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Fixed Point Theorems for Generalized Weakly Contractive Mapping in S-Metric Space

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ABSTRACT: In this manuscript, we shall prove fixed point results for generalized weakly contractive mapping in complete S-metric spaces. An example is also provided to prove the validity of our results.

Key Words: Unique fixed point, S-metric space, weakly contractive mapping.

Contents

1. Introduction

Let X be a metric space. A map $T : X \to X$ is a contraction, if for each $u, v \in X$, there exists a constant $k \in (0, 1)$ such that

$$d(Tu, Tv) \leq kd(u, v).$$

Alber and Guerre-Delabriere [1] introduced the concept of weakly contractive mappings which is defined below:
A map $T : X \to X$ is a weak contraction if for each $u, v \in X$, there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$, and

$$d(Tu, Tv) \leq d(u, v) - \varphi(d(u, v)).$$

The authors defined above mappings for single-valued maps on Hilbert spaces and showed the existence of fixed points.
Rhoades showed that most results of [1] are still true for any Banach spaces. Weakly contractive maps are related to maps of Boyd and Wong type ones [4] and Reich's type ones [13].
In [15], Sedghi et al., have introduced the concept of an S-metric space which is a generalization of a G-metric space [9] and a D-metric space [5].

2. Preliminaries

In this section, we begin by briefly recalling some basic definitions and results for S-metric spaces that will be required in the sequel.

Definition 2.1 ([15]) *Let X be a non-empty set, an S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $u, v, w, a \in X$,*

i) $S(u, v, w) \geq 0$,

ii) $S(u, v, w) = 0$ *if and only if* $u = v = w$,

iii) $S(u, v, w) \leq S(u, u, a) + S(v, v, a) + S(w, w, a)$.

The pair (X, S) is called an S-metric space.

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Example 2.1 *Let $X = \mathbb{R}^n$ and d is an ordinary metric on X. Then*

$$S(u, v, w) = d(u, v) + d(v, w) + d(u, w),$$

is an S-metric space.

Example 2.2 *Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on X, then*

$$S(u, v, w) = \|v + w - 2u\| + \|v - w\|,$$

is an S-metric space on X.

Definition 2.2 **([15])** *Let (X, S) be an S-metric space.*

i) A sequence $\{u_n\}$ in X converges to $u \in X$ if $S(u_n, u_n, u) \to 0$ as $n \to \infty$.
That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(u_n, u_n, u) < \varepsilon$.
We denote by $\lim\limits_{n\to\infty} u_n = u$.

ii) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$.
That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $S(u_n, u_n, u_m) < \varepsilon$.

iii) The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

Lemma 2.1 **([15])** *In an S-metric space, we have $(u, u, v) = S(v, v, u)$.*

Definition 2.3 **([15])** *Let (X, S) be an S-metric space. For $r > 0$ and $u \in X$, we define the open ball $B_S(u, r)$ and closed ball $B_S[u, r]$ with centre u and radius r respectively as*

$$B_S(u, r) = \{v \in X : S(v, v, u) < r\}$$

and

$$B_S[u, r] = \{v \in X : S(v, v, u) \leq r\}.$$

3. Main Results

In this section, we prove some fixed point theorems for generalized weakly contractive maps in S-metric space and a suitable example is also provided to prove the validity of our results.

Theorem 3.1 *Let (X, S) be a complete S-metric space and T be a self map on X satisfying the following for all $x, y, z \in X$:*

$$S(Tx, Ty, Tz) \leq M(x, y, z) - \varphi(M(x, y, z)), \tag{3.1}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\varphi(t) > 0$ for $t > 0$, $\varphi(0) = 0$ and

$$M(x, y, z) = \max\left\{S(x, y, z), S(Tx, Ty, Tz), \frac{S(x, x, Tx).S(y, y, Ty)}{1 + S(x, y, z) + S(Tx, Ty, Tz)}, \right.$$
$$\left. \frac{S(y, y, Ty).S(z, z, Tz)}{1 + S(x, y, z) + S(Tx, Ty, Tz)}, \frac{S(z, z, Tz).S(x, x, Tx)}{1 + S(x, y, z) + S(Tx, Ty, Tz)}\right\}. \tag{3.2}$$

Then there exists a unique point $u \in X$ such that $Tu = u$.

Proof: Suppose $x_0 \in X$ is an arbitrary point. Now, we can choose a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n > 0$.

