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Fixed Point Theorems for Several Contractions in Super Metric Space

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ABSTRACT: In this paper, fixed point results for several contractions in the setting of super metric spaces are proved. Some suitable examples are also provided to illustrate the main results.

Key Words: Fixed point, super metric space, asymptotically regular map.

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1. Introduction and Preliminaries

The metric fixed theory is most demanded and interesting in solving many problems in different areas. A lot of research has been performed in this field. In 1922, Banach [2] introduced Banach Contraction Principle in metric space. In 1994, Matthews [8] introduced the concept of partial metric space. Czerwik [3] extended the metric space in b-metric space and Jleli [5] extended the metric space in generalized metric space. In 2022, Karapinar and Khojasteh [7] introduced a new concept of super metric space as a generalization of metric space as follows.

Definition 1.1 *Let X be a non-empty set. A function $m : X \times X \to [0, +\infty)$ is called a super metric if it satisfies the following axioms:*

(m1) *for all $x, y \in X$, if $m(x, y) = 0$, then $x = y$;*

(m2) *$m(x, y) = m(y, x)$ for all $x, y \in X$;*

(m3) *there exists $s \geq 1$ such that for every $y \in X$, there exist distinct sequences $\{x_p\}$, $\{y_p\} \subset X$, with $m(x_p, y_p) \to 0$ when $p \to \infty$, such that*

$$\limsup_{p\to\infty} m(y_p, y) \leq s \limsup_{p\to\infty} m(x_p, y).$$

Here (X, m) is called a super metric space.

The notions of convergence and the Cauchy sequence with respect to completeness of a super metric space are defined as follows:

Definition 1.2 *On a super metric space (X, m), a sequence $\{x_p\}$:*

(a) *converges to x in X if and only if $\lim_{p\to\infty} m(x_p, x) = 0$.*

(b) *is a Cauchy sequence in X if and only if $\lim_{p\to\infty} \sup\{m(x_p, x_q) : q > p\} = 0$.*

Definition 1.3 *A super metric space (X, m) is said to be complete if and only if every Cauchy sequence in X is convergent.*

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Example 1.1 *Let $X = [0, 2]$ and define $m : X \times X \to [0, +\infty)$ by*

$$m(x, y) = \begin{cases} \frac{x+y}{2x+y+2}, & x \neq y,\, x \neq 0,\, y \neq 0, \\ 0, & x = y, \\ \max\left\{\frac{x}{4}, \frac{y}{4}\right\}, & \text{otherwise.} \end{cases}$$

For $y \in X$, we have $\{x_p\}$, $\{y_p\}$ are two distinct sequences such that $m(x, y) \to 0$ as $p \to \infty$.
For these sequences we have

$$m(x_p, y_p) = \frac{x_p + y_p}{2x_p + y_p + 2} \to 0 \quad \text{as} \quad p \to \infty \quad \text{and} \quad \lim_{p\to\infty} x_p = \lim_{p\to\infty} y_p = 0.$$

So, there exists any $q > 0$ such that for all $p \geq q$, we have

$$\begin{aligned} \limsup_{p\to\infty} m(y_p, y) &= \limsup_{p\to\infty} \frac{(y_p + y)}{(2y_p + y + 2)} \\ &= \frac{y}{y+2} \\ &\leq s\frac{y}{y+2} \\ &= s \limsup_{p\to\infty} \frac{(x_p + y)}{(2x_p + y + 2)}. \end{aligned}$$

Thus (X, m) is Super metric space.

Definition 1.4 *Let $F : X \to X$ be a mapping and $\{F^p x\}p \geq 0$ be the Picard iteration for initial point $x \in X$ where F^p denote the p-th iterates of F. The mapping F is asymptotically regular if*

$$\lim_{p\to\infty} m(F^p x, F^{p+1} x) = 0, \quad \text{for every } x \in X.$$

Further, Karapinar and Fulga [6] examined contractions in rational form in super metric spaces.

2. Main Results

In this section, we prove some fixed point theorems using several contractions in super metric spaces. Further, we shall provide some suitable examples also to prove the validity of our results.

