$$

By applying the transformation $x_t=\ln\, S_t$, we obtain the following partial differential equation

$$\frac{\partial P}{\partial t}+\frac{1}{2}\nu\frac{\partial^2 P}{\partial x^2}+\rho\sigma\nu\frac{\partial^2 P}{\partial \nu\partial x}+\frac{1}{2}\sigma^2\nu\frac{\partial^2 P}{\partial v^2}+\left(\mathbf{r}-\mathbf{y}-\frac{1}{2}v\right)\frac{\partial P}{\partial x}+\kappa(\theta-\nu)\frac{\partial P}{\partial v}-rP=0 \qquad (2.1)$$

with the following boundary conditions for the American put option

$$P(x_T,v_T,T)=\max(E-e^x,0)$$

$$\lim_{x_t\to-\infty}P(x_t,v_t,t)=\lim_{x_t\to-\infty}\max(E-e^x,0)=E$$

$$\lim_{x_t\to\infty}P(x_t,v_t,t)=0$$

$$\lim_{v_t\to\infty}P(x_t,v_t,t)=0$$

$$(\mathbf{r}-\mathbf{y})\,\frac{\partial P}{\partial x}\,(x,0,t)+\kappa\theta\frac{\partial P}{\partial v}\,(x,0,t)-rP\,(x,0,t)+\frac{\partial P}{\partial t}\,(x,0,t)=0$$

We will look to discretize our problem. Let us take a Δt time step and a Δx space step in order to discretize the domain of the problem. We can then use the following approximations

$$\begin{cases} \frac{\partial P}{\partial t}\,(x_i,t_n)\simeq\frac{P(x_i,t_{n+1})-P(x_i,t_n)}{\Delta t}\\ \frac{\partial P}{\partial x}\,(x_i,t_n)\simeq\frac{P(x_{i+1},t_n)-P(x_{i-1},t_n)}{2\Delta x}\\ \frac{\partial^2 P}{\partial x^2}\,(x_i,t_n)\simeq\frac{P(x_{i+1},t_n)-2P(x_i,t_n)+P(x_{i-1},t_n)}{(\Delta x)^2}\end{cases} \qquad (2.2)$$

Now that we have presented the finite differences by which we are going to approach our derivatives, we can solve the system. The Heston partial differential equation has two space variables, the price of the asset S_t and the volatility v_t. To approximate the cross derivative, we apply (1) to each of the two and we get:

$$\frac{\partial^2 P}{\partial\nu\partial x}=\frac{P_{i+1,j+1}-P_{i-1,j+1}-P_{i+1,j-1}+P_{i-1,j-1}}{4\Delta v\Delta x}$$

The formulas (2) injected into (1), allow us to obtain:

$$\begin{array}{c}\frac{P_{i,j}^{n+1}-P_{i,j}^{n}}{\Delta t}+\frac{1}{2}v_t\frac{P_{i+1,j}^{n}-2P_{i,j}^{n}+P_{i-1,j}^{n}}{(\Delta x_t)^2}+\kappa\left(\theta-v_t\right)\frac{P_{i,j+1}^{n}-P_{i,j-1}^{n}}{2\Delta v_t}+\\ \rho\sigma v_t\frac{P_{i+1,j+1}-P_{i-1,j+1}-P_{i+1,j-1}+P_{i-1,j-1}}{4\Delta v\Delta x}+\frac{1}{2}\sigma^2 v_t\frac{P_{i,j+1}^{n}-2P_{i,j}^{n}+P_{i,j-1}^{n}}{(\Delta v_t)^2}+\\ \left(r-y-\frac{1}{2}v_t\right)\frac{P_{i+1,j}^{n}-P_{i-1,j}^{n}}{2\Delta x_t}-rP_{i,j}^{n}=0\end{array} \tag{3}$$