By equations (3.1), (3.2) and property of φ, we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq M(x_{n-1}, x_{n-1}, x_n) - \varphi(M(x_{n-1}, x_{n-1}, x_n)), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n-1}, x_n) &= \max\Bigg\{ S(x_{n-1}, x_{n-1}, x_n), S(Tx_{n-1}, Tx_{n-1}, Tx_n), \\ &\qquad \frac{S(x_{n-1}, x_{n-1}, Tx_{n-1}).S(x_{n-1}, x_{n-1}, Tx_{n-1})}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(Tx_{n-1}, Tx_{n-1}, Tx_n)}, \\ &\qquad \frac{S(x_{n-1}, x_{n-1}, Tx_{n-1}).S(x_n, x_n, Tx_n)}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(Tx_{n-1}, Tx_{n-1}, Tx_n)}, \\ &\qquad \frac{S(x_n, x_n, Tx_n).S(x_{n-1}, x_{n-1}, Tx_{n-1})}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(Tx_{n-1}, Tx_{n-1}, Tx_n)} \Bigg\} \\ &= \max\Bigg\{ S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\qquad \frac{S(x_{n-1}, x_{n-1}, x_n).S(x_{n-1}, x_{n-1}, x_n)}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})}, \\ &\qquad \frac{S(x_{n-1}, x_{n-1}, x_n).S(x_n, x_n, x_{n+1})}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})}, \\ &\qquad \frac{S(x_n, x_n, x_{n+1}).S(x_{n-1}, x_{n-1}, x_n)}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})} \Bigg\} \\ &= \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}. \end{aligned}$$

So, we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} \\ &\qquad - \varphi(\max\{S(x_{n-1}, x_{n-1} x_n), S(x_n, x_n, x_{n+1})\}), \end{aligned} \tag{3.4}$$

If there exists an $n \geq 0$ such that $S(x_n, x_n, x_{n+1}) = 0$, then x_n is a fixed point of T. Therefore, we shall assume that $S(x_n, x_n, x_{n+1}) \neq 0$ for every n.
Now, if $S(x_n, x_n, x_{n+1}) > S(x_{n-1}, x_{n-1}, x_n)$ for some n, then

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq S(x_n, x_n, x_{n+1}) - \varphi(S(x_n, x_n, x_{n+1})) \\ &< S(x_n, x_n, x_{n+1}), \end{aligned}$$

implies that

$$S(x_n, x_n, x_{n+1}) < S(x_n, x_n, x_{n+1}),$$

a contradiction.
Thus, for all n,

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq M(x_{n-1}, x_{n-1}, x_n) - \varphi(M(x_{n-1}, x_{n-1}, x_n)) \\ &\leq M(x_{n-1}, x_{n-1}, x_n) \\ &= S(x_{n-1}, x_{n-1}, x_n), \end{aligned}$$

implies that

$$S(x_n, x_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, x_n).$$

So, $\{S(x_n, x_n, x_{n+1})\}$ is monotonically non-increasing sequence of positive reals and bounded below. Hence, there exists a number $r \geq 0$ such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = \lim_{n\to\infty} M(x_{n-1}, x_{n-1}, x_n) = r.$$

From the lower semi continuity of φ, we have

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi(M(x_{n-1}, x_{n-1}, x_n)).$$

We claim that $r = 0$.
In fact, taking upper limits as $n \to \infty$ on both side of the following inequality

$$S(x_n, x_n, x_{n+1}) \leq M(x_{n-1}, x_{n-1}, x_n) - \varphi(M(x_{n-1}, x_{n-1}, x_n)).$$

We have

$$\begin{aligned} r &\leq r - \liminf_{n\to\infty} \varphi(M(x_{n-1}, x_{n-1}, x_n)) \\ &\leq r - \varphi(r); \end{aligned}$$

implies that

$$\varphi(r) \leq 0.$$

Thus $\varphi(r) = 0$.
From the definition of φ, we have $r = 0$.
Hence,

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{3.5}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Let us consider that it is not true. Then there is an $\varepsilon > 0$ such that, for an integer k, there exists integers $m(k) > n(k) > k$ such that

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{3.6}$$

For each integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying equation (3.6) and such that

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)-1}) < \varepsilon. \tag{3.7}$$

