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# Translation surfaces generated by the spherical indicatrices of Frenet frame of regular curves of 3-dimensional Euclidean space

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ABSTRACT: In this paper, we are interested in translation surfaces generated by the principal normals of two regular curves provided with their alternative frames, and consequently, we generalize this study to confirm the results obtained. Moreover, we give the position vector with illustrations for these surfaces, in some special cases, where the two regular curves are respectively general helices, slant helices and slant-slant helices.

Key Words: Translation surfaces, Spherical indicatrices, Gaussian curvature, Mean curvature, Frenet frame, Alternative frame, Euclidean 3-space.

**Contents**



## 1. Introduction

The best known translation surface in 3-dimensional Euclidien space is generated by two planar curves lying in orthogonal planes and parametrized by

$$\psi(u,v) = (u,0,f(u)) + (0,v,g(v)).$$

A generalization of translation surface is the surface obtained by translating a curve $\alpha(u)$ parallel to itself along another curve $\beta(v)$. Therfore the parametric representation of such surface is given by

$$\psi(u,v) = \alpha(u) + \beta(v).$$

The theory of translation surfaces is always one of interesting topics in Euclidean space. Translation surfaces have been investigated by some diferential geometers. Verstraelen et al. have investigated minimal translation surfaces of plane type in n-dimensional Euclidean spaces [14]. Liu obtained some characterizations of translation surfaces with constant mean curvature or constant Gauss curvature in Euclidean 3-space $\mathbb{E}^3$ and Minkowski 3-space $\mathbb{E}^3_1$ [10]. In [2] Ali et al. gives some results on curvatures of some special points of the translation surfaces in $\mathbb{E}^3$, in the same regard, Muntenau and Nistor has studied the second fundamental form of the translation surfaces in Euclidean 3-space and they obtained some characterizations by using the second Gaussian curvature $K_{II}$ of the translation surfaces [11]. Recently, in [1] Neriman Acar et al. studied translation surfaces generated by the spherical indicatrices of space curves in $\mathbb{E}^3$, and obtained some characterizations based on the fact that these surfaces are developable

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or minimal. In [6] Cetin et al. have investigated geometric properties of surfaces that are parallel to translation surfaces in Euclidean 3-space. In [7,8] Cetin et al. studied translation surfaces in Euclidean 3-space generated by two space curves, and using non-planar space curves he expressed some properties of translation surfaces according to Frenet frames in Minkowski 3-space.

In [1] the authors have studied the translation surface generated by the tangent indicatrices of two regular curves of $\mathbb{E}^3$, provided with their respective Frenet frames. In this paper, we investigated the translation surfaces generated by the normals indicatrices of two regular curves provided with thier alternatives frames. We determine some properties concerning the developability and minimality of these surfaces. Subsequently, we generalize these results to the translation surfaces generated by the spherical k-indicatrices. Finally, we give the position vector with illustrations for these surfaces, in the some sepecial cases, where the two regular curves are respectively general helices, slant helices and slant-slant helices.

## 2. Preliminaries

Let $\mathbb{E}^3$ be a 3-dimensional euclidean space provided with the metric given by $\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2$, where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbb{E}^3$.
Let $I$ be an interval of $\mathbb{R}$ and $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ $(s \to \alpha(s))$ a regular curve of $\mathbb{E}^3$, parametrized by arc length, otherwise $\langle \alpha'(s), \alpha'(s) \rangle = 1$, for all $s \in I$.
The Serret-Ferent frame along the curve $\alpha$, is the moving frame, direct orthonormal, noted $(T(s), N(s), B(s))$ where

$$T(s) = \alpha'(s), \qquad N(s) = \frac{T'(s)}{\| T'(s) \|} \quad \text{and} \quad B(s) = T(s) \wedge N(s).$$

The derivative formulas of the Serret-Ferent frame are given as follows:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},$$

where $\kappa(s)$ and $\tau(s)$ are respectively the curvature and the torsion of the curve at the point $\alpha(s)$.
A curve $\alpha$ is called general helix if its unit tangent vector field $T(s)$ makes a constant angle with a fixed straight line. $\alpha$ is a general helix if and only if $\sigma_0 = \frac{\tau}{\kappa}$ is a constant function [13].
Denote by $(N(s), C(s), W(s))$ the alternative moving frame along the curve $\alpha$, where

$$N = \frac{T'}{\| T' \|}, \quad C = \frac{N'}{\| N' \|} \quad \text{and} \quad W = N \wedge C.$$

The derivative formulas of the alternative moving frame, are given as follows:

$$\begin{bmatrix} N'(s) \\ C'(s) \\ W'(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix}, \tag{2.1}$$

where

$$f = \sqrt{\kappa^2 + \tau^2} \qquad \text{and} \qquad g = \frac{\kappa^2}{(\kappa^2 + \tau^2)} \left(\frac{\tau}{\kappa}\right)'. \tag{2.2}$$

A curve $\alpha$ is called slant helix if its unit principal normal vector field $N(s)$ makes a cosntant angle with a fixed straight line. $\alpha$ is a slant helix if and only if $\sigma_1 = \frac{g}{f}$ is a constant function [12].
We denote by $C_0 = \alpha(s)$ therefore,

$$C_k(s) = \frac{C'_{k-1}(s)}{\| C'_{k-1}(s) \|}, \quad \text{and} \quad W_{k+1}(s) = C_k(s) \wedge C_{k+1}(s), \quad k \in \{1, 2, ...\},$$

accordingly, $(C_k, C_{k+1}, W_{k+1})$ is the Frenet frame of $s \to C_{k-1}(s)$. Then the derivative formulas of Frenet frame are given by:

$$\begin{pmatrix} C'_k(s) \\ C'_{k+1}(s) \\ W'_{k+1}(s) \end{pmatrix} = \begin{pmatrix} 0 & f_{k-1}(s) & 0 \\ -f_{k-1}(s) & 0 & g_{k-1}(s) \\ 0 & -g_{k-1}(s) & 0 \end{pmatrix} \begin{pmatrix} C_k(s) \\ C_{k+1}(s) \\ W_{k+1}(s) \end{pmatrix}, \tag{2.3}$$

where $f_{k-1}$ and $g_{k-1}$ are the Frenet invariants of $s \to C_{k-1}(s)$, such as $f_0 = \kappa$ and $g_0 = \tau$.
A curve $\alpha$ is called a k-slant helix if the unit vector $C_{k+1} = \frac{C'_k}{\|C'_k(s)\|}$ makes a constant angle with a fixed direction. $\alpha$ is a k-slant helix if and only if

$$\sigma_k = \frac{g_k}{f_k} = \frac{\sigma'_{k-1}}{f_{k-1}\left(1+\sigma_{k-1}^2\right)^{\frac{3}{2}}} \tag{2.4}$$

is a constant function [5].
Let $M : X = X(u,v) \subset \mathbb{E}^3$ be a regular surface. Then the unit normal vector field of the surface $M$ is identified by:

$$N(u,v) = \frac{X_u \wedge X_v}{\| X_u \wedge X_v \|},$$

where $X_u = \frac{\partial X(u,v)}{\partial u}$, $X_v = \frac{\partial X(u,v)}{\partial v}$.
The compenents of the first fundamental form and the second fundamental form of a regular surface $M$ are given by