Theorem 2.1 *Let (X, m) be a complete super metric space. Let F be a self-map on X such that $m(Fx, Fy) \leq Lm(x, y)\ \forall\ x, y \in X$, where L is a Lipschitz constant.*
Then F has a unique fixed point.
Further, for any $x \in X$, we have $\lim_{p\to\infty} F^p(x) = z$ with $m(F^p(x), z) \leq \frac{L^p}{1-L} m(x, F(x))$.

Proof: To prove uniqueness, Let us choose $x, y \in X$ with $F(x) = x$, $F(y) = y$ then

$$m(x, y) = m(F(x), F(y)) \leq Lm(x, y).$$

Therefore, $m(x, y) = 0$.
So, $x = y$.
Now we prove that $\{F^p(x)\}$ is a Cauchy sequence for $x \in X$.
Now

$$\begin{aligned} m(F^p(x), F^q(x)) &\leq m(F^p(x), F^{(p+1)}(x)) + m(F^{(p+1)}(x), F^{(p+2)}(x)) + \cdots + m(F^{(q-1)}(x), F^q(x)) \\ &\leq L^p m(x, F(x)) + \cdots + L^{(q-1)} m(x, F(x)) \\ &\leq L^p m(x, F(x))[1 + L + L^2 + \ldots] \\ &= \frac{L^p}{(1-L)} m(x, F(x)), \quad \text{for all } q > p. \end{aligned}$$

Therefore,

$$\limsup_{p\to\infty} m(F^p(x), F^q(x)) \leq s \limsup_{p\to\infty} m((x), F(x)) = 0.$$

Thus $\{F^p(x)\}$ is a Cauchy sequence and X is complete therefore there exists $z \in X$ with

$$\lim_{p\to\infty} F^p(x) = z.$$

Also,

$$z = \lim_{p\to\infty} F^{(p+1)}(x) = \lim_{p\to\infty} F(F^p(x)) = F(z).$$

So, z is a fixed point of F and $m(F^p(x), z) \leq \frac{L^p}{1-L} m(x, F(x))$. □

Theorem 2.2 *Let (X, m) be a complete super metric space and let*

$$m(F(x), F(y)) \leq \phi(m(x, y)) \quad \forall\ x, y \in X,$$

where $\phi : [0, \infty) \to [0, \infty)$ is any monotonic, non-decreasing function with $\lim_{p\to\infty} \phi^p(t) = 0$ for any fixed $t > 0$.
Then F has a unique fixed point $u \in X$ with $\lim_{p\to\infty} F^p(x) = u$ for each $x \in X$.

Proof: Suppose $t \leq \phi(t)$ for some $t > 0$. Then $\phi(t) \leq \phi(\phi(t))$ and therefore $t \leq \phi^2(t)$. By induction $t \leq \phi^p(t)$ for $p \in \{1, 2, \dots\}$.
A contradiction, since

$$\lim_{p\to\infty} \phi^p(t) = 0 \quad \text{for any fixed } t > 0.$$

Thus $\phi(t) < t$ for each $t > 0$.
Now,

$$m(F^p(x), F^{(p+1)}(x)) \leq \phi^p(m(x, F(x))), \quad \text{for } x \in X$$

and therefore

$$\limsup_{p\to\infty} m(F^p(x), F^{(p+1)}(x)) = 0, \quad \text{for each } x \in X.$$

Let $\varepsilon > 0$ and choose $\delta(\varepsilon) = \varepsilon - \phi(\varepsilon)$.
If $m(x, F(x)) < \delta(\varepsilon)$, then for any $u \in B(x, \varepsilon) = \{y \in X : m(x, y) < \varepsilon\}$, we have

$$\begin{aligned} m(F(u), x) &\leq m(F(u), F(x)) + m(F(x), x) \\ &\leq \phi(m(u, x)) + m(F(x), x) \\ &\leq \phi(m(u, x)) + \delta(\varepsilon) \\ &\leq \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

So, $F(u) \in B(x, \varepsilon) = \{y \in X : m(x, y) < \varepsilon\}$.
Then by Theorem 2.1 F has a fixed point u.
Thus F has only one fixed point u in X.
Since by Theorem 2.1 $\{F^p(x)\}$ is a cauchy sequence and X is complete therefore there exists $u \in X$ with $\lim_{p\to\infty} F^p(x) = u$. □

