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Fixed Point Theorems with PPF Dependence for (α, β)-F Contraction in Razumikhin Class

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ABSTRACT: In this paper, we provide a novel idea of (α, β)-F contractive, weak (α, β)-F contractive and generalized (α, β)-F contractive nonself mappings. We establish the existence of fixed point results with PPF dependence in Razumikhin class. Some examples are also provided to support our conclusions.

Key Words: Fixed point with PPF dependence, Razumikhin class, (α, β)-admissible nonself mapping, (α, β)-F contractive nonself mapping.

Contents

1. Introduction

Problems in various fields of mathematics can be transformed to fixed point problem $Tx = x$ for self mapping. One of the pillars of the development of fixed point theory is the well-known Banach's contraction principle. After him, many authors expanded this conclusion in various ways (for detail see [1], [2], [4], [9], [10], [15]). Samet et al. [13], established the idea of α-admissible self mapping and demonstrated some fixed point results for this mapping. Salimi et al. [14] modified alightly the notation of α-ψ contractive and founded some fixed point results to generalise the results given by Sametet al. [13]. In 2015, Chandok [7] introduced the concept of (α, β)-admissible Geraghty type contractive mappings and obtained fixed point results for the same.

Bernfeld et al. [6], in 1977, established the notion of Past-Present-Future (PPF) dependent fixed point which is kind of fixed point for nonself mappings. They established the existence of PPF dependent fixed point results for Banach type contraction in the Razumikhin class. In 2014, Kutbi et al. [12] introduced Ciric rational type contraction and demonstrated several fixed point results with PPF dependence. These conclusions are highly valuable for showing the solution of nonlinear functional differential and integral equations that depend on past history, present facts and future consideration. In 2015, Kutbi et al. [11] introduced notions of a Suzuki type GF-contractions, weak α_c-GF-contractive, an α_c-GF-contractive and a generalized α_c-GF-contractive nonself mappings and they also proved various fixed point results with PPF dependence.

Wardowski [16] has developed a novel contractive mapping and demonstrated some fixed point solutions for this contraction. For $k \in (0, 1)$, Δ_k represents the family of all functions $F : \mathbb{R}_0^+ \to \mathbb{R}$ which satisfies:
(1) F be strictly increasing;
(2) $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$ for any sequence α_n in $\mathbb{R}_0^+$;
(3) $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
Family Δ denotes $\cup\{\Delta_k : k \in (0, 1)\}$. A Wardowski function is an element F of Δ family.

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Inspired by the findings of Bernfeld et al. [6], Kutbi et al. [11] and Salimi et al. [14], we introduce the idea of (α, β)-F contractive, weak (α, β)-F contractive and generalized (α, β)-F contractive nonself mappings. For these contractive nonself mappings, we show certain fixed point results with PPF dependence in the Razumikhin class. Also, we give some examples related to these theorems.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ represents the set of natural numbers and for each $i \in \mathbb{N}$, $\mathbb{N}_i = \{n \in \mathbb{N} : i \leq n\}$, E is a Banach space with norm $\|.\|_E$, $E_0 = C(I, E)$ represents the set of all continuous functions from I to E, with the supremum norm $\|.\|_{E_0}$ defined as

$$\|\psi\|_{E_0} = \sup_{t \in I} \|\psi(t)\|_E$$

where $\psi \in E_0$ and I is a closed interval [a,b] in $\mathbb{R}$.

Definition 2.1 [6] "A function $\psi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of a nonself mapping S if $S\psi = \psi(c)$ for some $c \in I$."

Definition 2.2 [8] (i) "The Razumikhin class (minimal class) of functions in E_0 is defined as

$$R_c = \{\psi \in E_0 : \|\psi\|_{E_0} = \|\psi(c)\|_E\}.$$

This class R_c is said to be algebraically closed if it is closed with respect to difference that is $\psi - \xi \in R_c$ whenever $\psi, \xi \in R_c$ and topologically closed with respect to the topology on E_0 generated by norm $\|.\|_{E_0}$."
(ii) "R_c^0 is the class of all constant functions $\psi \in R_c$, which is referred to as the constant Razumikhin class."
H[u] is the constant function of E_0 defined as $H[u](t) = u$ for all $u \in E$, $t \in I$.
Thus, $\|H[u]\|_{E_0} = \|u\|_E, H[u](c) = u$, where $H[u] \in R_c$.