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle,$$
$$l = \langle X_{uu}, N \rangle, \quad m = \langle X_{uv}, N \rangle, \quad n = \langle X_{vv}, N \rangle.$$

The Gaussian curvature K and the mean curvature H of the surface M are expressed as follows:

$$K = \frac{ln - m^2}{EG - F^2}, \tag{2.5}$$

$$H = \frac{En + Gl - 2Fm}{2(EG - F^2)}. \tag{2.6}$$

**Definition 2.1** *A regular surface in $\mathbb{E}^3$ is called developable if $K = 0$ and called minimal if $H = 0$.*

**Definition 2.2** *A constant angle surface in $\mathbb{E}^3$ is a surface whose unit normal vector makes a constant angle with an assigned direction.*

## 3. Translation surfaces generated by principal normal indicatrices of regular curves of $\mathbb{E}^3$

Let us denote by $u \to \alpha(u)$ and $v \to \beta(v)$ two non degenerate curves of class $C^3$ of $\mathbb{E}^3$. Denote by $(N_\alpha, C_\alpha, W_\alpha, f_\alpha, g_\alpha)$ and $(N_\beta, C_\beta, W_\beta, f_\beta, g_\beta)$ the alternative frames of the curves $\alpha$ and $\beta$, respectively. The translation surface generated by principal normal indicatrices of the curves $\alpha$ and $\beta$ is defined by:

$$M_N : X(u,v) = N_\alpha(u) + N_\beta(v). \tag{3.1}$$

By calculating the partial derivatives with respect to $u$ and $v$ of the translation surface (3.1) and using the derivative formulas (2.1), we get:

$$X_u = f_\alpha C_\alpha, \quad X_v = f_\beta C_\beta.$$

Then, we obtain the components of the first fundamental form of the surface $M_N$:

$$E = f_\alpha^2, \tag{3.2}$$

$$F = f_\alpha f_\beta \cos[\phi_N(u,v)], \tag{3.3}$$

$$G = f_\beta^2, \tag{3.4}$$

where $\phi_N = \phi_N(u,v)$ is the angle function between $C_\alpha$ and $C_\beta$. The unit normal vector of the translation surface $M_N$ is given by:

$$N(u,v) = \frac{C_\alpha \wedge C_\beta}{\sin[\phi_N(u,v)]}, \tag{3.5}$$

The vector $C_\alpha$ of the curve $\alpha$ can be expressed in the frame $\{N_\beta, C_\beta, W_\beta\}$ as follows:

$$C_\alpha = \mu_1 N_\beta + \mu_2 C_\beta + \mu_3 W_\beta, \tag{3.6}$$

where

$$\begin{aligned} \mu_1 = \langle C_\alpha, N_\beta\rangle &= \sin[\phi_N(u,v)]\cos[\gamma_N(u,v)], \\ \mu_2 = \langle C_\alpha, C_\beta\rangle &= \cos[\phi_N(u,v)], \\ \mu_3 = \langle C_\alpha, W_\beta\rangle &= \sin[\phi_N(u,v)]\sin[\gamma_N(u,v)]. \end{aligned} \tag{3.7}$$

Similarly, the vector $C_\beta$ of the curve $\beta$ can be expressed as a linear combination of the vectors $N_\alpha$, $C_\alpha$ and $W_\alpha$ as follows:

$$C_\beta = \lambda_1 N_\alpha + \lambda_2 C_\alpha + \lambda_3 W_\alpha, \tag{3.8}$$

where

$$\begin{aligned} \lambda_1 = \langle C_\beta, N_\alpha\rangle &= \sin[\phi_N(u,v)]\cos[\theta_N(u,v)], \\ \lambda_2 = \langle C_\beta, C_\alpha\rangle &= \cos[\phi_N(u,v)], \\ \lambda_3 = \langle C_\beta, W_\alpha\rangle &= \sin[\phi_N(u,v)]\sin[\theta_N(u,v)]. \end{aligned} \tag{3.9}$$

Therefore, we can express the unit normal vector $N$ of the surface $M_N$ in each of the two frames. By using (3.5) and (3.8), it is determined by

$$N_1(u,v) = \sin[\theta_N(u,v)]N_\alpha - \cos[\theta_N(u,v)]W_\alpha. \tag{3.10}$$

Similarly, using (3.5) and (3.6), we obtain

$$N_2(u,v) = -\sin[\gamma_N(u,v)]N_\beta + \cos[\gamma_N(u,v)]W_\beta. \tag{3.11}$$

The components of the second fundamental form of the $M_N$ surface are given by:

$$l = -f_\alpha^2 \left[\cos[\theta_N(u,v)]\frac{g_\alpha}{f_\alpha} + \sin[\theta_N(u,v)]\right], \tag{3.12}$$

$$m = 0, \tag{3.13}$$

$$n = f_\beta^2 \left[\cos[\gamma_N(u,v)]\frac{g_\beta}{f_\beta} + \sin[\gamma_N(u,v)]\right]. \tag{3.14}$$

**Proposition 3.1** *The Gaussian curvature $K$ and the mean curvature $H$ of the translation surface $M_N$ are respectively:*

$$K = -\frac{\left[\cos[\theta_N(u,v)]\frac{g_\alpha}{f_\alpha} + \sin[\theta_N(u,v)]\right]\left[\cos[\gamma_N(u,v)]\frac{g_\beta}{f_\beta} + \sin[\gamma_N(u,v)]\right]}{\sin^2[\phi_N(u,v)]}, \tag{3.15}$$

$$H = \frac{-\left[\cos[\theta_N(u,v)]\frac{g_\alpha}{f_\alpha} + \sin[\theta_N(u,v)]\right] + \left[\cos[\gamma_N(u,v)]\frac{g_\beta}{f_\beta} + \sin[\gamma_N(u,v)]\right]}{2\sin^2[\phi_N(u,v)]}. \tag{3.16}$$

**Proof:** By substituting (3.2), (3.3), (3.4), (3.12), (3.13), (3.14) in (2.5) and (2.6), we obtain the result as desired. □

**Theorem 3.1** *The surface $M_N$ is developable, if and only if*

$$\sigma_{1_\alpha} = -\tan[\theta_N(u,v)] \quad or \quad \sigma_{1_\beta} = -\tan[\gamma_N(u,v)]. \tag{3.17}$$