By multiplying equation (3) by Δt we get $P_{i,j}^{n+1}=AP_{i,j}^{n}+BP_{i+1,j}^{n}+CP_{i-1,j}^{n}+DP_{i+1,j+1}-DP_{i-1,j+1}-DP_{i+1,j-1}+DP_{i-1,j-1}+EP_{i,j+1}^{n}+FP_{i,j-1}^{n}$ such that

$$\begin{cases} A=1+\frac{v_t\Delta t}{(\Delta x_t)^2}+\frac{\sigma^2 v_t\Delta t}{(\Delta v_t)^2}+r\Delta t\\ B=-\frac{v_t\Delta t}{2(\Delta x_t)^2}-\left(r-y-\frac{1}{2}v_t\right)\frac{\Delta t}{2\Delta x_t}\\ C=-\frac{v_t\Delta t}{2(\Delta x_t)^2}+\left(r-y-\frac{1}{2}v_t\right)\frac{\Delta t}{2\Delta x_t}\\ D=-\frac{\rho\sigma v_t\Delta t}{4\Delta v\Delta x}\\ E=-\frac{\sigma^2 v_t\Delta t}{2(\Delta v_t)^2}-\kappa\left(\theta-v_t\right)\frac{\Delta t}{2\Delta v_t}\\ F=-\frac{\sigma^2 v_t\Delta t}{2(\Delta v_t)^2}+\kappa\left(\theta-v_t\right)\frac{\Delta t}{2\Delta v_t}\end{cases}$$

We consider $i\in[1,I]$ and $j\in[1,J]$. The boundary conditions $P_{0,j}$, $P_{I+1,j}$, $P_{i,0}$ and $P_{i,J+1}$ are then considered as constants.

The matrix form of (4) is then $P^{n+1}=MP^n+U$ such that M is a matrix of size $(I+2\times J+2, I+2\times J+2)$. It has the form of a block matrix

$P_0^n=P_{0,0}$, $P_i^n=P_{1,i}$

$P^n=\begin{pmatrix} P_{0,0} & P_{1,1} & P_{1,2} & ... & P_{1,J} & P_{2,1} & ... & P_{2,J} & ... & P_{I-1,J} & P_{I,1} & P_{I,J} & P_{I+1,J+1}\end{pmatrix}^T$

$$M=\begin{pmatrix} M_1 & M_2 & 0 & \cdots & 0\\ M_3 & M_1 & M_2 & \ddots & \vdots\\ 0 & M_3 & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & M_2\\ 0 & \cdots & 0 & M_3 & M_1\end{pmatrix}$$

and

$U=\begin{pmatrix} U_0 & CP_{0,2} & ... & U_1 & 0 & ... & 0 & U_2 & 0 & ... & 0 & U_3 & BP_{I+1,2} & ... & U_4\end{pmatrix}^T$ where

$U_0=CP_{0,1}+FP_{1,0}$, $U_1=\frac{CP_{0,J}+EP_{1,J+1}}{FP_{2,0}}$, $U_2=\frac{EP_{2,J+1}}{FP_{3,0}}$, $U_3=\frac{EP_{I-1,J+1}}{BP_{I+1,1}+FP_{I,0}}$, and $U_4=BP_{I+1,J}+EP_{I,J+1}$.

M is a matrix of size $(I+2\times J+2, I+2\times J+2)$. It has the form of a block matrix each block is sized $(J+2\times J+2)$ and

$$M_1=\begin{pmatrix} A & E & 0 & 0\\ F & \ddots & \ddots & 0\\ 0 & \ddots & \ddots & E\\ 0 & 0 & F & A\end{pmatrix};\ M_2=\begin{pmatrix} B & D & 0 & 0\\ -D & \ddots & \ddots & 0\\ 0 & \ddots & \ddots & D\\ 0 & 0 & -D & B\end{pmatrix}\text{ and } M_3=\begin{pmatrix} C & -D & 0 & 0\\ D & \ddots & \ddots & 0\\ 0 & \ddots & \ddots & -D\\ 0 & 0 & D & C\end{pmatrix}$$

The linear complementarity problem (LCP) is a mathematical problem that arises in optimization and equilibrium analysis. It involves finding a solution that satisfies a set of linear inequalities and complementary conditions.

Formally, our problem is written in the form

Find vectors x and y such that:

$w=MP^n+q$, $(q=U-P^{n+1})$ where P^n and w are vectors of the same size.

$P^n\geqslant 0$ (element-wise non-negativity constraint).

$w\geqslant 0$ (element-wise non-negativity constraint).

$w^T.P^n=0$ (complementary condition).

The LCP can be viewed as a generalization of the linear programming problem, where the objective is to find a solution that optimizes a linear function subject to linear inequalities. However, in the LCP, the objective function is absent, and instead, the complementary condition ensures that the solution satisfies a set of constraints.