Proposition 2.1 *[8]* "Under the above conventions,

1. $H[u + v] = H[u] + H[v]$ for all $u, v \in E$;
2. $H[\lambda u] = \lambda H[u]$ for all $\lambda \in \mathbb{R}$ and $u \in E$;
3. $\|u\|_E = \|H[u]\|_{E_0}$ for all $u \in E$;
4. the mapping $u \mapsto H[u]$ is an algebraic, topological isomorphism between $(E, \|.\|_E)$ and $(R_c^0, \|.\|_{E_0})$.

where H[u] denotes the constant function of E_0 defined by $H[u](t) = u \; \forall \; t \in I$ and $\|H[u]\|_{E_0} = \|u\|_E$ and $H[u](c) = u$. Hence $H[u] \in R_c$."

Note: In R_c, any constant fuction ξ may be expressed as $\xi = H[u]$ for some $u \in E$.

3. The Main Results

Definition 3.1 [7] Let E be a non empty set. $S : E_0 \to E$ and $\alpha, \beta : E \times E \to \mathbb{R}_0^+$ are nonself mappings. Then S is called an (α, β)-admissible if, for any $\psi, \xi \in E_0$,

$$\alpha(\psi(c), \xi(c)) \geq 1 \text{ and } \beta(\psi(c), \xi(c)) \geq 1 \implies \alpha(S\psi, S\xi) \geq 1 \text{ and } \beta(S\psi, S\xi) \geq 1.$$

Example 3.1 For $E = \mathbb{R}$, $I = [0, 1]$ and $c = \frac{1}{2}$. We define a mapping $S : E_0 \to E$ as

$$S\psi = \frac{4}{7}\psi(\frac{1}{2}) + \frac{3}{28}$$

where ψ, ξ are mappings from I to E defined by

$$\psi(x) = \begin{cases} x^2 & \text{if } x \in [0, \frac{1}{2}] \\ \frac{1}{4} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$\xi(x) = 0, \ \forall\, x \in [0,1].$$

Now, we define mappings $\alpha, \beta : E \times E \to \mathbb{R}_0^+$ by

$$\alpha(\psi(c), \xi(c)) = \begin{cases} 2 & \text{if } \psi(c) \geq \xi(c) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta(\psi(c), \xi(c)) = \begin{cases} 3 & \text{if } \psi(c) \geq \xi(c) \\ 0 & \text{otherwise} \end{cases}$$

Thus considered mapping S is (α,β)-admissible.

Definition 3.2 Suppose $\alpha, \beta : E \times E \to \mathbb{R}_0^+$ are nonself mappings and $F \in \Delta$ is a Wardowski function. The mapping S is said to be

(i) (α,β)-F contractive if $\exists\, \tau > 0$ such that for all $\psi, \xi \in E_0$ and $\|S\psi - S\xi\|_E > 0$,
$$\tau + \alpha(\psi(c), \xi(c)) + \beta(\psi(c), \xi(c)) + F(\|S\psi - S\xi\|_E) \leq F\|\psi - \xi\|_{E_0};$$

(ii) weak (α,β)-F contractive if $\exists\, \tau > 0$ such that for all $\psi, \xi \in E_0$ and $\|S\psi - S\xi\|_E > 0$,
$$\tau + \alpha(\psi(c), \xi(c)) + \beta(\psi(c), \xi(c)) + F(\|S\psi - S\xi\|_E) \leq F(\max\{\|\psi - \xi\|_{E_0}, \|\psi(c) - S\psi\|_E, \|\xi(c) - S\xi\|_E\});$$

(iii) generalized (α,β)-F contractive if $\exists\, \tau > 0$ such that for all $\psi, \xi \in E_0$ and $\|S\psi - S\xi\|_E > 0$,
$$\tau + \alpha(\psi(c), \xi(c)) + \beta(\psi(c), \xi(c)) + F(\|S\psi - S\xi\|_E) \leq F(\max\{\|\psi - \xi\|_{E_0}, \|\psi(c) - S\psi\|_E, \|\xi(c) - S\xi\|_E, \frac{\|\psi(c) - S\xi\|_E + \|\xi(c) - S\psi\|_E}{2}\}).$$

Definition 3.3 Suppose $\alpha, \beta : E \times E \to \mathbb{R}_0^+$ are nonself mappings.

(i) The mapping S is called $(R_c, (\alpha,\beta))$-starting if $\exists\, \psi_0 \in R_c$ such that
$$\alpha(\psi_0(c), S\psi_0) \geq 0 \text{ and } \beta(\psi_0(c), S\psi_0) \geq 0.$$

(ii) The mapping S is called $(R_c^0, (\alpha,\beta))$-starting if $\exists\, \psi_0 \in R_c^0$ such that
$$\alpha(\psi_0(c), S\psi_0) \geq 0 \text{ and } \beta(\psi_0(c), S\psi_0) \geq 0.$$

Remark: If S is $(R_c^0, (\alpha,\beta))$-starting, it is also $(R_c, (\alpha,\beta))$-starting.