**Corollary 3.1** *If the surface $M_N$ is developable, then the angle $\theta_N[(u,v)]$ is a function that depends only on $u$ or the angle $\gamma_N[(u,v)]$ is a function that depends only on $v$.*

**Corollary 3.2** *Let the surface $M_N$ be developable, if the curves $\alpha$ and $\beta$ are planar curves then the angles $\theta_N = k\pi$ or $\gamma_N = k\pi$, $(k \in \mathbb{Z})$.*

**Proof:** If $\alpha$ and $\beta$ are planar curves, then $\tau_\alpha = \tau_\beta = 0$, and according to (2.2), $\sigma_{1_\alpha} = 0$ and $\sigma_{1_\beta} = 0$. As we suppose $M_N$ developable, according to (3.17), it comes:

$$\sin[\theta_N(u,v)] = 0 \quad or \quad \sin[\gamma_N(u,v)] = 0,$$

hence $\theta_N = k\pi$, $(k \in \mathbb{Z})$ or $\gamma_N = k\pi$, $(k \in \mathbb{Z})$. □

**Corollary 3.3** *Let the surface $M_N$ be developable, if the curves $\alpha$ and $\beta$ are slant helices, then one of the angles $\theta_N$ or $\gamma_N$ is constant.*

**Corollary 3.4** *Let the surface $M_N$ be developable, if the curves $\alpha$ and $\beta$ are slant helices, then the surface $M_N$ is a constant angle surface.*

**Proof:** We assume that the surface $M_N$ is developable and that the curves $\alpha$ and $\beta$ are slant helices. According to the corollary (3.1$C$), $\gamma_N = \gamma_0$ or $\theta_N = \theta_0$ are constant angles.
Without loss of generality, we assume that $\theta_N$ is constant.
Since $\alpha$ is a slant helix, there is a constant unit direction $u_\alpha$ which makes a constant angle with the unit principal normal vector $N_\alpha$ of the curve $\alpha$. Then

$$\langle N_\alpha, u_\alpha \rangle = \cos\delta_0 = \ constant.$$

We can therefore define $u_\alpha$ as follows

$$u_\alpha = \cos\delta_0 N_\alpha + \sin\delta_0 W_\alpha. \tag{3.18}$$

Using (3.10) and (3.18), we obtain

$$\begin{aligned}\langle N_1, u_\alpha \rangle &= \sin\theta_0\cos\delta_0 - \cos\theta_0\sin\delta_0 \\ &= constant,\end{aligned}$$

which completes the proof. □

**Theorem 3.2** *If the surface $M_N$ is minimal and the curves $\alpha$ and $\beta$ are planar curves, then the angles between the vectors $C_\alpha$ and $W_\beta$ and the vectors $C_\beta$ and $W_\alpha$ coincide.*

**Proof:** According to (3.16) the surface $M_N$ is minimal, if and only if

$$\cos[\theta_N(u,v)]\frac{g_\alpha}{f_\alpha} + \sin[\theta_N(u,v)] = \cos[\gamma_N(u,v)]\frac{g_\beta}{f_\beta} + \sin[\gamma_N(u,v)].$$

The curves $\alpha$ and $\beta$ are planar curves, then $\tau_\alpha = \tau_\beta = 0$, and using (2.2), $g_\alpha = g_\beta = 0$. Since the surface $M_N$ is minimal, we get:

$$\sin[\theta_N(u,v)] = \sin[\gamma_N(u,v)],$$

for using (3.7) and (3.9), we get

$$\langle C_\beta, W_\alpha \rangle = \langle C_\alpha, W_\beta \rangle.$$

□

**Example 3.1** *Let $\alpha$ and $\beta$ be two curves defined by*:

$$\alpha(u) = \big(u, \frac{u^2}{2}, \frac{u^3}{6}\big),$$

$$\beta(v) = \big(6v+2, 5v^2, -8v\big).$$

*The translation surface generated by the principal normal indicatrices of the curves $\alpha$ and $\beta$ is given by*:

$$M_N(u,v) = \big(-\frac{2u}{u^2+2} - \frac{3v}{5\sqrt{v^2+1}}, -\frac{u^2-2}{u^2+2} + \frac{1}{\sqrt{v^2+1}}, \frac{2u}{u^2+2} + \frac{4v}{5\sqrt{v^2+1}}\big).$$

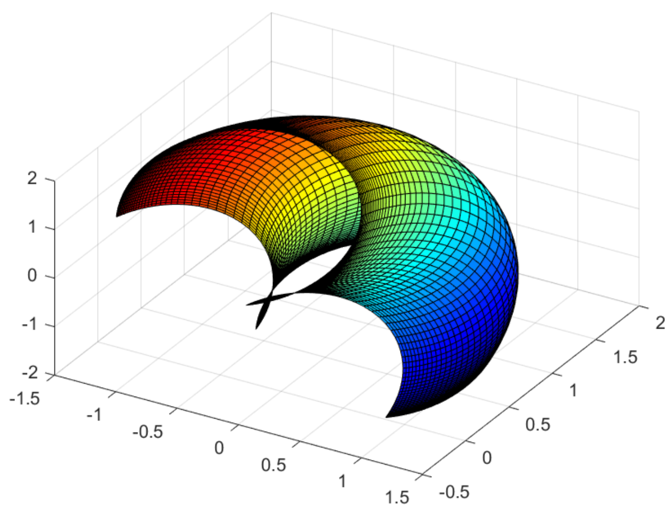


Figure 1: Translation surface $M_N$ generated by the principal normal indicatrices

**Example 3.2** *Let $\alpha$ and $\beta$ be two planar curves parametrized by arc length and defined by*:

$$\alpha(u) = \big(2\cos(\frac{u}{2}), 2\sin(\frac{u}{2}), 0\big),$$

$$\beta(v) = \big(0, 3\cos(\frac{v}{3}), 3\sin(\frac{v}{3})\big).$$

*The translation surface generated by the principal normal indicatrices of the curves $\alpha$ and $\beta$ is given by*:

$$M_N(u,v) = \big(-\cos(\frac{u}{2}), -\sin(\frac{u}{2}) - \cos(\frac{v}{3}), -\sin(\frac{v}{3})\big).$$

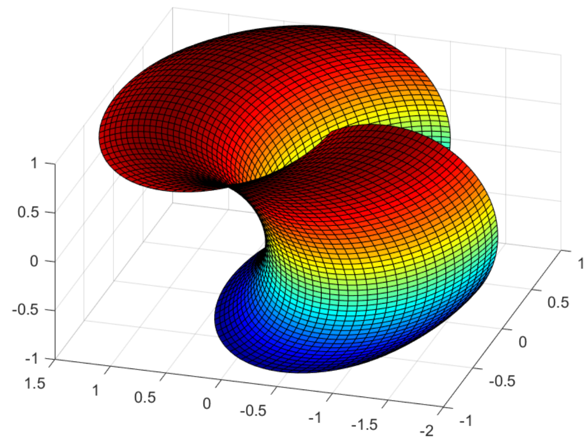


Figure 2: Translation surface $M_N$ generated by the principal normal indicatrices