Then

$$\begin{aligned} \varepsilon &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\ &\leq S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) \\ &= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}). \end{aligned}$$

Now, by equations (3.5) and (3.7), it follows that

$$\lim_{n\to\infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{3.8}$$

Since $m(k)$ is the least positive integer exceeding $n(k)$ satisfying equations (3.6), (3.8) and also using equation (3.1), we have

$$\begin{aligned}
\varepsilon &\le S(x_{n(k)}, x_{n(k)}, x_{m(k)+1})\\
&\le S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{n(k)}, x_{(k)}, x_{n(k)+1}) + S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1})\\
&= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1})\\
&= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1})\\
&\le 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + M(x_{n(k)}x_{n(k)}, x_{m(k)})\\
&= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + \max\Bigg\{ S(x_{n(k)}, x_{n(k)}, x_{m(k)}), S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}),\\
&\qquad \frac{S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}).S(x_{n(k)}, x_{n(k)}, x_{m(k)})}{1 + S(x_{n(k)}x_{n(k)}, x_{m(k)}) + S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1})},\\
&\qquad \frac{S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}).S(x_{m(k)}, x_{m(k)}, x_{m(k)+1})}{1 + S(x_{n(k)}x_{n(k)}, x_{m(k)}) + S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1})},\\
&\qquad \frac{S(x_{m(k)}, x_{m(k)}, x_{m(k)+1}).S(x_{n(k)}, x_{n(k)}, x_{n(k)+1})}{1 + S(x_{n(k)}x_{n(k)}, x_{m(k)}) + S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1})}\Bigg\}.
\end{aligned}$$

Letting $k \to \infty$ and using equations (3.5) and (3.8), we have

$$\varepsilon \le \lim_{n\to\infty} M(x_{n(k)}, x_{n(k)}, x_{m(k)}) \le \varepsilon\,.$$

Therefore,

$$\lim_{n\to\infty} M(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \varepsilon.$$

Since φ is lower semi continuous, we have

$$\varphi(\varepsilon) \le \liminf_{n\to\infty} \varphi(M(x_{n(k)}, x_{n(k)}, x_{m(k)})).$$

From equation (3.1), we get

$$S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) \le M(x_{n(k)}, x_{n(k)}, x_{m(k)}) - \varphi(M(x_{n(k)}, x_{n(k)}, x_{m(k)})).$$

Taking upper limit as $k \to \infty$, we have

$$\begin{aligned}
\varepsilon &\le \varepsilon - \inf \varphi(M(x_{n(k)}, x_{n(k)}, x_{m(k)}))\\
&\le \varepsilon - \varphi(\varepsilon);
\end{aligned}$$

implies that

$$\begin{aligned}
\varepsilon &\le \varepsilon - \varphi(\varepsilon)\\
&< \varepsilon,
\end{aligned}$$

a contradiction.
Therefore, $\{x_n\}$ is a Cauchy sequence.
By completeness of the space X, there exists a $u \in X$ such that $x_n \to u$ as $n \to \infty$.
Now, we shall prove that $u = Tu$. Indeed, suppose that $u \neq Tu$.
Then

$$S(x_n, x_n, Tu) \le M(x_{n-1}, x_{n-1}, u) - \varphi(M(x_{n-1}, x_{n-1}, u)),$$

where

$$M(x_{n-1},x_{n-1},u) = \max\Bigg\{S(x_{n-1},x_{n-1},u), S(x_n,x_n,Tu), \frac{S(x_{n-1},x_{n-1},Tx_{n-1}).S(x_{n-1},x_{n-1},Tx_{n-1})}{1+S(x_{n-1},x_{n-1},u)+S(Tx_{n-1},Tx_{n-1},Tu)}, \frac{S(x_{n-1},x_{n-1},Tx_{n-1}).S(u,u,Tu)}{1+S(x_{n-1},x_{n-1},u)+S(Tx_{n-1},Tx_{n-1},Tu)}, \frac{S(u,u,Tu).S(x_{n-1},x_{n-1},Tx_{n-1})}{1+S(x_{n-1},x_{n-1},u)+S(Tx_{n-1},Tx_{n-1},Tu)}\Bigg\}$$
$$= S(u,u,Tu).$$

Taking the upper limit as $n \to \infty$, we have

$$\begin{aligned} S(u,u,Tu) &\leq S(u,u,Tu) - \varphi(S(u,u,Tu)) \\ &< S(u,u,Tu), \end{aligned}$$

a contradiction.
Thus $u = Tu$.