Proposition 3.1 Suppose $\alpha, \beta : E \times E \to \mathbb{R}_0^+$ are nonself mappings which satisfies the following

(i) S is (α,β)-admissible;

(ii) S is $(R_c, (\alpha,\beta))$-starting.

Then, S is $(R_c^0, (\alpha,\beta))$-starting.

Proof: From (ii), $\exists\, \psi_0 \in R_c$ such that
$$\alpha(\psi_0(c), S\psi_0) \geq 0 \text{ and } \beta(\psi_0(c), S\psi_0) \geq 0.$$
Since $S\psi_0 \in E$, we can assume the element $\xi_0 = H[S\psi_0]$ from R_c^0 (constant Razumikhin class). So, $H[S\psi_0](t) = S\psi_0 \ \forall\, t \in I \implies \xi_0(t) = H[S\psi_0](t) \ \forall\, t \in I$
$$\xi_0(c) = S\psi_0. \tag{3.1}$$
That implies $\alpha(\psi_0(c), \xi_0(c)) \geq 0$.
As S is (α,β)-admissible, we obtain
$$\alpha(S\psi_0, S\xi_0) \geq 0 \text{ and } \beta(S\psi_0, S\xi_0) \geq 0.$$
By equation (3.1), $\alpha(\xi_0(c), S\xi_0) \geq 0$ and $\beta(\xi_0(c), S\xi_0) \geq 0$,
which completes the proof. □

Theorem 3.1 Let $\alpha, \beta : E \times E \to \mathbb{R}_0^+$ be the nonself mappings with F as a Wardowski function. Consider

(i) S is (α, β)-admissible;

(ii) S is generalized (α, β)-F contraction;

(iii) $\exists\ \psi_0 \in R_c$ such that $\alpha(\psi_0(c), S\psi_0) \geq 0$ and $\beta(\psi_0(c), S\psi_0) \geq 0$.

In addition suppose that:
S has no PPF dependent fixed point in R_c^0, i.e., $S\psi \neq \psi(c)$ for all $\psi \in R_c^0$.
Then, $\exists$ a sequence $\{\psi_n\}$ in R_c^0, $\psi^* \in R_c^0$ and $k \in \mathbb{N}$ such that:

(k1) $S\psi_n = \psi_{n+1}(c)$ and $\alpha(\psi_n(c), \psi_{n+1}(c)) \geq 0$ and $\beta(\psi_n(c), \psi_{n+1}(c)) \geq 0$;

(k2) $\psi_n \to \psi^*$ as $n \to \infty$;

(k3) $S\psi_n \neq S\psi^*$, hence $\psi_n \neq \psi^*$ for all $n \in \mathbb{N}_k$.

Proof: From proposition (3.5) and conditions (i) and (iii), $\exists\ \psi_0 \in R_c^0$ such that

$$\alpha(\psi_0(c), S\psi_0) \geq 0 \text{ and } \beta(\psi_0(c), S\psi_0) \geq 0. \tag{3.2}$$

Since $S\psi_0 \in E$, we can take an element as $\psi_1 = H[S\psi_0]$ from R_c^0 which means

$$S\psi_0 = \psi_1(t)\ \forall\ t \in I.$$

Thus, $S\psi_0 = \psi_1(c)$.
Now, since $S\psi_1 \in E$, so we can consider an element $\psi_2 = H[S\psi_1]$ from R_c^0, which means $S\psi_1 = \psi_2(t)\ \forall\ t \in I$. Hence, $S\psi_1 = \psi_2(c)$.
Continuing like this, we have a sequence $\{\psi_n\}$ in R_c^0 such that

$$S\psi_{n-1} = \psi_n(t)\ \forall\ t \in I \text{ and } n \in \mathbb{N}. \tag{3.3}$$

Hence $S\psi_{n-1} = \psi_n(c)$.
Since constant Razumikhin class R_c^0, has the algebraic topological properties, so $\|\psi_{n-1} - \psi_n\|_{E_0} = \|\psi_{n-1}(c) - \psi_n(c)\|_E$.
Because of condition (i)

$$\alpha(\psi_0(c), \psi(c)) = \alpha(\psi_0(c), S\psi_0) \geq 0 \text{ and } \beta(\psi_0(c), \psi(c)) = \beta(\psi_0(c), S\psi_0) \geq 0$$

implies that

$$\alpha(\psi_1(c), \psi_2(c)) = \alpha(S\psi_0, S\psi_1) \geq 0 \text{ and } \beta(\psi_1(c), \psi_2(c)) = \beta(S\psi_0, S\psi_1) \geq 0.$$