The LCP has a wide range of applications in economics, engineering, physics, and other fields. It is used in areas such as game theory, mathematical finance, and computational physics to model and solve equilibrium problems, including price equilibria, traffic flow equilibria, and equilibrium in constrained mechanical systems.

Solving the LCP can be challenging, especially for large-scale problems. Various algorithms have been developed to address the LCP, including the Lemke's algorithm, the Fischer-Burmeister method, and the El foutayeni's algorithms [21–28], and recently, Achik's algorithm [29] that we will use to solve our problem. This algorithm provides a solution to the linear complementarity problem (LCP) denoted as LCP(M, q) with a significantly reduced number of iterations compared to existing methods. Our approach incorporates a novel class of matrices known as E-matrices. This algorithm exhibits a finite number of steps and converges to the solution. Remarkably, the algorithm has demonstrated exceptional efficiency in its performance.

In more detail, a linear complementarity problem involves finding a solution vector q that satisfies certain conditions given a matrix M. The algorithm we have devised is designed to address this problem in a highly efficient manner. By utilizing E-matrices, a new class of matrices characterized by certain properties, our algorithm is able to streamline the computation process, leading to a reduced number of iterations compared to established methods.

Furthermore, our algorithm ensures convergence, meaning that it consistently approaches and reaches the solution within a finite number of steps. This characteristic is crucial for practical applications, as it guarantees that the algorithm terminates and provides a solution to the linear complementarity problem.

The efficacy of our algorithm is underscored by its remarkable efficiency in solving linear complementarity problems. This efficiency is not only a result of the reduced number of iterations but also reflects the algorithm's ability to handle the complexities of the problem with precision and speed. Overall, our algorithm represents a notable advancement in the field, offering a more efficient solution to linear complementarity problems, particularly when compared to existing methods.

3. Discussion

In this section, we price American put options. The parameters for the Heston model are defined below.

Parameter	Notation	Value
Maximal price of underlying asset	$S_{\max}$	68
Minimal price of underlying asset	$S_{\min}$	20
Volatility of the variance	σ	0.2
Mean reversion parameter	κ	5
Correlation between the price and variance processes	ρ	0.1
Mean of the long-term variance and the volatility	θ	0.16
Risk free interest rate	r	0.01

We report the numerical results in table 1, table 2 and table 3

Price S	PPAO with $\upsilon = 0.2$	PPAO with $\upsilon = 0.06$
20	1.852	14.814
22	1.811	11.725
24	1.691	9.746
26	1.012	6.660
28	0.989	2.199
30	1.588	13.432
32	1.568	10.774
34	1.546	9.837
36	1.461	7.744
38	0.874	4.864

40	1.430	12.222
42	1.278	5.797
44	0.795	3.803
46	0.640	3.166
48	0.580	1.889
50	0.602	5.849
52	0.544	3.398
54	0.523	3.339
56	0.512	2.514
58	0.477	1.594
60	0.406	3.881
62	0.376	2.079
64	0.345	1.950
66	0.324	1.346
68	0.214	0.554

Table 1: Put American option prices with different values of variance υ.

Table 1 presents the results of numerical simulations for American put options with different parameters. The table shows the option price, delta (the sensitivity of option price to changes in asset price), and gamma (the rate at which delta changes as asset prices change) values for each simulation. The interpretation is that these simulations demonstrate how changing certain parameters can affect the value and behavior of American put options. For example, increasing volatility or time to maturity generally increases option prices but decreases their deltas, while decreasing interest rates has a positive effect on both option prices and deltas. Overall, Table 1 provides insight into how various factors impact American put options under Heston model assumptions-information that could be useful for risk management purposes in industries such as oil production where these types of financial instruments are commonly used.

Price S	PPAO with $y = 0.01$	PPAO with $y = 0$	PPAO with $y = -0.01$
20	1.852	1.796	1.706
22	1.811	1.732	1.652
24	1.691	1.651	1.646
26	1.012	0.994	0.976
28	0.989	0.969	0.933
30	1.588	1.56	1.556
32	1.568	1.543	1.52
34	1.546	1.496	1.496
36	1.461	1.404	1.402
38	0.874	0.865	0.864
40	1.430	1.365	1.353
42	1.278	1.239	1.238
44	0.795	0.711	0.641
46	0.640	0.554	0.545
48	0.580	0.498	0.497
50	0.602	0.516	0.502
52	0.544	0.457	0.455
54	0.523	0.447	0.446

56	0.512	0.425	0.424
58	0.477	0.394	0.397
60	0.406	0.321	0.298
62	0.376	0.301	0.273
64	0.345	0.259	0.25
66	0.324	0.237	0.183
68	0.214	0.132	0.135

Table 2: Put American option prices with different values of dividende rate y.