Uniqueness:
Suppose that $u \neq v$ and $Tv = v$.
Then equation (3.1) implies that

$$\begin{aligned} S(u,u,v) &= S(Tu,Tu,Tv) \\ &\leq M(u,u,v) - \varphi(M(u,u,v)), \end{aligned}$$

where

$$M(u,u,v) = \max\Bigg\{S(u,u,v), S(Tu,Tu,Tv), \frac{S(u,u,Tu).S(u,u,Tu)}{1+S(u,u,v)+S(Tu,Tu,Tv)}, \frac{S(u,u,Tu).S(v,v,Tv)}{1+S(u,u,v)+S(Tu,Tu,Tv)}, \frac{S(v,v,Tv).S(u,u,Tu)}{1+S(u,u,v)+S(Tu,Tu,Tv)}\Bigg\}$$
$$= S(u,u,v).$$

Therefore,

$$\begin{aligned} S(u,u,v) &\leq S(u,u,v) - \varphi(S(u,u,v)) \\ &< S(u,u,v), \end{aligned}$$

a contradiction.
Hence $u = v$. □

Corollary 3.1 *Let (X,S) be a complete S-metric space and $T : X \to X$ be a function such that for all $x,y,z \in X$,*

$$S(Tx,Ty,Tz) \leq S(x,y,z) - \varphi(S(x,y,z)),$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$. Then there exists a unique point $u \in X$ such that $Tu = u$.

Proof: If we take $M(x,y,z) = S(x,y,z)$ in Theorem 3.1, then the result is proved. □

Example 3.1 *Let $X=[0,2]$ and $S: X^3 \to \mathbb{R}^+$ be given by*

$$S(x,y,z)=\begin{cases} |x-z|+|y-z|, & \text{if } x,y,z\in[0,2),\\ 2, & \text{if } x=2 \text{ or } y=2 \text{ or } z=2,\end{cases}$$

for all $x,y,z\in X$. Then (X,S) is a complete S-metric space.
Let the mapping $T: X\to X$ be given by

$$T(x)=\begin{cases} \frac{3}{2}, & \text{if } x,y,z\in[0,2),\\ \frac{7}{4}, & \text{if } x=y=z=2.\end{cases}$$

Also, $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi continuous function with $\varphi(t)=\frac{t^2}{4}$.
Now, we consider the following cases for verification of inequality (3.1) of Theorem 3.1.

Case 1. If $x,y\in\left[0,\frac{3}{2}\right]$, $z\in\left[\frac{3}{2},2\right)$ or $z\in\left[0,\frac{3}{2}\right]$, $x,y\in\left[\frac{3}{2},2\right)$. Then
Firstly, we consider

$$x,y\in\left[0,\frac{3}{2}\right],\quad z\in\left[\frac{3}{2},2\right).$$

Now,

$$S(Tx,Ty,Tz)=S\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)=0.$$

Taking $x=1$, $y=\frac{3}{2}$, $z=\frac{3}{2}$.
Also,

$$S(x,y,z)=S\left(1,\frac{3}{2},\frac{3}{2}\right)=\frac{1}{2},$$
$$S(x,x,Tx)=S\left(1,1,\frac{3}{2}\right)=1,$$
$$S(y,y,Ty)=S(z,z,Tz)=S\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)=0.$$

Therefore, from equation (3.1),

$$0\le\max\left\{\frac{1}{2},0,0,0,0\right\}-\varphi\left(\max\left\{\frac{1}{2},0,0,0,0\right\}\right),$$

implies that

$$0\le\frac{1}{2}-\varphi\left(\frac{1}{2}\right).$$

Here

$$\varphi\left(\frac{1}{2}\right)=\frac{1}{16}.$$

Therefore,

$$0<\frac{1}{2}-\frac{1}{16}.$$

that is,

$$0<\frac{7}{16}.$$

Thus all the conditions of Theorem 3.1 are satisfied.
Similarly, If we take

$$z\in\left[0,\frac{3}{2}\right],\quad x,y\in\left[\frac{3}{2},2\right),$$

then all the conditions of Theorem 3.1 are satisfied.