Again

$$\alpha(\psi_1(c), \psi_2(c)) \geq 0 \text{ and } \beta(\psi_1(c), \psi_2(c)) \geq 0 \implies \alpha(\psi_2(c), \psi_3(c)) \geq 0 \text{ and } \beta(\psi_2(c), \psi_3(c)) \geq 0.$$

Continuing in the same manner, we get

$$\alpha(\psi_{n-1}(c), \psi_n(c)) \geq 0 \text{ and } \beta(\psi_{n-1}(c), \psi_n(c)) \geq 0 \text{ for all } n \in \mathbb{N}$$

which proves condition (k1).
S is a non self mapping, thus $\nexists$ any $k \in \mathbb{N}$ such that $S(\psi_{k+1}) = \psi_{k+1} = S\psi_k$
So, $S\psi_n \neq S\psi_{n+1}$.
Hence $\psi_n \neq \psi_{n+1}\ \forall\ n \in \mathbb{N}$.
Now,

$$\begin{aligned}
\tau + F(\|\psi_n - \psi_{n+1}\|_{E_0}) &\leq \tau + \alpha(\psi_{n-1}(c), \psi_n(c)) + \beta(\psi_{n-1}(c), \psi_n(c)) + F(\|\psi_n - \psi_{n+1}\|_{E_0}) \\
&= \tau + \alpha(\psi_{n-1}(c), \psi_n(c)) + \beta(\psi_{n-1}(c), \psi_n(c)) + F(\|\psi_n(c) - \psi_{n+1}(c)\|_E) \\
&= \tau + \alpha(\psi_{n-1}(c), \psi_n(c)) + \beta(\psi_{n-1}(c), \psi_n(c)) + F(\|S\psi_{n-1} - S\psi_n\|_E).
\end{aligned}$$

As S is generalized (α,β)-F contraction. So,

$$\begin{aligned}\tau+F(\|\psi_n-\psi_{n+1}\|_{E_0}) &\le F(\max\{\|\psi_{n-1}-\psi_n\|_{E_0},\|\psi_{n-1}(c)-S\psi_{n-1}\|_E,\|\psi_n(c)-S\psi_n\|_E,\\ &\quad \frac{\|\psi_{n-1}(c)-S\psi_n\|_E+\|\psi_n(c)-S\psi_{n-1}\|_E}{2}\})\\ &= F(\max\{\|\psi_{n-1}-\psi_n\|_{E_0},\|\psi_{n-1}(c)-\psi_n(c)\|_E,\|\psi_n(c)-\psi_{n+1}(c)\|_E,\\ &\quad \frac{\|\psi_{n-1}(c)-\psi_{n+1}(c)\|_E}{2}\})\\ &= F(\max\{\|\psi_{n-1}-\psi_n\|_{E_0},\|\psi_{n-1}-\psi_n\|_{E_0},\|\psi_n-\psi_{n+1}\|_{E_0},\\ &\quad \frac{\|\psi_{n-1}-\psi_{n+1}\|_{E_0}}{2}\})\\ &= F(\max\{\|\psi_{n-1}-\psi_n\|_{E_0},\|\psi_n-\psi_{n+1}\|_{E_0},\frac{\|\psi_{n-1}-\psi_{n+1}\|_{E_0}}{2}\})\\ &\le F(\max\{\|\psi_{n-1}-\psi_n\|_{E_0},\|\psi_n-\psi_{n+1}\|_{E_0},\\ &\quad \frac{\|\psi_{n-1}-\psi_n\|_{E_0}+\|\psi_n-\psi_{n+1}\|_{E_0}}{2}\})\\ &= F(\max\{\|\psi_{n-1}-\psi_n\|_{E_0},\|\psi_n-\psi_{n+1}\|_{E_0}\}).\end{aligned}$$

This implies that for each $n\in\mathbb{N}$,

$$\tau+F(\|\psi_n-\psi_{n+1}\|_{E_0})\le F(\max\{\|\psi_{n-1}-\psi_n\|_{E_0},\|\psi_n-\psi_{n+1}\|_{E_0}\}).$$