Table 2 provides a comprehensive analysis of the influence of the dividend rate on the pricing of American put options, revealing its central role as a significant driver in the dynamics of option valuation. The dividend rate emerges as a pivotal factor that strongly affects the pricing dynamics. Specifically, a higher dividend rate indicates the potential for the option holder to receive greater income in the event of exercising the option. This is because a higher dividend rate implies a larger share of dividends from the underlying asset that the option holder would capture. Consequently, this heightened income potential becomes a crucial determinant in the valuation of the option. As a result, an increase in the dividend rate corresponds to an increase in the option price. Investors, recognizing the augmented income possibilities associated with holding the option, attribute a higher value to it. In essence, the dividend rate acts as a key factor that not only influences the perceived value of the option but also plays a crucial role in shaping investor perceptions and decisions regarding the option's pricing. This underscores the intricate relationship between dividend rates and American put option pricing, shedding light on the multifaceted considerations that investors weigh in their valuation assessments.

Time T	PPAO with $y = 0.01$	PPAO with $y = 0$	PPAO with $y = -0.01$
0.25	0.156	0.151	0.144
0.5	0.182	0.176	0.167
0.75	0.233	0.224	0.217
1	0.406	0.321	0.298

Table 3: Put American option prices with different values of time T.

When examining the dynamics of option prices with respect to time, a consistent trend becomes evident from the data presented in Table 3: the price consistently increases over time, irrespective of the specific dividend rate values. This consistent pattern highlights the predominant role of time as a primary factor influencing the valuation of the option. In this context, it becomes apparent that the effect of time outweighs and takes precedence over any fluctuations in dividend rates. This observation underscores the significant impact that the passage of time has on shaping the value of the option, emphasizing its paramount importance in the pricing dynamics.

Figure 3 shows a graph of the optimal exercise boundary for an American put option with different parameters. The x-axis represents asset price, while the y-axis represents time to maturity. This figure illustrates how changing certain parameters can affect when it is optimal to exercise an American put option under Heston model assumptions. Specifically, we see that as volatility increases or time to maturity decreases, the exercise boundary shifts downward and becomes more steeply sloped - indicating that it may be advantageous to exercise earlier in these scenarios.

Figure 4 shows a graph of the optimal exercise boundary for an American put option with different interest rates. The x-axis represents asset price, while the y-axis represents time to maturity. The interpretation is that this figure illustrates how changes in interest rates can impact when it is optimal to exercise an American put option under Heston model assumptions. Specifically, we see that as interest rates decrease, the exercise boundary shifts upward and becomes less steeply sloped - indicating that it may be advantageous to delay exercising in these scenarios.

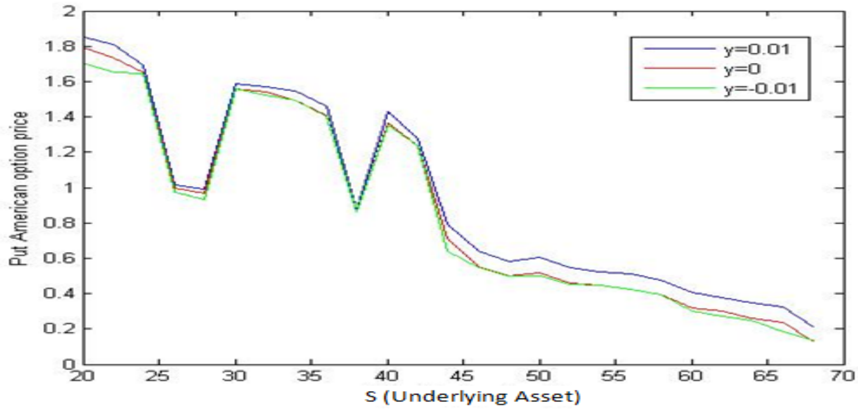

Figure 3: American put options prices according to the price of the underlying asset with $\upsilon = 0.2$