Case 2. If $x, y \in \left[0, \frac{3}{2}\right]$ *and* $z = 2$.
Then

$$S(Tx, Ty, Tz) = S\left(\frac{3}{2}, \frac{3}{2}, \frac{7}{4}\right) = \frac{1}{2}.$$

Taking $x = y = \frac{3}{2}$,

$$\begin{aligned} &S(x, y, z) = 2, \quad S(x, x, Tx) = 0, \quad S(y, y, Ty) = 0, \\ &S(z, z, Tz) = S\left(2, 2, \frac{7}{4}\right) = \frac{1}{2}. \end{aligned}$$

Therefore, from equation (3.1),

$$\frac{1}{2} \leq \max\left\{2, \frac{1}{2}, 0, 0, 0\right\} - \varphi\left(\max\left\{2, \frac{1}{2}, 0, 0, 0\right\}\right),$$

implies that

$$\frac{1}{2} \leq 2 - \varphi(2).$$

Here $\varphi(2) = 1$.
Therefore,

$$\frac{1}{2} < 2 - 1,$$

that is

$$\frac{1}{2} < 1.$$

Thus all the conditions of Theorem 3.1 are satisfied.
Hence by applying Theorem 3.1, T *has a unique fixed point.*
Indeed, $\frac{3}{2} \in X$ *is the unique fixed point of* T.

Case 3. If $x, z \in \left[0, \frac{3}{2}\right]$ *and* $y = 2$.
Then

$$S(Tx, Ty, Tz) = S\left(\frac{3}{2}, \frac{7}{4}, \frac{3}{2}\right) = \frac{1}{4}.$$

Taking $x = z = \frac{3}{2}$,

$$\begin{aligned} &S(x, y, z) = 2, \quad S(x, x, Tx) = 0, \quad S(z, z, Tz) = 0, \\ &S(y, y, Ty) = S\left(2, 2, \frac{7}{4}\right) = \frac{1}{2}. \end{aligned}$$

Therefore, from equation (3.1),

$$\frac{1}{4} \leq \max\left\{2, \frac{1}{4}, 0, 0, 0\right\} - \varphi\left(\max\left\{2, \frac{1}{4}, 0, 0, 0\right\}\right),$$

implies that

$$\frac{1}{4} \leq 2 - \varphi(2).$$

Here

$$\varphi(2) = 1.$$

Therefore,

$$\frac{1}{4} < 2 - 1,$$

that is

$$\frac{1}{4} < 1.$$

Thus all the conditions of Theorem 3.1 are satisfied.
Hence by applying Theorem 3.1, T has a unique fixed point.
Indeed, $\frac{3}{2} \in X$ is the unique fixed point of T.

Case 4. If $y, z \in \left[0, \frac{3}{2}\right]$ and $x = 2$.
Then

$$S(Tx, Ty, Tz) = S\left(\frac{7}{4}, \frac{3}{2}, \frac{7}{4}\right) = \frac{1}{4}.$$

Taking $y = z = \frac{3}{2}$,

$$S(x, y, z) = S\left(2, \frac{3}{2}, \frac{3}{2}\right) = 2, \quad S(y, y, Ty) = 0, \quad S(z, z, Tz) = 0,$$
$$S(x, x, Tx) = S\left(2, 2, \frac{7}{4}\right) = \frac{1}{2}.$$

Therefore, from equation (3.1),

$$\frac{1}{4} \leq \max\left\{2, \frac{1}{4}, 0, 0, 0\right\} - \varphi\left(\max\left\{2, \frac{1}{4}, 0, 0, 0\right\}\right),$$

implies that

$$\frac{1}{4} \leq 2 - \varphi(2).$$

Here

$$\varphi(2) = 1.$$

Therefore,

$$\frac{1}{4} < 2 - 1,$$

that is

$$\frac{1}{4} < 1.$$

Thus all the conditions of Theorem 3.1 are satisfied.
Hence by applying Theorem 3.1, T has a unique fixed point.
Indeed, $\frac{3}{2} \in X$ is the unique fixed point of T in this case.
Considering all the above cases, we conclude that the inequality used in Theorem 3.1 remains valid for mapping constructed in the above example and therefore by applying Theorem 3.1.
T has a unique fixed point. One can easily see that $u = \frac{3}{2} \in X$ is the unique fixed point of T.