Now, consider $\max\{\|\psi_{n-1}-\psi_n\|_{E_0},\|\psi_n-\psi_{n+1}\|_{E_0}\}=\|\psi_n-\psi_{n+1}\|_{E_0}$.
Then $\tau+F(\|\psi_n-\psi_{n+1}\|_{E_0})\le F(\|\psi_n-\psi_{n+1}\|_{E_0})$ where $\tau>0$
which is a contradiction. Therefore, $\tau+F(\|\psi_n-\psi_{n+1}\|_{E_0})\le F(\|\psi_{n-1}-\psi_n\|_{E_0})$.
So, $\forall\ n\in\mathbb{N}$

$$\begin{aligned}F(\|\psi_n-\psi_{n+1}\|_{E_0}) &\le F(\|\psi_{n-1}-\psi_n\|_{E_0})-\tau\\ &\le F(\|\psi_{n-2}-\psi_{n-1}\|_{E_0})-2\tau\\ &\le \ldots\\ &\le F(\|\psi_0-\psi_1\|_{E_0})-n\tau.\end{aligned}$$

Clearly,

$$F(\|\psi_n-\psi_{n+1}\|_{E_0})\le F(\|\psi_0-\psi_1\|_{E_0})-n\tau. \tag{3.4}$$

Hence, $\lim_{n\to\infty}F(\|\psi_n-\psi_{n+1}\|_{E_0})=-\infty$.
Since $F\in\Delta$, implies

$$\lim_{n\to\infty}\|\psi_n-\psi_{n+1}\|_{E_0}=0.$$

Again since $F\in\Delta$, $\exists\ p\in(0,1)$ such that

$$\lim_{n\to\infty}\|\psi_n-\psi_{n+1}\|^p_{E_0}F(\|\psi_n-\psi_{n+1}\|_{E_0})=0.$$

From inequation(3.4), $\forall\ n\in\mathbb{N}$
$0\le n\tau\|\psi_n-\psi_{n+1}\|^p_{E_0}\le\|\psi_n-\psi_{n+1}\|^p_{E_0}[F(\|\psi_0-\psi_1\|_{E_0})-F(\|\psi_n-\psi_{n+1}\|_{E_0})]$.
Applying $n\to\infty$, we obtain

$$\lim_{n\to\infty}n\|\psi_n-\psi_{n+1}\|^p_{E_0}=0.$$

Therefore, $\exists\ i\in\mathbb{N}$ such that

$$n\|\psi_n-\psi_{n+1}\|^p_{E_0}\le 1\ \forall\ n\in\mathbb{N}_i.$$

This implies

$$\|\psi_n-\psi_{n+1}\|_{E_0}\le\frac{1}{n^{\frac{1}{p}}}\ \forall\ n\in\mathbb{N}_i.$$

Thus for each $n>m\ge i$, we obtain

$$\|\psi_m-\psi_n\|_{E_0}\le\sum_{j=m}^{n-1}\|\psi_j-\psi_{j+1}\|_{E_0}\le\sum_{j=m}^{n-1}\frac{1}{j^{\frac{1}{p}}}.$$

Now, $\sum_{j\ge1}\frac{1}{j^{\frac{1}{p}}}$ converges ($\because 0<p<1$), and so, $\|\psi_m-\psi_n\|_{E_0}\to0$ when $m,n\to\infty$.

Thus ψ_n is a Cauchy sequence. Since R_c^0 is complete, $\exists\ \psi^* \in R_c^0$ such that $\psi_n \to \psi^*$ as $n \to \infty$. Hence condition (k2) proved.
(iii) Finally, let if possible condition (k3) is not true, i.e., $\forall\ \in \mathbb{N},\ \exists\ m > n$ such that

$$S\psi^* = S\psi_m = \psi_{m+1}(c)$$

which implies that $\exists$ an infinite sequence $\{p(n)\}$ in such a way that

$$S\psi^* = \psi_{p(n)}(c)\ \forall\ n \in \mathbb{N}.$$

As $n \to \infty$, we obtain $S\psi^* = \psi^*(c)$, which is a contradiction. Hence condition (k3) holds. □

Theorem 3.2 Supppose $\alpha, \beta : E \times E \to \mathbb{R}_0^+$ are nonself mappings with F as a Wardowski function. Assume

(i) S is (α, β)-admissible;

(ii) S is an (α, β)-F contraction;

(iii) $\exists\ \psi_0 \in R_c$ such that $\alpha(\psi_0(c), S\psi_0) \geq 0$ and $\beta(\psi_0(c), S\psi_0) \geq 0$;

(iv) if $\{\psi_n\}$ is a sequence in E_0 s.t. $\psi_n \to \psi$ when $n \to \infty$ and $\alpha(\psi_n(c), \psi_{n+1}(c)) \geq 0$, $\beta(\psi_n(c), \psi_{n+1}(c)) \geq 0$ for each $n \in \mathbb{N}$, then $\alpha(\psi_n(c), \psi(c)) \geq 0$ and $\beta(\psi_n(c), \psi(c)) \geq 0$.