Example 2.1 *Let $X = [0, 4]$ and define $m : X \times X \to [0, +\infty)$ by*

$$m(x, y) = \begin{cases} x + y, & x \neq y \\ 0, & x = y \end{cases}$$

Let $\{x_p\}$ and $\{y_p\}$ are two distinct sequences such that $m(x,y) \to 0$ as $p \to \infty$.
Since sequences are distinct we have $m(x_p, y_p) = x_p + y_p \to 0$ and we choose $y_p \to 0$ and $x_p \to 0$ as $p \to \infty$.
For $y \in X$, we have

$$\limsup_{p\to\infty} m(y_p, y) = \limsup_{p\to\infty}(y_p + y) = y \leq s \limsup_{p\to\infty} m(x_p, y) = y.$$

From here, we can say that (X, m) is a super metric space.
Now, let us consider $F : X \to X$ as

$$F(x) = \begin{cases} 1, & x \neq 2, \\ \frac{1}{2}, & x = 2. \end{cases}$$

Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t$, where $0 < t < 1$. Thus ϕ is monotonic non decreasing such that $\lim_{p\to\infty} \phi^p(t) = 0$ which satisfy following condition:

$$m(F(x), F(y)) \leq \phi(m(x,y)) \quad \forall\ x, y \in X.$$

Here 1 is fixed point of F.

Example 2.2 *Let $X = [0, 7]$ and define $m : X \times X \to [0, +\infty)$ by*

$$m(x,y) = \begin{cases} \frac{xy}{x+y+1}, & x \neq y, \\ 0, & x = y. \end{cases}$$

Let $\{x_p\}$ and $\{y_p\}$ are two distinct sequences such that $m(x,y) \to 0$ as $p \to \infty$.
Since sequences are distinct we have $m(x_p, y_p) = \frac{x_p y_p}{x_p+y_p+1} \to 0$ and we choose $y_p \to 0$ and $x_p \to 0$ as $p \to \infty$.
For $y \in X$

$$\limsup_{p\to\infty} m(y_p, y) = \limsup_{p\to\infty} \frac{y_p y}{y_p + y + 1} = 0 \leq s \limsup_{p\to\infty} m(x_p, y) = 0.$$

From here, we can say that (X, m) is a super metric space.
Now, let us consider $F : X \to X$ as

$$F(x) = \begin{cases} 1, & x \neq 3, \\ \frac{1}{4}, & x = 3 \end{cases}$$

and $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t$, where $0 < t < 1$. Thus ϕ is monotonic non decreasing such that $\lim_{p\to\infty} \phi^p(t) = 0$ which satisfy following condition:

$$m(F(x), F(y)) \leq \phi(m(x,y)) \quad \forall\ x, y \in X.$$

Here 1 is fixed point of F.

Theorem 2.3 *Let (X, m) be a complete super metric space. Let $F : X \to X$ be a asymptotically mapping and $\phi : X \times X \to [0,\infty)$ is a continuous finite map such that*

$$m(F(x), F(y)) \leq \phi(x,y) - \phi(F(x), F(y)) \quad \forall\ x, y \in X.$$

Then F has a fixed point in X.

Proof: Define $x_{p+1} = Fx_p$ such that $x_p \neq x_{p+1}$.
So,

$$m(x_p, x_{p+1}) > 0 \quad \forall\ p \in N.$$

So, we have

$$m(x_p, x_{p+1}) \leq \phi(x_{p-1}, x_p) - \phi(Fx_{p-1}, Fx_p).$$

So,

$$\sum_{p=1}^{k} m(x_p, x_{p+1}) \leq \sum_{p=1}^{k} \phi(x_{p-1}, x_p) - \phi(x_p, x_{p+1}) = \phi(x_0, x_1).$$