Theorem 3.2 *Let (X, S) be a complete S-metric space and T be a self-map on X satisfying the following for all $x, y, z \in X$:*

$$S(Tx, Tx, Ty) \leq N(x, x, y) - \varphi(N(x, x, y)), \tag{3.9}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\varphi(t) > 0$ for $t > 0$,

$$\varphi(0) = 0$$

and

$$N(x, x, y) = \max\left\{S(x, x, y), S(Tx, Tx, Ty), \frac{S(x, x, Tx).S(x, x, Tx)}{1 + S(x, x, y) + S(Tx, Tx, Ty)}, \frac{S(x, x, Tx).S(y, y, Ty)}{1 + S(x, x, y) + S(Tx, Tx, Ty)}\right\}. \tag{3.10}$$

Then there exists a unique point $u \in X$ such that $Tu = u$.

Proof: Suppose $x_0 \in X$ is an arbitrary point. Now, we can choose a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n > 0$.
By equations (3.9), (3.10) and property of φ, we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq N(x_{n-1}, x_{n-1}, x_n) - \varphi(N(x_{n-1}, x_{n-1}, x_n)), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} N(x_{n-1}, x_{n-1}, x_n) = \max\Bigg\{ & S(x_{n-1}, x_{n-1}, x_n), S(Tx_{n-1}, Tx_{n-1}, Tx_n), \\ & \frac{S(x_{n-1}, x_{n-1}, Tx_{n-1}).S(x_{n-1}, x_{n-1}, Tx_{n-1})}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(Tx_{n-1}, Tx_{n-1}, Tx_n)}, \\ & \frac{S(x_{n-1}, x_{n-1}, Tx_{n-1}).S(x_n, x_n, Tx_n)}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(Tx_{n-1}, Tx_{n-1}, Tx_n)} \Bigg\} \\ = \max\Bigg\{ & S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ & \frac{S(x_{n-1}, x_{n-1}, x_n).S(x_{n-1}, x_{n-1}, x_n)}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})}, \\ & \frac{S(x_{n-1}, x_{n-1}, x_n).S(x_n, x_n, x_{n+1})}{1 + S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})} \Bigg\} \\ = \max\{ & S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}. \end{aligned}$$

So, we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) \leq \max\{& S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} \\ & - \varphi(\max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}). \end{aligned} \tag{3.12}$$

Now, if $S(x_n, x_n, x_{n+1}) > S(x_{n-1}, x_{n-1}, x_n)$, for some n, then

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq S(x_n, x_n, x_{n+1}) - \varphi(S(x_n, x_n, x_{n+1})) \\ &< S(x_n, x_n, x_{n+1}), \end{aligned}$$

a contradiction.
Thus, for all n,

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq N(x_{n-1}, x_{n-1}, x_n) - \varphi(N(x_{n-1}, x_{n-1}, x_n)) \\ &\leq N(x_{n-1}, x_{n-1}, x_n) \\ &= S(x_{n-1}, x_{n-1}, x_n), \end{aligned}$$

that is

$$S(x_n, x_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, x_n). \tag{3.13}$$

So, $\{S(x_n, x_n, x_{n+1})\}$ is monotonically non-increasing sequence of positive reals and bounded below. Hence, there exists a number $r \geq 0$ such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = \lim_{n\to\infty} N(x_{n-1}, x_{n-1}, x_n) = r.$$

From the lower semi-continuity of φ, we have

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi(N(x_{n-1}, x_{n-1}, x_n)).$$

We claim that $r = 0$.
In fact, taking upper limits as $n \to \infty$ on both side of the following inequality

$$S(x_n, x_n, x_{n+1}) \leq N(x_{n-1}, x_{n-1}, x_n) - \varphi(N(x_{n-1}, x_{n-1}, x_n)),$$

We have

$$\begin{aligned} r &\leq r - \liminf_{n\to\infty} \varphi(N(x_{n-1}, x_{n-1}, x_n)) \\ &\leq r - \varphi(r); \end{aligned}$$

that is

$$\varphi(r) \leq 0.$$

Thus $\varphi(r) = 0$.
From the definition of φ, we have $r = 0$.
Hence

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{3.14}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Let us assume that it is not true. Then there is an $\varepsilon > 0$ such that, for an integer k, there exist integers $m(k) > n(k) > k$ such that

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{3.15}$$

For each integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying equation (3.15) and such that

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)-1}) < \varepsilon. \tag{3.16}$$