Then, in R_c^0, S has a PPF dependent fixed point.

Proof: Let if possible the result does not hold. Since F is strictly increasing, every (α, β)-F contraction is generalized (α, β)-F contraction. Then, all the criteria of Theorem 1 satisfies and so $\exists$ a sequence $\{\psi_n\}$ in R_c^0, a $\psi^* \in R_c^0$ and $l \in \mathbb{N}$ s.t.
(k1) $S\psi_n = \psi_{n+1}(c)$ and $\alpha(\psi_n(c), \psi_{n+1}(c)) \geq 0$, $\beta(\psi_n(c), \psi_{n+1}(c)) \geq 0\ \ \forall n \in \mathbb{N}$;
(k2) $\psi_n \to \psi^*$ as $n \to \infty$;
(k3) $S\psi_n \neq S\psi^*$ and so $\psi_n \neq \psi^*$ for all $n \in \mathbb{N}_l$.
S is an (α, β)-F contraction. So, $\forall\ n \in \mathbb{N}_l$

$$F(\|S\psi_n - S\psi^*\|_E \leq \tau + \alpha(\psi_n(c), \psi^*(c)) + \beta(\psi_n(c), \psi^*(c)) + F(\|S\psi_n - S\psi^*\|_E) \leq F(\|\psi_n - \psi^*\|_{E_0}).$$

Since $F \in \Delta$, so, for each $n \in \mathbb{N}_l$,

$$\|S\psi_n - S\psi^*\|_E \leq \|\psi_n - \psi^*\|_{E_0}.$$

Hence,

$$\begin{aligned}\|S\psi^* - \psi^*(c)\|_E &\leq \|S\psi^* - S\psi_n\|_E + \|S\psi_n - \psi^*(c)\|_E \\ &= \|S\psi^* - S\psi_n\|_E + \|\psi_{n+1}(c) - \psi^*(c)\|_E \\ &\leq \|\psi^* - \psi_n\|_{E_0} + \|\psi_{n+1} - \psi^*\|_{E_0}.\end{aligned}$$

With $n \to \infty$, $\|S\psi^* - \psi^*(c)\|_E = 0, \quad i.e., \quad S\psi^* = \psi^*(c)$
This is a contradiction. Hence it completes the proof. □

Example 3.2 Suppose $(E, \|.\|_E)$ is a Banach space, where $\|x\|_E = |x|,\ E = \mathbb{R}$ and $E_0 = C([0,1], E)$ denotes the set of all continuous function from [0,1] to E equipped the supremum norm $\|.\|_{E_0}$ defined as

$$\|\psi\|_{E_0} = \sup_{t \in I} \|\psi(t)\|_E.$$

Define $\alpha, \beta : E \times E \to \mathbb{R}_0^+,\ S : E_0 \to E$ and $F : \mathbb{R}_0^+ \to \mathbb{R}$ by

$$S\psi = \begin{cases} [\psi(1)]^2 + 1 & \text{if } \psi(1) < 0 \\ \frac{e^{-\tau-5}}{8}[\psi(1)]^2 & \text{if } 0 \leq \psi(1) \leq 1 \\ \psi(1) + 5 & \text{if } \psi(1) > 0 \end{cases}$$

where $\tau > 0$.

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in [0,1] \\ 0 & \text{othewise} \end{cases},$$

$$\beta(x,y) = \begin{cases} 3 & \text{if } x,y \in [0,1] \\ 0 & \text{othewise} \end{cases}$$

and $F(u) = \log u$.
Now assume that $\alpha(\psi(1), \xi(1)) > 0$ and $\beta(\psi(1), \xi(1)) > 0$. Then, $0 \le \psi(1) \le 1$ and $0 \le \xi(1) \le 1$ resulting in $0 \le S\psi = \frac{e^{-\tau-5}}{8}[\psi(1)]^2 \le 1$ and $0 \le S\xi = \frac{e^{-\tau-5}}{8}[\xi(1)]^2 \le 1$, i.e.,