As $p \to \infty$ we have

$$\limsup_{p\to\infty} m(x_p, x_{p+1}) = 0.$$

Now suppose that $\alpha, \beta \in N$ and $\beta > \alpha$. If $x_\alpha = x_\beta$ we have

$$F^{\beta}(x_0) = F^{\alpha}(x_0).$$

Thus

$$F^{\beta-\alpha}(F^{\alpha}(x_0)) = F^{\beta}(x_0) = F^{\alpha}(x_0).$$

Thus we have $F^{\alpha}(x_0)$ is the fixed point of $F^{\beta-\alpha}$.
Also

$$F(F^{\beta-\alpha}(F^{\alpha}(x_0))) = F^{\beta-\alpha}(F(F^{\alpha}(x_0))) = F(F^{\alpha}(x_0)).$$

It means that $F((F^{\alpha}(x_0))$ is the fixed point of $F^{\beta-\alpha}$.
Thus $F(F^{\alpha}(x_0)) = F^{\alpha}(x_0)$.
So $F^{\alpha}(x_0)$ is the fixed point of F.
Therefore, $x_\beta \neq x_\alpha$.
Therefore,

$$\limsup_{p\to\infty} m(x_p, x_{p+2}) \leq sm(x_{p+1}, x_{p+2}).$$

Thus since

$$\limsup_{p\to\infty} m(x_p, x_{p+1}) \to 0.$$

So we have

$$\limsup_{p\to\infty} m(x_p, x_{p+3}) \leq sm(x_{p+2}, x_{p+3}).$$

Thus we get

$$\limsup_{p\to\infty} m(x_p, x_q) : q > p = 0.$$

Therefore, $\{x_p\}$ is a Cauchy sequence.
Since (X, m) is complete super metric space therefore $\{x_p\}$ converges to $z \in X$ (say).
Since F is continuous mapping. So, $F(z) = z$ and hence z is the fixed point of F in X. □

Example 2.3 *Let $X = [0, 6]$ and define $m : X \times X \to [0, +\infty)$ by*

$$m(x, y) = \begin{cases} x + 2y, & x \neq y, \\ 0, & x = y. \end{cases}$$

Let $\{x_p\}$ and $\{y_p\}$ are two distinct sequences such that $m(x_p, y_p) \to 0$ as $p \to \infty$.
Since sequences are distinct we have $m(x_p, y_p) = x_p + 2y_p \to 0$ and we choose $y_p \to 0$ and $x_p \to 0$ as $p \to \infty$.
For $y \in X$

$$\begin{aligned} \limsup_{p\to\infty} m(y_p, y) &= \limsup_{p\to\infty} (y_p + 2y) \\ &= 2y \\ &\leq s \limsup_{p\to\infty} m(x_p, y) \\ &= s(2y). \end{aligned}$$

From here, we can say that (X, m) is super metric space.
Now let us consider $F : X \to X$ be asymptotically mapping defined as

$$F(x) = \begin{cases} 2, & x \neq 4, \\ \frac{1}{2}, & x = 4. \end{cases}$$

and $\phi : X \times X \to [0, \infty)$ by $\phi(x, y) = xy$ is a continuous finite map.
It satisfy the property

$$m(F(x), F(y)) \leq \phi(x, y) - \phi(F(x), F(y)) \quad \forall\, x, y \in X.$$

Then F has a fixed point 2 in X.

Theorem 2.4 *Let (X, m) be a complete super metric space and $F : X \to X$ be a mapping such that there exists $k \in [0, 1)$ and that*

$$m(F(x), F(y)) \geq k \min\left\{m(x, y), m(x, F(x)), m(y, F(y)), m(F(x), F(y)), \frac{m(F(x), y) + m(x, F(y))}{m(x, y) + 1}\right\}.$$

Then F has a unique fixed point.

Proof: Let $x \in X$ and $\{x_p\}$ be the Picard iteration of mapping F then $Fx_p = x_{p+1}$.
If $x_p = x_{p+1}$, then

$$Fx_{p_0} = x_{p_0+1} = x_{p_0}$$

and hence x_{p_0} is a fixed point of F.
So, let us suppose that

$$x_p \neq x_{p+1} \quad \forall\, p \in N.$$

Hence $m\left(x_{p_0}, x_{p_{0+1}}\right) > 0$ and

$$\begin{aligned} m(x_p, x_{p+1}) &= m(Fx_{p-1}, Fx_p) \\ &\geq k \min\left\{m(x_{p-1}, x_p), m(x_{p-1}, x_p) m(x_p, x_{p+1}), m(x_p, x_{p+1}), \frac{m(x_p, x_p) + m(x_{p-1}, x_p)}{m(x_{p-1}, x_p) + 1}\right\} \\ &= k \min\{m(x_{p-1}, x_p), m(x_p, x_{p+1})\}. \end{aligned}$$