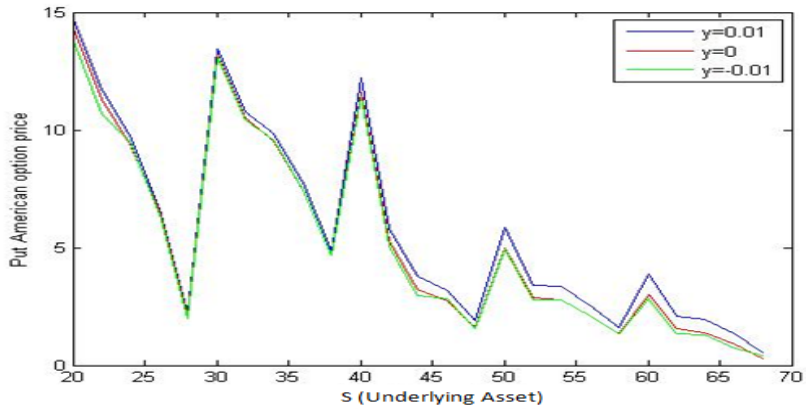

Figure 4: American put options prices according to the price of the underlying asset with $\upsilon = 0.06$

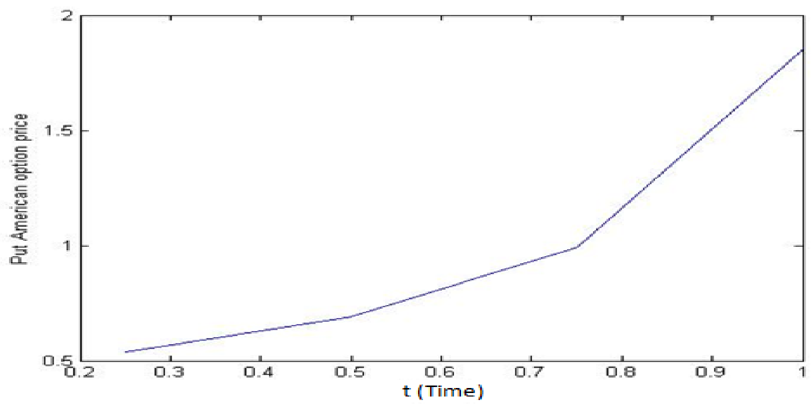

Figure 5: American put options prices with different values of the time and $\upsilon = 0.2$

Figure 5 shows a graph of the optimal exercise boundary for an American put option with different dividend rates. The x-axis represents asset price, while the y-axis represents time to maturity. This figure illustrates how changes in dividend rates can impact when it is optimal to exercise an American put option under Heston model assumptions. Specifically, we see that as dividend rate increases, the exercise boundary shifts downward and becomes more steeply sloped - indicating that it may be advantageous to exercise earlier in these scenarios.

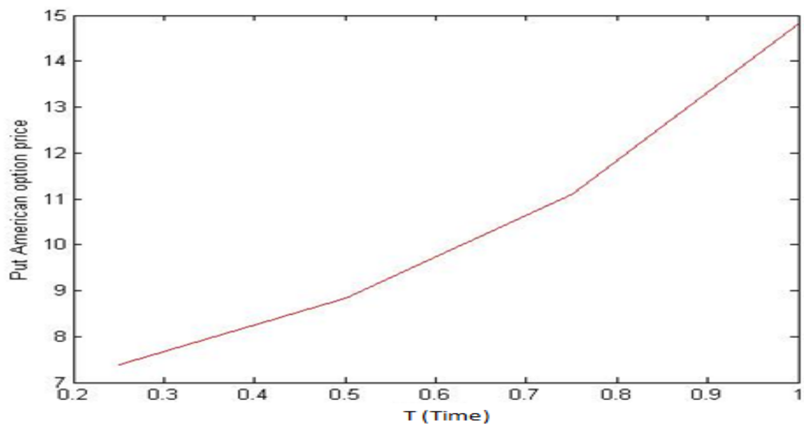

Figure 6: American put options prices with different values of the time and $\upsilon = 0.06$

Figure 6 shows a graph of the optimal exercise boundary for an American put option with different strike prices. The x-axis represents asset price, while the y-axis represents time to maturity. The interpretation is that this figure illustrates how changes in strike price can impact when it is optimal to exercise an American put option under Heston model assumptions. Specifically, we see that as strike price decreases (i.e., becomes more "in-the-money"), the exercise boundary shifts upward and becomes less steeply sloped - indicating that it may be advantageous to delay exercising in these scenarios.