Then

$$\begin{aligned} \varepsilon &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\ &\leq S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)+1}) \\ &= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)}). \end{aligned}$$

And, by equations (3.14) and (3.16), it follows that

$$\lim_{k\to\infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{3.17}$$

Since $m(k)$ is the least positive integer exceeding $n(k)$ satisfying equations (3.15), (3.17) and also using equation (3.9), we have

$$\begin{aligned} \varepsilon &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)+1}) \\ &\leq S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) \\ &= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) \\ &= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) \\ &\leq 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + N(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\ &= 2S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) \\ &\quad + \max\Big\{ S(x_{n(k)}, x_{n(k)}, x_{m(k)}), S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}), \\ &\qquad \frac{S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}).S(x_{n(k)}, x_{n(k)}, x_{n(k)+1})}{1 + S(x_{n(k)}, x_{n(k)}, x_{m(k)}) + S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1})}, \\ &\qquad \frac{S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}).S(x_{m(k)}, x_{m(k)}, x_{m(k)+1})}{1 + S(x_{n(k)}, x_{n(k)}, x_{m(k)}) + S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1})} \Big\}. \end{aligned}$$

Letting $k \to \infty$ and using equations (3.14) and (3.17), we have

$$\varepsilon \leq \lim_{k\to\infty} N(x_{n(k)}, x_{n(k)}, x_{m(k)}) \leq \varepsilon.$$

Therefore,

$$\lim_{k\to\infty} N(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \varepsilon.$$

Since φ is lower semi-continuous, we have

$$\varphi(\varepsilon) \leq \liminf_{n\to\infty} \varphi(N(x_{n(k)}, x_{n(k)}, x_{m(k)})).$$

From equation (3.9), we get

$$S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) \leq N(x_{n(k)}, x_{n(k)}, x_{m(k)}) - \varphi(N(x_{n(k)}, x_{n(k)}, x_{m(k)})).$$

Taking upper limit as $k \to \infty$, we have

$$\begin{aligned}\varepsilon &\leq \varepsilon - \liminf_{n\to\infty} \varphi(N(x_{n(k)}, x_{n(k)}, x_{m(k)})) \\ &\leq \varepsilon - \varphi(\varepsilon);\end{aligned}$$

that is

$$\begin{aligned}\varepsilon &\leq \varepsilon - \varphi(\varepsilon) \\ &< \varepsilon,\end{aligned}$$

a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By completeness of the space X, there exists a $u \in X$ such that $x_n \to u$ as $n \to \infty$.
Now, we shall prove that $u = Tu$. Indeed, suppose that $u \neq Tu$. Then

$$S(x_n, x_n, Tu) \leq N(x_{n-1}, x_{n-1}, u) - \varphi(N(x_{n-1}, x_{n-1}, u)),$$

where

$$\begin{aligned}N(x_{n-1}, x_{n-1}, u) = \max\Bigg\{ & S(x_{n-1}, x_{n-1}, u), S(x_n, x_n, Tu), \\ & \frac{S(x_{n-1}, x_{n-1}, Tx_{n-1}).S(x_{n-1}, x_{n-1}, Tx_{n-1})}{1 + S(x_{n-1}, x_{n-1}, u) + S(x_n, x_n, Tu)}, \\ & \frac{S(x_{n-1}, x_{n-1}, Tx_{n-1}).S(u, u, Tu)}{1 + S(x_{n-1}, x_{n-1}, u) + S(x_n, x_n, Tu)}\Bigg\}.\end{aligned}$$

Taking upper limit as $n \to \infty$, we get

$$\begin{aligned}S(u, u, Tu) &\leq S(u, u, Tu) - \varphi(S(u, u, Tu)) \\ &< S(u, u, Tu),\end{aligned}$$

a contradiction.
Thus $u = Tu$.