$$\alpha(S\psi, S\xi) = 2, \ \beta(S\psi, S\xi) = 3$$

which implies that $\alpha(S\psi, S\xi) \ge 0$ and $\beta(S\psi, S\xi) \ge 0$. So S is (α, β)-admissible.
Assume $\{\psi_n\}$ is a sequence in E_0 s.t. $\psi_n \to \psi$ as $n \to \infty$
and $\alpha(\psi_n(1), \psi_{n+1}(1)) \ge 0, \ \beta(\psi_n(1), \psi_{n+1}(1)) \ge 0 \ \forall \ n \in \mathbb{N}$.
Now,
$0 \le \psi(1) \le 1$ $[\because 0 \le \psi_n, \xi_n \le 1 \ \forall \ n \in \mathbb{N}$ and $\psi_n \to \psi$ as $n \to \infty]$
which means $\alpha(\psi_n(1), \psi(1)) \ge 0, \ \beta(\psi_n(1), \psi(1)) \ge 0 \ \forall \ n \in \mathbb{N}$.
Clearly, $\alpha(0, S0) \ge 0$ and $\beta(0, S0) \ge 0$.
Now, let $0 \le \psi(1) \le 1$ & $0 \le \xi(1) \le 1$. Then $\alpha(\psi(1), \xi(1)) = 2$ & $\beta(\psi(1), \xi(1)) = 3$
and $0 \le S\psi = \frac{e^{-\tau-5}}{8}[\psi(1)]^2 \le 1$ & $0 \le S\xi = \frac{e^{-\tau-5}}{8}[\xi(1)]^2 \le 1$
and

$$\begin{aligned} \|S\psi - S\xi\|_E &= \frac{e^{-\tau-5}}{8}|[\psi(1)]^2 - [\xi(1)]^2| \\ &= \frac{e^{-\tau-5}}{8}|\psi(1) + \xi(1)| \ |\psi(1) - \xi(1)| \\ &\le e^{-\tau-5}|\psi(1) - \xi(1)| \\ &\le e^{-\tau-5} \sup_{t \in [0,1]} |\psi(t) - \xi(t)| \\ &= e^{-\tau-5}\|\psi - \xi\|_{E_0}. \end{aligned}$$

Therefore,

$$\begin{gathered} \tau + \alpha(\psi(1), \xi(1)) + \beta(\psi(1), \xi(1)) + F(\|S\psi - S\xi\|_E) - F\|\psi - \xi\|_{E_0} \le 0 \\ \Rightarrow \tau + 2 + 3 + \log(\|S\psi - S\xi\|_E) - \log\|\psi - \xi\|_{E_0} \le 0 \\ \Rightarrow \tau + 5 + \log e^{-\tau-5}\|\psi - \xi\|_{E_0} - \log\|\psi - \xi\|_{E_0} \le 0 \\ \Rightarrow 0 \le 0, \end{gathered}$$

which is true.
Otherwise $\alpha(\psi(1), \xi(1)) = 0$ and $\beta(\psi(1), \xi(1)) = 0$.

$$\begin{gathered} \tau + \alpha(\psi(1), \xi(1)) + \beta(\psi(1), \xi(1)) + F(\|S\psi - S\xi\|_E) - F\|\psi - \xi\|_{E_0} \le 0 \\ \Rightarrow \tau + 0 + 0 + \log(\|S\psi - S\xi\|_E) - \log\|\psi - \xi\|_{E_0} \le 0 \\ \Rightarrow \tau + \log e^{-\tau-5}\|\psi - \xi\|_{E_0} - \log\|\psi - \xi\|_{E_0} \le 0. \end{gathered}$$

Again it is true.
So, S is an (α, β)-F contraction. Hence all conditions of theorem (3.7) hold.
Here we can see that S has a fixed point wth PPF dependence. Here $\psi \equiv 0$ is fixed point with PPF dependence of mapping S.

Theorem 3.3 Assume $\alpha, \beta : E \times E \to \mathbb{R}_0^+$ are nonself mappings with F as a Wardowski function

(i) S is (α, β)-admissible;

(ii) F is continuous and S is a generalized (α, β)-F contraction;

(iii) $\exists \ \psi_0 \in R_c$ such that $\alpha(\psi_0(c), S\psi_0) \ge 0$ and $\beta(\psi_0(c), S\psi_0) \ge 0$;

(iv) if $\{\psi_n\}$ is a sequence in E_0 s.t. $\psi_n \to \psi$ as $n \to \infty$ and $\alpha(\psi_n(c), \psi_{n+1}(c)) \geq 0$, $\beta(\psi_n(c), \psi_{n+1}(c)) \geq 0 \ \forall \ n \in \mathbb{N}$, then $\alpha(\psi_n(c), \psi(c)) \geq 0$ and $\beta(\psi_n(c), \psi(c)) \geq 0$.

Then S has a fixed point with PPF dependence in R_c^0.

Proof: Let if possible the result is not true. Here S is a generalized (α, β)-F contraction. So, from Theorem (3.6) $\exists$ a sequence $\{\phi_n\}$ in R_c^0, a $\psi^* \in R_c^0$ and $l \in \mathbb{N}$ s.t.