Thus

$$m(x_p, x_{p+1}) \geq m(x_{p-1}, x_p).$$

If

$$\min\{m(x_{p-1}, x_p), m(x_p, x_{p+1})\} = m(x_p, x_{p+1})$$

then

$$m(x_p, x_{p+1}) \leq km(x_p, x_{p+1}) < m(x_p, x_{p+1})$$

a contradiction
And hence

$$0 < m(x_p, x_{p+1}) \leq km(x_{p-1}, x_p) \leq k^2 m(x_{p-2}, x_{p-1}) \leq \cdots \leq k^k m(x_0, x_1).$$

And in taking limit from inequality we get,

$$\limsup_{p \to \infty} m(x_p, x_{p+1}) = \limsup_{p \to \infty} m(F^{p-1}(x), F^p(x)) = 0.$$

Thus F is asymptotically regular and Picard iteration $\{F^p(x)\}$ is convergent sequence. Thus there exists $w \in X$ such that $\lim_{p \to \infty} m(x_p, w) = 0$. □

Example 2.4 *Let $X = [0, 8]$ and define $m : X \times X \to [0, +\infty)$ by*

$$m(x, y) = \begin{cases} xy, & x \neq y, \\ 0, & x = y. \end{cases}.$$

Let $\{x_p\}$ and $\{y_p\}$ are two distinct sequences such that $m(x_p, y_p) \to 0$ as $p \to \infty$.
Since sequences are distinct we have $m(x_p, y_p) = (x_p y_p) \to 0$ and we choose $y_p \to 0$ and $x_p \to 0$ as $p \to \infty$.
For $y \in X$,

$$\limsup_{p\to\infty} m(y_p, y) = \limsup_{p\to\infty}(y_p y) = 0 \leq s \limsup_{p\to\infty} m(x_p, y) = s(0) = 0.$$

From here, we can say that (X, m) is super metric space.
Now let us consider $F : X \to X$ be asymptotically mapping defined as

$$F(x) = \begin{cases} 1, & x \neq 2, \\ \frac{1}{4}, & x = 2. \end{cases}$$

and $k \in [0, 1)$.
And it satisfy

$$m(F(x), F(y)) \geq k \min\left\{ m(x, y), m(x, F(x)), m(y, F(y)), m(F(x), F(y)), \frac{m(F(x), y) + m(x, F(y))}{m(x, y) + 1} \right\}$$

for $k \in [0, 1)$.
Then F has a fixed point 1 in X.

Theorem 2.5 *Let (X, m) be a complete super metric space and define $F : X \to X$ as follows:*

$$\begin{aligned} m(F(x), F(y)) &\leq \alpha m(x, y) + \beta[m(x, F(x)) + m(y, F(y))] + \gamma[m(x, F(y)) + m(y, F(x))] \\ &\quad + \delta \left[\frac{m(x, F(x)) m(y, F(y))}{m(x, y) + m(y, F(y))} \right] \quad \forall\ x, y \in X,\ x \neq y \end{aligned}$$

and $\alpha, \beta, \gamma, \delta \in [0, 1)$ and such that $\alpha + 2\beta + 2\gamma + \delta < 1$ and $\alpha + 2\gamma < 1$.
Then F contains a fixed point which is also unique in X.