## 4. Translation surfaces generated by spherical k-indicatrices of regular curves of $\mathbb{E}^3$

Let us denote by $u \to \alpha(u)$ and $v \to \beta(v)$ two non-degenerate curves of class $C^n \quad (n \geq k+1)$ of $\mathbb{E}^3$. Denote by $(C_{k_\alpha}, C_{k+1_\alpha}, W_{k+1_\alpha}, f_{k-1_\alpha}, g_{k-1_\alpha})$ and $(C_{k_\beta}, C_{k+1_\beta}, W_{k+1_\beta}, f_{k-1_\beta}, g_{k-1_\beta})$ the Serret-Frenet frames respectively of the curve $u \to C_{k-1_\alpha}(u)$ and the curve $v \to C_{k-1_\beta}(v)$ .
The translation surface generated by the curves $C_{k_\alpha}$ and $C_{k_\beta}$, associated to the curves $\alpha$ and $\beta$ is defined by:

$$M_{C_k} : \; X(u,v) = \; C_{k_\alpha}(u) \; + \; C_{k_\beta}(v). \tag{4.1}$$

By using the derivative formulas (2.3), we obtain the partial derivatives with respect to $u$ and $v$ of the translation surface given by the parametrization (4.1):

$$X_u \;=\; f_{k-1_\alpha} C_{k+1_\alpha}, \qquad X_v \;=\; f_{k-1_\beta} C_{k+1_\beta}.$$

Thus, the components of the first fundamental form of the surface $M_{C_k}$ are given by:

$$\begin{aligned} E &= f^2_{k-1_\alpha}, \\ F &= f_{k-1_\alpha} f_{k-1_\beta} \cos[\phi_{C_k}(u,v)], \\ G &= f^2_{k-1_\beta}, \end{aligned}$$

where $\phi_{C_k} = \phi_{C_k}(u,v)$ is the angle function between the vectors $C_{k+1_\alpha}$ and $C_{k+1_\beta}$. The unit normal vector of the translation surface $M_{C_k}$ is given by:

$$N(u,v) \;=\; \frac{C_{k+1_\alpha} \wedge C_{k+1_\beta}}{\sin[\phi_{C_k}(u,v)]}. \tag{4.2}$$

As in the previous paragraph, we express the normal vector $N(u,v)$ in the two frames: $(C_{k_\alpha}, C_{k+1_\alpha}, W_{k+1_\alpha})$ associated to $\alpha$ and $(C_{k_\beta}, C_{k+1_\beta}, W_{k+1_\beta})$ associated to $\beta$. To achieve this, we express the vector $C_{k+1_\alpha}$ (resp. $C_{k+1_\beta}$) in the frame $(C_{k_\beta}, C_{k+1_\beta}, W_{k+1_\beta})$ (resp. $(C_{k_\alpha}, C_{k+1_\alpha}, W_{k+1_\alpha})$). We can write

$$C_{k+1_\alpha} = \mu_1 C_{k_\beta} + \mu_2 C_{k+1_\beta} + \mu_3 W_{k+1_\beta}, \tag{4.3}$$

where

$$\mu_1 = \langle C_{k+1_\alpha}, C_{k_\beta} \rangle \;=\; \sin[\phi_{C_k}(u,v)] \cos[\gamma_{C_k}(u,v)],$$

$$\mu_2 = \langle C_{k+1_\alpha}, C_{k+1_\beta} \rangle \; = \; \cos[\phi_{C_k}(u,v)],$$

$$\mu_3 = \langle C_{k+1_\alpha}, W_{k+1_\beta} \rangle \; = \; \sin[\phi_{C_k}(u,v)] \sin[\gamma_{C_k}(u,v)]. \tag{4.4}$$

Similarly, we have

$$C_{k+1_\beta} = \lambda_1 C_{k_\alpha} + \lambda_2 C_{k+1_\alpha} + \lambda_3 W_{k+1_\alpha}, \tag{4.5}$$

with

$$\lambda_1 = \langle C_{k+1_\beta}, C_{k_\alpha} \rangle \; = \; \sin[\phi_{C_k}(u,v)] \cos[\theta_{C_k}(u,v)],$$

$$\lambda_2 = \langle C_{k+1_\beta}, C_{k+1_\alpha} \rangle \; = \; \cos[\phi_{C_k}(u,v)],$$

$$\lambda_3 = \langle C_{k+1_\beta}, W_{k+1_\alpha} \rangle \; = \; \sin[\phi_{C_k}(u,v)] \sin[\theta_{C_k}(u,v)]. \tag{4.6}$$

Using (4.2) and (4.5), it comes:

$$N_1(u,v) = \sin[\theta_{C_k}(u,v)] C_{k_\alpha} - \cos[\theta_{C_k}(u,v)] W_{k+1_\alpha}. \tag{4.7}$$

Similarly, by using (4.2) and (4.3), we get:

$$N_2(u,v) = -\sin[\gamma_{C_k}(u,v)] C_{k_\beta} + \cos[\gamma_{C_k}(u,v)] W_{k+1_\beta}. \tag{4.8}$$

It follows, the components of the second fundamental form of the surface $M_{C_k}$:

$$l = \; -f_{k-1_\alpha}^2 \; \Big[ \cos[\theta_{C_k}(u,v)] \frac{g_{k-1_\alpha}}{f_{k-1_\alpha}} + \sin[\theta_{C_k}(u,v)] \Big],$$

$$m = 0,$$

$$n = \; f_{k-1_\beta}^2 \; \Big[ \cos[\gamma_{C_k}(u,v)] \frac{g_{k-1_\beta}}{f_{k-1_\beta}} + \sin[\gamma_{C_k}(u,v)] \Big].$$

**Proposition 4.1** *The Gaussian curvature $K$ and the mean curvature $H$ of the translation surface $M_{C_k}$ are respectively:*

$$K = -\frac{\Big[ \cos[\theta_{C_k}(u,v)] \frac{g_{k-1_\alpha}}{f_{k-1_\alpha}} + \sin[\theta_{C_k}(u,v)] \Big] \Big[ \cos[\gamma_{C_k}(u,v)] \frac{g_{k-1_\beta}}{f_{k-1_\beta}} + \sin[\gamma_{C_k}(u,v)] \Big]}{\sin^2[\phi_{C_k}(u,v)]}, \tag{4.9}$$

$$H = \frac{-\Big[ \cos[\theta_{C_k}(u,v)] \frac{g_{k-1_\alpha}}{f_{k-1_\alpha}} + \sin[\theta_{C_k}(u,v)] \Big] + \Big[ \cos[\gamma_{C_k}(u,v)] \frac{g_{k-1_\beta}}{f_{k-1_\beta}} + \sin[\gamma_{C_k}(u,v)] \Big]}{2\sin^2[\phi_{C_k}(u,v)]}. \tag{4.10}$$