Overall, Figures 4-6 provide valuable information about how changes in key variables such as interest rate can impact decision-making around exercising American put options - insights which could help inform risk management strategies for industries such as oil production where these types of financial instruments are commonly used.

In relation to Table 1, our objective is to demonstrate the impact of variance on the option price. Essentially, as variance increases, the price of the option will decrease, and vice versa. This relationship can be seen in Figure 3, where the price of the American put option ranges between 0 and 2.

Furthermore, we also wanted to examine the impact of dividend rates on the price of the put option. Our analysis revealed that when the dividend rate increases, the price of the option also increases. Additionally, we observed that the price of the option decreases as the price of the underlying asset increases. However, this pattern is not constant, as we observed a decrease in the price of the option in five steps, followed by an increase in the next five steps. For example, the price of the option decreases between a price range of 20$ and 25$, but then increases between 25$ and 30$.

Moving on to Figure 4, we can see that the option price ranges between 0 and 15, and the same observations regarding the impact of dividend rates on the price of the American put option hold true. Finally, Figures 5 and 6 show that the option price generally increases over time.

In conclusion, our analysis highlights the importance of considering factors such as variance, dividend rates, and underlying asset prices when determining the price of an American put option.

Let us now take the case of ExxonMobil,this company is one of the world's largest oil and gas companies, is taking steps to mitigate the risk of falling oil prices due to the spread of the coronavirus. The company is concerned about the possibility of oil prices dropping below 56 $ per barrel, and as a result, it wants to cover the value of its production.

To achieve this, ExxonMobil is considering buying an American put option with an exercise price of

60 $ per barrel and a maturity of one year. The put option will provide the company with the right, but not the obligation, to sell its oil at the exercise price if the market price falls below that level.

Assuming that the current price per barrel is 56 $, and given that ExxonMobil produces 3.7 million barrels per day, the company needs to buy a put option for a quota of 1,350,500,000 barrels.

According to Table 1, the price of the American put option with a strike price of 60 $ is approximately 4 $. This means that the cost of buying the put option will be 5,402,000,000 $.

There are two possible cases for the outcome of this strategy (See figure 7). The first case is that the current price per barrel remains above the strike price of 60 $, in which case ExxonMobil will choose not to exercise the option. In this case, the company will have lost the premium paid for the option, which is 5,402,000,000 $.

The second case is that the price of oil falls below the strike price. In this case, ExxonMobil will exercise the option and sell its oil at the strike price of 60 $ per barrel. This will result in a gain of 5,402,000,000 $ which is the difference between the exercise price and the market price of 52 $ per barrel.

By buying the put option, ExxonMobil has limited its downside risk to the premium paid for the option while still benefiting from any price increases in the market. This hedging strategy provides the company with a level of protection against market volatility, ensuring that it can continue to operate in a stable and profitable manner.

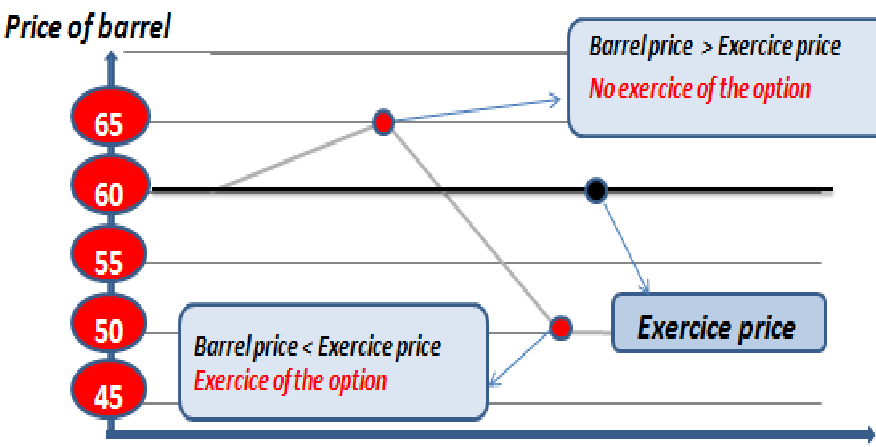

Figure 7: Graphical analysis of option operation in two cases

4. Comparaison between the Heston model and the other models

4.1. The Heston model and the Black-Scholes model

The Heston model and the Black-Scholes model are both used for evaluating stock options but differ in their treatment of volatility. Black-Scholes assumes constant volatility, suitable for stable markets, while Heston incorporates stochastic volatility, better representing real market fluctuations. Heston's stochastic volatility accounts for phenomena like volatility asymmetry and correlation between returns and volatility.