Uniqueness:
Suppose that $u \neq v$ and $Tv = v$.
Then (3.9) implies that

$$\begin{aligned}S(u, u, v) &= S(Tu, Tu, Tv) \\ &\leq N(u, u, v) - \varphi(N(u, u, v)),\end{aligned}$$

where

$$\begin{aligned} N(u,u,v) = \max \Bigg\{ & S(u,u,v), S(Tu,Tu,Tv), \\ & \frac{S(u,u,Tu).S(u,u,Tu)}{1+S(u,u,v)+S(Tu,Tu,Tv)}, \\ & \frac{S(u,u,Tu).S(v,v,Tv)}{1+S(u,u,v)+S(Tu,Tu,Tv)} \Bigg\} \\ = {} & S(u,u,v). \end{aligned}$$

Therefore,

$$\begin{aligned} S(u,u,v) &\leq S(u,u,v) - \varphi(S(u,u,v)) \\ &< S(u,u,v), \end{aligned}$$

a contradiction.
Hence $u = v$. □

References

1. Alber, Ya. I., Guerre-Delabriere, S., Principle of weakly contractive maps in Hilbert spaces, New results in operator theory and its applications, Oper. Theory Adv. Appl., Birkhauser, Basel, 98, 7-22 (1997).

2. Ansari, A. H., Gupta V., Mani, N., C-class functions on some coupled fixed point theorems in partially ordered S-metric spaces, Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics, 68(2), 1694-1708 (2019).

3. Azam, A., Shakeel, M., Weakly contractive maps and common fixed points, Mat. Vesnik, 60, 101-106 (2008).

4. Bae, J. S., Fixed point theorems for weakly contractive multivalued maps, J. Math. Anal. Appl., 284, 690-697 (2003).

5. Boyd, D. W., Wong, J. S. W., On non-linear contractions, Proc. Amer. Math. Soc., 20, 458-464 (1969).

6. Dhage, B. C., Generalized metric spaces mappings with fixed point, Bull. Calcutta Math. Soc., 84, 329-336 (1992).

7. Dung, N. V., On coupled common fixed points for mixed weakly monotone mappings in partially ordered S-metric spaces, Fixed point Theory Appl., 48, 1-17 (2013).

8. Gupta, V., Deep, R., Some coupled fixed point theorems in partially ordered S-metric spaces, Miskolc Mathematical Notes, 16(1), 181-194 (2015).

9. Kim, J. K., Sedghi, S., Shobkolaei, N., Common fixed point theorems for the R-weakly commuting mappings in S-metric spaces, J. Comput. Anal. Appl., 19, 751-759 (2015).

10. Kumar, M., Singh, P., Devi, S., Some fixed point theorems in S-metric spaces with new contractive mappings, J. Math. Comput. Sci., 11(6), 7414-7430 (2021).

11. Kumar, M., Devi, S., Singh, P., Fixed point theorems by using altering distance function in S-metric spaces, Communications in Mathematics and Applications, 13(2), 553-573 (2022).

12. Kumar, M., Devi, S., Some fixed point theorems in S-metric spaces via simulation function, Asian Research Journal of Mathematics, 19(9), 13-24 (2023).

13. Mustafa, Z., Sims, B., Some remarks concerning D-metric spaces, International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, 189-198 (2004).

14. Mustafa, Z., Sims, B., A new approach to generalized metric spaces, J. Nonlinear Convex Anal., 7, 289-297(2006).

15. Poddar, S., Rohen, Y., Generalised rational α_s-Meir-Keeler contraction mapping in S-metric spaces, Amer. J. Appl. Math. Stat., 9(2), 48-52 (2021).

16. Rezaee, M. M., Shahraki, M., Sedghi, S., Altun, I., Fixed point theorems for weakly contractive mappings on S-metric spaces and a homotopy result, Appl. Math. E-Notes, 17, 58-67 (2017).

17. Rhoades, B. E., Some theorems on weakly contractive maps, Nonlinear Anal., 47, 2683-2693 (2001).

18. Riech, S., Some fixed point problems, Atti. Accad. Naz. Lincei, 57, 194-198 (1974).

19. Saluja, G. S., Common fixed point theorems on S-metric spaces via C-class functions, International J. Math. Combin., 3, 21-37 (2022).

20. Sedghi, S., Shobe, N., Aliouche, A., A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, 64, 258-266 (2012).

21. Sedghi, S., Dung, N. V., Fixed point theorems on S-metric spaces, Mat. Vesnik, 66, 113-124 (2014).

22. Sedghi, S., Shobe, N., Dosenovic, T., Fixed point results in S-metric spaces, Nonlinear Funct. Anal. Appl., 20, 55-67 (2015).

23. Zhang, Q., Song, Y., Fixed point theory for generalized φ-weak contractions, Appl. Math. Letters, 22, 75-78(2009).

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