**Theorem 4.1** *The surface $M_{C_k}$ is developable, if and only if*

$$\sigma_{k-1_\alpha} \; = \; - \; \tan[\theta_{C_k}(u,v)] \quad \texttt{or} \quad \sigma_{k-1_\beta} \; = \; - \; \tan[\gamma_{C_k}(u,v)]. \tag{4.11}$$

**Corollary 4.1** *If the surface $M_{C_k}$ is developable, then the angle $\theta_{C_k}[(u,v)]$ is a function that depends only on $u$ or the angle $\gamma_{C_k}[(u,v)]$ is a function that depends only on $v$.*

**Corollary 4.2** *If the surface $M_{C_k}$ is developable and if the curves $\alpha$ and $\beta$ are planar curves then the angles $\theta_{C_k} = k\pi$ or $\gamma_{C_k} = k\pi$, $(k \in \mathbb{Z})$.*

**Proof:** If $\alpha$ and $\beta$ are planar curves, then $\tau_\alpha = \tau_\beta = 0$, and from (2.2) and (2.4) we have $\sigma_{k-1_\alpha} = 0$ and $\sigma_{k-1_\beta} = 0$. As we suppose $M_{C_k}$ is developable, we obtain from (4.11) that:

$$\sin[\theta_{C_k}(u,v)] \; = \; 0 \quad \textit{or} \quad \sin[\gamma_{C_k}(u,v)] = 0,$$

and therefore $\theta_{C_k} \; = \; k\pi, \; (k \in \mathbb{Z})$ or $\gamma_{C_k} \; = \; k\pi, \; (k \in \mathbb{Z})$. □

**Corollary 4.3** *If the surface $M_{C_k}$ is developable and if the curves $\alpha$ and $\beta$ are (k-1)-slant helices, then one of the angles $\theta_{C_k}$ or $\gamma_{C_k}$ is constant.*

**Corollary 4.4** *If the surface $M_{C_k}$ is developable, and if the curves $\alpha$ and $\beta$ are (k-1)-slant helices, then the surface $M_{C_k}$ is a constant angle surface.*

**Proof:** *By the corollary (4.1C), $\gamma_{C_k} = \gamma_0$ or $\theta_{C_k} = \theta_0$ are constant angles. Without loss of generality, we assume that $\theta_{C_k} = \theta_0$ is constant.*
*Since $\alpha$ is a (k-1)-slant helix, then there exists a constant vector $u_\alpha$ verifying*

$$\langle C_{k_\alpha}, u_\alpha \rangle = \cos\delta_0 = \text{ constant}$$

*We can then define $u_\alpha$ as follows*

$$u_\alpha = \cos\delta_0 C_{k_\alpha} + \sin\delta_0 W_{k+1_\alpha}. \tag{4.12}$$

*By using (4.7) and (4.12), we obtain*

$$\begin{aligned} \langle N_1, u_\alpha \rangle = & \ \sin\theta_0 \cos\delta_0 - \cos\theta_0 \sin\delta_0 \\ = & \ \text{constant}. \end{aligned}$$

□

**Theorem 4.2** *If the surface $M_{C_k}$ is minimal and the curves $\alpha$ and $\beta$ are planer curves, then the angles between the vectors $C_{k+1_\alpha}$ and $W_{k+1_\beta}$ and the vectors $C_{k+1_\beta}$ and $W_{k+1_\alpha}$ coincide.*

**Proof:** Since $\alpha$ and $\beta$ are planar curves, we have $\tau_\alpha = \tau_\beta = 0$, and according to (2.2) and (2.4), we obtain $\sigma_{k-1_\alpha} = 0$ and $\sigma_{k-1_\beta} = 0$.
As the surface $M_{C_k}$ is assumed to be minimal, we have $H = 0$, and therefore, according to (4.10), it comes

$$\sin[\theta_{C_k}(u,v)] = \sin[\gamma_{C_k}(u,v)].$$

By using (4.4) and (4.6), it follows

$$\langle C_{k+1_\alpha}, W_{k+1_\beta} \rangle = \langle C_{k+1_\beta}, W_{k+1_\alpha} \rangle.$$

□

## 5. Applications

In this paragraph, we propose to give the parametric representations of the translation surfaces $M_{C_k} = C_{k_\alpha}(u) + C_{k_\beta}(v)$, in the case where $\alpha$ and $\beta$ are (k-1)-slant helices for $k = 1, 2$ *and* $3$ followed by some illustratives examples.

### 5.1. Case k = 1

**Theorem 5.1** *[3] The position vector of the general helix c, is expressed in the natural representation form as follows:*

$$c(s) = \frac{1}{\sqrt{1+m^2}} \int \Big( \cos\big(\sqrt{1+m^2}\int \kappa(s)ds\big), \sin\big(\sqrt{1+m^2}\int \kappa(s)ds\big), m \Big)\, ds,$$

*where $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos(\phi)$, $\phi$ is the angle between the tangent vector of the curve c and a fixed direction.*

Let $\alpha,\ \beta$ be two general helices of $\mathbb{E}^3$, with curvatures $\kappa_\alpha, \kappa_\beta$ and admitting the values $m_1, m_2$ respectively. The components of the translation surface $M_T = (M_{T_1}, M_{T_2}, M_{T_3})$ generated by the tangent vectors $T_\alpha$ and $T_\beta$ are given by:

$$\begin{cases} M_{T_1}(u,v) &= \frac{1}{\sqrt{1+m_1^2}}\cos\left(\sqrt{1+m_1^2}\int\kappa_\alpha(u)\,du\right) + \frac{1}{\sqrt{1+m_2^2}}\cos\left(\sqrt{1+m_2^2}\int\kappa_\beta(v)\,dv\right),\\ M_{T_2}(u,v) &= \frac{1}{\sqrt{1+m_1^2}}\sin\left(\sqrt{1+m_1^2}\int\kappa_\alpha(u)\,du\right) + \frac{1}{\sqrt{1+m_2^2}}\sin\left(\sqrt{1+m_2^2}\int\kappa_\beta(v)\,dv\right),\\ M_{T_3}(u,v) &= \frac{m_1}{\sqrt{1+m_1^2}} + \frac{m_2}{\sqrt{1+m_2^2}}. \end{cases}$$

**Example 5.1** *Let $\alpha$ and $\beta$ be two circular helices defined by the intrinsic equations*:

$$\begin{cases} \kappa_\alpha(u) &= \kappa_1, \quad \kappa_\beta(v) = \kappa_2,\\ \tau_\alpha(u) &= m_1\kappa_1, \quad \tau_\beta(v) = m_2\kappa_2, \end{cases}$$