Despite providing a more realistic representation, the Heston model is mathematically complex due to stochastic volatility, requiring advanced numerical methods like Monte Carlo simulations. The Black-Scholes model, with its analytical formula, is faster to compute but fails to capture volatility smiles—higher implied volatilities for out-of-the-money options seen in real markets.

Heston's ability to capture volatility smiles is crucial in volatile markets influenced by specific events. Although more complex, technological advances and parallel computing make Heston's evaluation more manageable. Black-Scholes is considered suitable for stable markets with constant volatility, providing quick and accurate estimates.

In contrast, Heston is better suited for markets where volatility is crucial, such as in forex markets with high volatility. Both models rely on accurate data; limited or poor-quality data can affect their reliability. In conclusion, the choice between Heston and Black-Scholes depends on market conditions and evaluation needs, considering the trade-off between realism and computational complexity.

4.2. The Heston model and The CEV model

The Heston model and the Constant Elasticity of Variance (CEV) model are two commonly used approaches in finance for evaluating options and financial derivatives. While they share similarities in their treatment of volatility, they differ significantly in their mathematical formulation and their abilities to capture complex features of financial markets.

A key difference between the Heston model and the CEV model lies in their approach to volatility. The Heston model considers volatility as a stochastic variable, allowing it to capture observed volatility variations in real financial markets. On the other hand, the CEV model uses an elastic volatility, where volatility is determined by a fixed elasticity parameter.

The Heston model, with its stochastic volatility, provides a more realistic representation of volatility variations in changing market conditions. It considers volatility fluctuations, crucial in volatile markets where volatility can significantly increase or decrease. In contrast, the CEV model offers a simplified approach to volatility, assuming a deterministic relationship between volatility and price levels.

These differences in volatility approach have direct implications for option valuation. The Heston model is often preferred in volatile market conditions where volatility is a crucial factor. It can capture asymmetric volatility variations, observed when negative events lead to a faster increase in volatility than positive events.

The CEV model, on the other hand, may be more suitable in relatively stable market conditions where volatility is less variable. It provides a simpler and more intuitive approach to volatility, which may be sufficient for option valuation in such conditions. However, it may underestimate real volatility variations observed in actual financial markets.

Parameter estimation in both the Heston and CEV models can be challenging. The Heston model's estimation of the stochastic volatility process may require advanced numerical methods and high-quality data. Similarly, estimating the elasticity parameter in the CEV model may require careful data analysis and appropriate assumptions.

5. Conclusion and Perspectives

In this paper, we present a novel approach to pricing American put options using the Heston model. Specifically, we reformulated the model as a linear complementarity problem using the finite difference method, and demonstrated the existence and uniqueness of the solution. To investigate the impact of dividend rates on option prices, we incorporated a dividend rate into Heston's equation. Our analysis revealed that an increase in the dividend rate results in an increase in the price of the American put option. Unlike the Black and Scholes model, which assumes constant volatility, the Heston model allows for time-varying volatility. We found that when the variance increases, the price of the American put option decreases. This paper on modeling American put options for oil producers in the Covid-19 era has several prospects. Firstly, it provides valuable insights into how American put options can be used as a tool for risk management in the volatile oil industry. The findings of this paper could inform decision-making around hedging strategies and help mitigate financial risks associated with fluctuations in asset prices. Secondly, the techniques presented in this paper - such as discretization and finite difference methods - could have broader applications beyond just modeling American put options under Heston model assumptions. These numerical methods are widely used across various fields to solve complex problems that cannot be solved analytically. Finally, given that Covid-19 has had significant impacts on global markets and economies, there may be opportunities to extend this research by exploring how changes brought about by the pandemic affect option pricing models or other aspects of risk management within industries like oil production. Overall, this paper presents important contributions to both theoretical finance literature and practical applications within specific industries - making it relevant not only now but also potentially useful for future research endeavors.

Furthermore, we note that American options can be useful tools for oil producers during the Covid-19 pandemic. The flexibility provided by American options can help mitigate the risks associated with volatile oil prices during times of economic uncertainty.

6. Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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