(k1) $S\psi_n = \psi_{n+1}(c)$ and $\alpha(\psi_n(c), \psi_{n+1}(c)) \geq 0$, $\beta(\psi_n(c), \psi_{n+1}(c)) \geq 0 \ \forall \ n \in \mathbb{N}$;

(k2) $\psi_n \to \psi^*$ as $n \to \infty$;

(k3) $S\psi_n \neq S\psi^*$, and so $\psi_n \neq \psi^*$ for all $n \in \mathbb{N}_l$.

Now, S is an (α, β)-F contraction. So, for each $n \in \mathbb{N}_l$, we get

$$\begin{aligned}
\tau + F(\|\psi_{n+1}(c) - S\psi*\|_E) &\leq \tau + \alpha(\psi_n(c), \psi^*(c)) + \beta(\psi_n(c), \psi^*(c)) + F(\|\psi_{n+1}(c) - S\psi^*\|_E) \\
&= \tau + \alpha(\psi_n(c), \psi^*(c)) + \beta(\psi_n(c), \psi^*(c)) + F(\|S\psi_n - S\psi^*\|_E) \\
&\leq F(\max\{\|\psi_n - \psi^*\|_{E_0}, \|\psi_n(c) - S\psi_n\|_E, \|\psi^*(c) - S\psi^*\|_E, \\
&\quad \frac{\|\psi_n(c) - S\psi^*\|_E + \psi^*(c) - S\psi_n\|_E}{2}\}) \\
&\leq F(\max\{\|\psi_n - \psi^*\|_{E_0}, \|\psi_n(c) - \psi_{n+1}\|_E, \|\psi^*(c) - S\psi^*\|_E, \\
&\quad \frac{\|\psi_n(c) - S\psi^*\|_E + \psi^*(c) - \psi_{n+1}\|_E}{2}\}).
\end{aligned}$$

Now, taking $n \to \infty$,

$$\tau + F(\|\psi^*(c) - S\psi^*\|_E) \leq F(\|\psi^*(c) - S\psi^*\|_E)$$

{because F is continuous}

Which will be hold only if $\|\psi^*(c) - S\psi^*\|_E = 0$. So, $\psi^*(c) = S\psi^*$. That is a contradiction.

Hence, the proof. □

Note: If we take $\beta(\psi, \xi) = 0 \ \forall \ \psi, \xi \in E_0$ in our results, then we find the following corollaries:

Corollary 3.1 "Let $\alpha : E \times E \to \mathbb{R}_0^+$ be the nonself mappings and F be the Wardowski function with the following conditions:

(i) S is α-admissible;

(ii) S is generalized α-F contraction;

(iii) $\exists \ \psi_0 \in R_c$ such that $\alpha(\psi_0(c), S\psi_0) \geq 0$.

In addition suppose:
S has no PPF dependent fixed point in R_c^0, i.e., $S\psi \neq \psi(c)$ for all $\psi \in R_c^0$.
Then, $\exists$ a sequence $\{\psi_n\}$ in R_c^0, $\psi^* \in R_c^0$ and $k \in \mathbb{N}$ such that:

(k1) $S\psi_n = \psi_{n+1}(c)$ and $\alpha(\psi_n(c), \psi_{n+1}(c)) \geq 0$;

(k2) $\psi_n \to \psi^*$ as $n \to \infty$;

(k3) $S\psi_n \neq S\psi^*$, hence $\psi_n \neq \psi^*$ for all $n \in \mathbb{N}_k$".

Corollary 3.2 "Let $\alpha : E \times E \to \mathbb{R}_0^+$ be the nonself mappings and F be the Wardowski function with the following conditions:

(i) S is α-admissible;

(ii) S is an α-F contraction;

(iii) $\exists \ \psi_0 \in R_c$ such that $\alpha(\psi_0(c), S\psi_0) \geq 0$;

(iv) if $\{\psi_n\}$ is a sequence in E_0 such that $\psi_n \to \psi$ as $n \to \infty$ and $\alpha(\psi_n(c), \psi_{n+1}(c)) \geq 0$ for each $n \in \mathbb{N}$, then $\alpha(\psi_n(c), \psi(c)) \geq 0$.

Then S has a PPF dependent fixed point in R_c^0."

4. Conclusion

Inspired by the work of Bernfeld et al. [6] and Kutbi et al. [11], we introduced the idea of (α, β)-F contractive, weak (α, β)-F contractive and generalized (α, β)-F contractive nonself mappings. We proved some fixed point theorems with PPF dependence for these contractive nonself mappings and provided some related examples.
Finally, we anticipate that our primary findings will make a significant contribution to the advancement of PPF dependent fixed point theory.

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