*The components of the translation surface $M_T = (M_{T_1}, M_{T_2}, M_{T_3})$ generated by the tangent vectors $T_\alpha$ and $T_\beta$ are given by*:

$$\begin{cases} M_{T_1}(u,v) &= \frac{1}{\sqrt{1+m_1^2}}\cos\left(\sqrt{1+m_1^2}\kappa_1 u\right) + \frac{1}{\sqrt{1+m_2^2}}\cos\left(\sqrt{1+m_2^2}\kappa_2 v\right),\\ M_{T_2}(u,v) &= \frac{1}{\sqrt{1+m_1^2}}\sin\left(\sqrt{1+m_1^2}\kappa_1 u\right) + \frac{1}{\sqrt{1+m_2^2}}\sin\left(\sqrt{1+m_2^2}\kappa_2 v\right),\\ M_{T_3}(u,v) &= \frac{m_1}{\sqrt{1+m_1^2}} + \frac{m_2}{\sqrt{1+m_2^2}}. \end{cases}$$

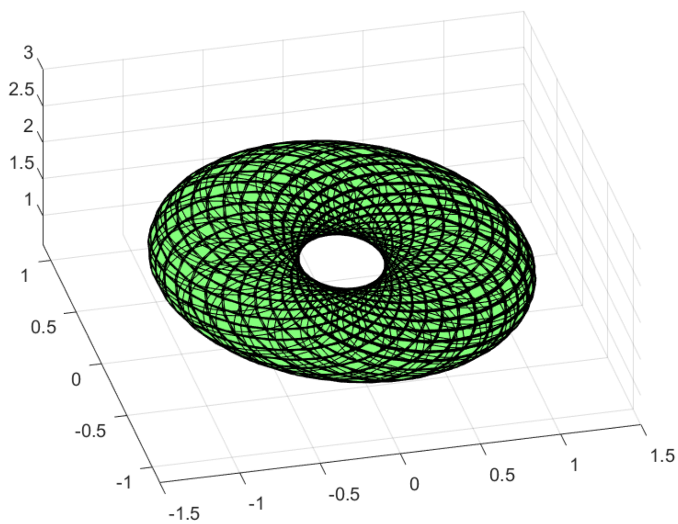


Figure 3: Translation surface $M_T$, with $m_1 = \kappa_2 = 1$, $\kappa_1 = m_2 = 2$

**Example 5.2** *Let $\alpha$ be a general helix defined by the intrinsic equations*

$$\kappa_\alpha(u) = \frac{a_1}{u}, \quad \tau_\alpha(u) = \frac{m_1 a_1}{u},$$

*and let $\beta$ be a spherical helix defined by the intrinsic equations*

$$\kappa_\beta(v) = \frac{a_2}{\sqrt{1-m_2^2v^2}}, \quad \tau_\beta(v) = \frac{a_2 m_2}{\sqrt{1-m_2^2v^2}}.$$

*The components of the translation surface $M_T(u,v) = (M_{T_1}, M_{T_2}, M_{T_3})$ generated by the tangent vectors*

*$T_\alpha$ and $T_\beta$ are given by*:

$$\begin{cases} M_{T_1}(u,v) &= \frac{1}{\sqrt{1+m_1^2}}\cos\left(\sqrt{1+m_1^2}\ a_1\ln(u)\right) + \frac{1}{\sqrt{1+m_2^2}}\cos\left(\frac{a_2\ \sqrt{1+m_2^2}}{m_2}\ \arcsin(m_2 v)\right),\\ M_{T_2}(u,v) &= \frac{1}{\sqrt{1+m_1^2}}\sin\left(\sqrt{1+m_1^2}\ a_1\ln(u)\right) + \frac{1}{\sqrt{1+m_2^2}}\sin\left(\frac{a_2\ \sqrt{1+m_2^2}}{m_2}\ \arcsin(m_2 v)\right),\\ M_{T_3}(u,v) &= \frac{m_1}{\sqrt{1+m_1^2}} + \frac{m_2}{\sqrt{1+m_2^2}}. \end{cases}$$

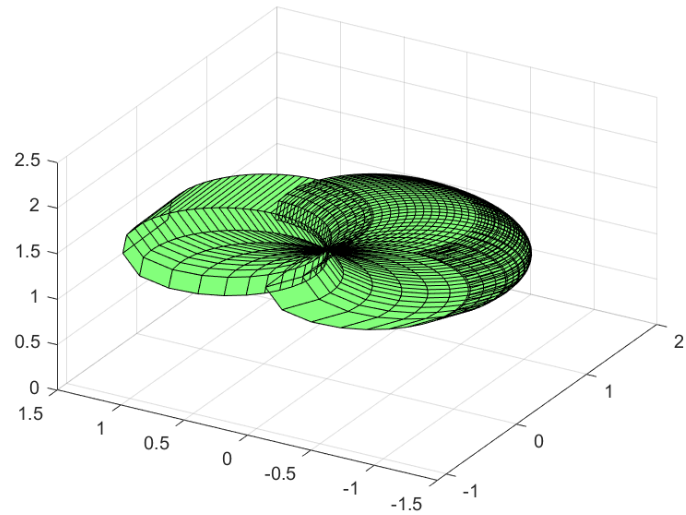


Figure 4: Translation surface $M_T$, with $m_1 = a_1 = 1$, $a_2 = m_2 = 1$

### 5.2. Case k = 2

**Theorem 5.2** *[4] The position vector of the slant helix $c(s) = (c_1(s), c_2(s), c_3(s))$ is expressed in the natural representation form as follows*:

$$\begin{cases} c_1(s) &= \frac{n}{m}\int\left[\int \kappa(s)\cos\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\ ds\right)\right]\ ds\right]\ ds,\\ c_2(s) &= \frac{n}{m}\int\left[\int \kappa(s)\sin\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\ ds\right)\right]\ ds\right]\ ds,\\ c_3(s) &= n\int\left[\int\kappa(s)\ ds\right]\ ds, \end{cases}$$

*where $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos(\phi)$, $\phi$ is the angle between the principal normal vector of the curve $c$ and a fixed direction.*

Let $\alpha$, $\beta$ be two slant helices of $\mathbb{E}^3$, with curvatures $\kappa_\alpha, \kappa_\beta$ and admitting the values $n_1, m_1$ *and* $n_2, m_2$ respectively. The components of the translation surface $M_N = (M_{N_1}, M_{N_2}, M_{N_3})$ generated by the principal normals vectors $N_\alpha$ and $N_\beta$ are given by:

$$\begin{cases} M_{N_1}(u,v) &= \frac{n_1}{m_1}\ \cos\left[\frac{1}{n_1}\arcsin\left(m_1\int\kappa_\alpha(u)\ du\right)\right] + \frac{n_2}{m_2}\ \cos\left[\frac{1}{n_2}\arcsin\left(m_2\int\kappa_\beta(v)\ dv\right)\right],\\ M_{N_2}(u,v) &= \frac{n_1}{m_1}\ \sin\left[\frac{1}{n_1}\arcsin\left(m_1\int\kappa_\alpha(u)\ du\right)\right] + \frac{n_2}{m_2}\ \sin\left[\frac{1}{n_2}\arcsin\left(m_2\int\kappa_\beta(v)\ dv\right)\right],\\ M_{N_3}(u,v) &= n_1 + n_2 \end{cases}$$

**Example 5.3** *Let $\alpha$ and $\beta$ be two Salkowski curves defined by the intrinsic equations*:

$$\begin{cases} \kappa_\alpha(u) &= 1, \quad \kappa_\beta(v) = 1,\\ \tau_\alpha(u) &= \frac{m_1 u}{\sqrt{1-m_1^2u^2}}, \quad \tau_\beta(v) = \frac{m_2 v}{\sqrt{1-m_2^2v^2}}. \end{cases}$$

*The components of the translation surface* $M_N(u,v) = (M_{N_1}, M_{N_2}, M_{N_3})$ *generated by the principal normals vectors* $N_\alpha$ *and* $N_\beta$ *are given by*:

$$\begin{cases} M_{N_1}(u,v) &= \frac{n_1}{m_1}\cos\left[\frac{1}{n_1}\arcsin\left(m_1 u\right)\right] + \frac{n_2}{m_2}\cos\left[\frac{1}{n_2}\arcsin\left(m_2 v\right)\right], \\ M_{N_2}(u,v) &= \frac{n_1}{m_1}\sin\left[\frac{1}{n_1}\arcsin\left(m_1 u\right)\right] + \frac{n_2}{m_2}\sin\left[\frac{1}{n_2}\arcsin\left(m_2 v\right)\right], \\ M_{N_3}(u,v) &= n_1 + n_2. \end{cases}$$

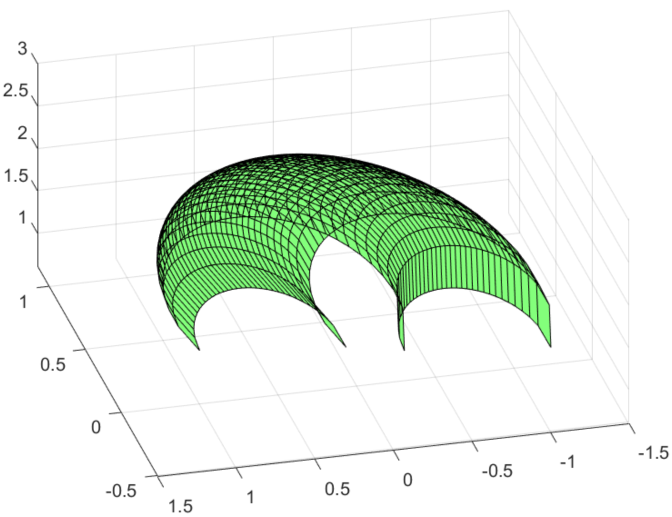


Figure 5: Translation surface $M_N$, with $m_1 = 1$, $m_2 = 2$

**Example 5.4** *Let* $\alpha$ *be an anti-Salkowski curve defined by the intrinsic equations*

$$\kappa_\alpha(u) = \frac{m_1 u}{\sqrt{1 - m_1^2 u^2}}, \quad \tau_\alpha(u) = 1,$$

*and let* $\beta$ *be a circular slant helix defined by the intrinsic equations*

$$\kappa_\beta(v) = \frac{\mu}{m_2}\cos(\mu v), \quad \tau_\beta(v) = \frac{\mu}{m_2}\sin(\mu v).$$

*The components of the translation surface* $M_N(u,v) = (M_{N_1}, M_{N_2}, M_{N_3})$ *generated by the principal normals vectors* $N_\alpha$ *and* $N_\beta$ *are given by*:

$$\begin{cases} M_{N_1}(u,v) &= \frac{n_1}{m_1}\cos\left[\frac{1}{n_1}\arcsin\left(-\sqrt{1 - m_1^2 u^2}\right)\right] + \frac{n_2}{m_2}\cos\left[\frac{1}{n_2}\arcsin\left(\sin(\mu v)\right)\right], \\ M_{N_2}(u,v) &= \frac{n_1}{m_1}\sin\left[\frac{1}{n_1}\arcsin\left(-\sqrt{1 - m_1^2 u^2}\right)\right] + \frac{n_2}{m_2}\sin\left[\frac{1}{n_2}\arcsin\left(\sin(\mu v)\right)\right], \\ M_{N_3}(u,v) &= n_1 + n_2. \end{cases}$$

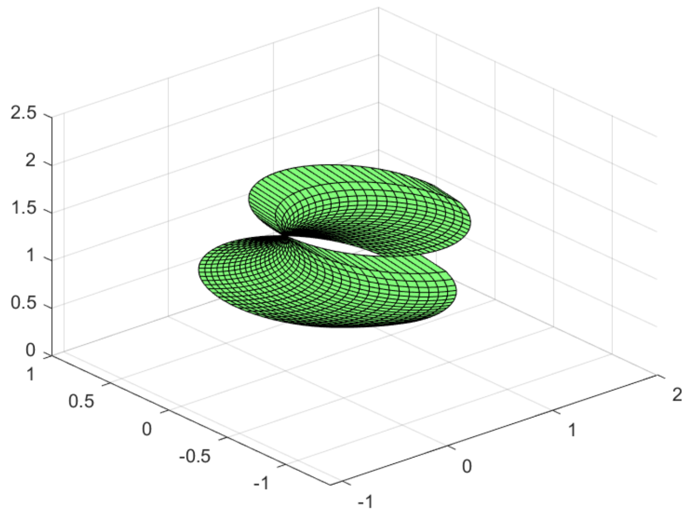


Figure 6: Translation surface $M_N$, with $m_1 = m_2 = \mu = 1$

### 5.3. Case k = 3

**Theorem 5.3** *[9] The position vector of the slant-slant helix (or 2-slant helix) $c(s) = (c_1(s), c_2(s), c_3(s))$ is expressed in the natural representation form as follows:*

$$\left\{\begin{array}{lcl} c_1(s) &=& \frac{n}{m}\int \left[\int \kappa(s) \left[\int f(s)\cos\left[\frac{1}{n}\arcsin\left(m\int f(s)\ ds\right)\right] ds\right] ds\right] ds, \\ c_2(s) &=& \frac{n}{m}\int \left[\int \kappa(s) \left[\int f(s)\sin\left[\frac{1}{n}\arcsin\left(m\int f(s)\ ds\right)\right] ds\right] ds\right] ds, \\ c_3(s) &=& n\int \left[\int \kappa(s) \left[\int f(s)\ ds\right] ds\right] ds, \end{array}\right.$$

*where $f = \sqrt{\tau^2 + \kappa^2}$, $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos(\phi)$, $\phi$ is the angle between the vector $C$ of the curve $c$ and a fixed direction.*

Considering the family of curves $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ with curvature $\kappa(s)$ and torsion $\tau(s)$ taking the following values:

$$\left\{\begin{array}{ll} \kappa(s) &= \frac{\mu}{m}\cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right), \\ \tau(s) &= -\frac{\mu}{m}\cos(\mu s)\sin\left(\frac{1}{m}\cos(\mu s)\right). \end{array}\right.$$

Then the position vector of the slant-slant helix $\gamma$ is given by [9]:

$$\left\{\begin{array}{lcl} \gamma_1(s) &=& \frac{n\mu}{2m^3}\int \left[\int \cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right)\left[\frac{n}{n+1}\sin\left(\frac{n+1}{n}\mu s\right) + \frac{n}{n-1}\sin\left(\frac{n-1}{n}\mu s\right)\right] ds\right] ds, \\ \gamma_2(s) &=& -\frac{n\mu}{2m^3}\int \left[\int \cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right)\left[\frac{n}{n+1}\cos\left(\frac{n+1}{n}\mu s\right) + \frac{n}{1-n}\cos\left(\frac{1-n}{n}\mu s\right)\right] ds\right] ds, \\ \gamma_3(s) &=& \frac{n\mu}{m^2}\int \left[\int \cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right)\sin(\mu s)\right] ds\ ds. \end{array}\right.$$

It follows that the vector $T(s) = (T_1(s), T_2(s), T_3(s))$ of the curve $\gamma$ is given by:

$$\left\{\begin{array}{lcl} T_1(s) &=& \frac{n\mu}{2m^3}\int \cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right)\left[\frac{n}{n+1}\sin\left(\frac{n+1}{n}\mu s\right) + \frac{n}{n-1}\sin\left(\frac{n-1}{n}\mu s\right)\right] ds, \\ T_2(s) &=& \frac{-n^2\mu}{2m^3}\int \cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right)\left[\frac{n}{n+1}\cos\left(\frac{n+1}{n}\mu s\right) + \frac{n}{1-n}\cos\left(\frac{1-n}{n}\mu s\right)\right] ds, \\ T_3(s) &=& \frac{n\mu}{m^2}\int \cos(\mu s)\cos\left(\frac{1}{m}\cos(\mu s)\right)\sin(\mu s)\ ds, \end{array}\right.$$

then the vectors $N(s) = (N_1(s), N_2(s), N_3(s))$ of $\gamma$ is given by:

$$\left\{\begin{array}{lcl} N_1(s) &=& \frac{n}{2m^2}\left[\frac{n}{n+1}\sin\left(\frac{n+1}{n}\mu s\right) + \frac{n}{n-1}\sin\left(\frac{n-1}{n}\mu s\right)\right], \\ N_2(s) &=& -\frac{n}{2m^2}\left[\frac{n}{n+1}\cos\left(\frac{n+1}{n}\mu s\right) + \frac{n}{1-n}\cos\left(\frac{1-n}{n}\mu s\right)\right], \\ N_3(s) &=& \frac{n}{m}\sin(\mu s), \end{array}\right.$$

it follows the parametric representation of the vector $C(s) = \frac{N'(s)}{\|N'(s)\|}$ of the curve $\gamma$:

$$\begin{cases} C_1(s) &= \frac{n}{m} \cos(\frac{\mu}{n}s), \\ C_2(s) &= \frac{n}{m} \sin(\frac{\mu}{n}s), \\ C_3(s) &= n. \end{cases}$$

**Example 5.5** *Let $\alpha = \alpha(u)$ and $\beta = \beta(v)$ be two curves of preceding family of slant-slant helices with curvatures $\kappa_\alpha(u)$, $\kappa_\beta(v)$ and torsions $\tau_\alpha(u)$, $\tau_\beta(v)$ having the following values:*

$$\begin{cases} \kappa_\alpha(u) &= \frac{\mu_1}{m_1}\cos(\mu_1 u)\cos\left(\frac{1}{m_1}\cos(\mu_1 u)\right) \quad \kappa_\beta(v) = \frac{\mu_2}{m_2}\cos(\mu_2 v)\cos\left(\frac{1}{m_2}\cos(\mu_2 v)\right), \\ \tau_\alpha(u) &= -\frac{\mu_1}{m_1}\cos(\mu_1 u)\sin\left(\frac{1}{m_1}\cos(\mu_1 u)\right) \quad \tau_\beta(v) = -\frac{\mu_2}{m_2}\cos(\mu_2 v)\sin\left(\frac{1}{m_2}\cos(\mu_2 v)\right). \end{cases}$$

*The components of the translation surface $M_C = (M_{C_1}, M_{C_2}, M_{C_3})$ generated by the vectors $C_\alpha$ and $C_\beta$ are given by:*

$$\begin{cases} M_{C_1}(u,v) &= \frac{n_1}{m_1} \cos(\frac{\mu_1}{n_1}u) + \frac{n_2}{m_2} \cos(\frac{\mu_2}{n_2}v) \\ M_{C_2}(u,v) &= \frac{n_1}{m_1} \sin(\frac{\mu_1}{n_1}u) + \frac{n_2}{m_2} \sin(\frac{\mu_2}{n_2}v) \\ M_{C_3}(u,v) &= n_1 + n_2 \end{cases}$$

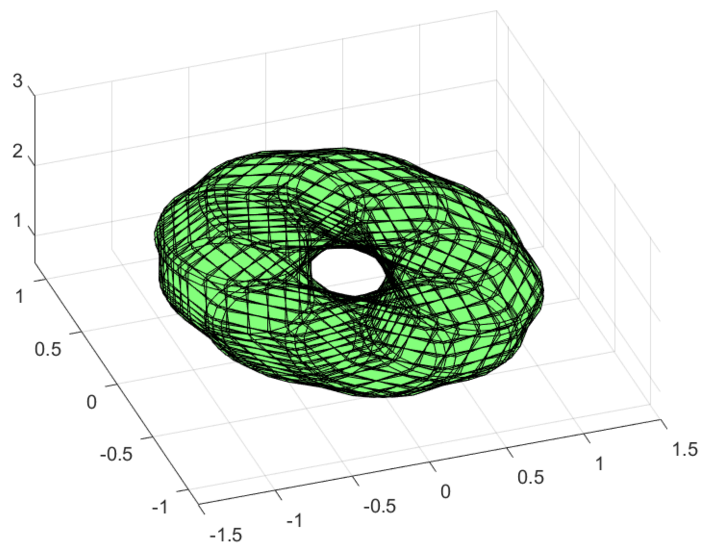


Figure 7: Translation surface $M_C$, with $m_1 = \mu_2 = 1$, $m_2 = \mu_1 = 2$

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