Proof: Let $\{x_p\}$ be sequence in X and $Fx_p = x_{p+1}$ for all $p = 0, 1, 2, \ldots$.
Now, we have

$$\begin{aligned} m(x_p, x_{p+1}) &= m(Fx_{p-1}, Fx_p) \\ &\leq \alpha[m(x_{p-1}, x_p)] + \beta[m(x_{p-1}, Fx_{p-1}) + m(x_p, Fx_p)] + \gamma[m(x_{p-1}, Fx_p) + m(x_p, Fx_{p-1})] \\ &\quad + \delta \left[\left(\frac{m(x_{p-1}, Fx_p) m(x_{p-1}, Fx_p)}{m(x_{p-1}, x_p) + m(x_p, Fx_p)} \right) \right] \\ &\leq \alpha[m(x_{p-1}, x_p)] + \beta[m(x_{p-1}, x_p) + m(x_p, x_{p+1})] + \gamma[m(x_{p-1}, x_p) + m(x_p, x_p)] \\ &\quad + \delta \left[\frac{m(x_{p-1}, x_p) m(x_{p-1}, x_{p+1})}{m(x_{p-1}, x_p) + m(x_p, x_{p+1})} \right] \\ &\leq \alpha[m(x_{p-1}, x_p)] + \beta[m(x_{p-1}, x_p) + m(x_p, x_{p+1})] \\ &\quad + \gamma[m(x_{p-1}, x_p) + m(x_p, x_{p+1})] + \delta[m(x_{p-1}, x_p)] \end{aligned}$$

that is,

$$(1 - \beta - \gamma)[m(x_p, x_{p+1})] \leq (\alpha + \beta + \gamma + \delta)[m(x_{p-1}, x_p)].$$

$$K = \left(\frac{\alpha + \beta + \gamma + \delta}{1{-}\beta - \gamma} \right) < 1$$

Since $\alpha + 2\beta + 2\gamma + \delta < 1$.
We have,

$$m(x_p, x_{p+1}) \leq k[m(x_{p-1}, x_p)] \leq k^2[m(x_{p-2}, x_{p-1})] \ldots .$$

Thus we have,

$$m(x_p, x_{p+1}) \leq k^p[m(x_0, x_1)].$$

Also $0 \leq k < 1$, so for $p \to \infty$, $k^p \to 0$ and hence $m(x_p, x_{p+1}) \to 0$.
So $\sup m(x_p, x_{p+1}) \to 0$ as $p \to \infty$.
Thus $\{x_p\}$ is a Cauchy sequence so there is a point x^* such that $x_p \to x^*$.
As F is continuous,

$$F(x^*) = \lim_{p\to\infty} F(x_p) = \lim_{p\to\infty} (x_{p+1}) = x^*.$$

Thus F has a fixed point.
Let y^* be another fixed point.
So $m(x^*, y^*) = m(Fx^*, Fy^*)$,

$$\begin{aligned}
m(x^*, y^*) &\leq \alpha[m(x^*, y^*)] + \beta[m(x^*, Fx^*) + m(y^*, Fy^*)] + \gamma[m(x^*, Fy^*) + m(y^*, Fx^*)] \\
&\quad + \delta \left[\frac{m(x^*, y^*)m(x^*, Fx^*)}{m(x^*, y^*) + m(y^*, Fy^*)}\right] \\
&\leq \alpha[m(x^*, y^*)] + \beta[m(x^*, x^*) + m(y^*, y^*)] + \gamma[m(x^*, y^*) + m(y^*, x^*)] \\
&\quad + \delta \left[\frac{m(x^*, y^*)m(x^*, x^*)}{m(x^*, y^*) + m(y^*, y^*)}\right] \\
&\leq \alpha[m(x^*, y^*)] + \beta(0) + \gamma[2m(x^*, y^*)] + \delta(0).
\end{aligned}$$

Thus

$$m(x^*, y^*) \leq (\alpha + 2\gamma)m(x^*, y^*)$$

that is,

$$(1 - \alpha - 2\gamma)m(x^*, y^*) \leq 0$$

is a contradiction.
Hence $m(x^*, y^*) = 0$.
So, $x^* = y^*$.
Hence F contains the unique fixed point. □

Example 2.5 *Let $X = [0, 2]$ and define $m : X \times X \to [0, +\infty)$ by*

$$m(x, y) = \begin{cases} x + y, & x \neq y, \\ 0, & x = y. \end{cases}$$

Let $\{x_p\}$ and $\{y_p\}$ are two distinct sequences such that $m(x_p, y_p) \to 0$ as $p \to \infty$.
Since sequences are distinct we have $m(x_p, y_p) = x_p + y_p \to 0$ and we choose $y_p \to 0$ and $x_p \to 0$ as $p \to \infty$.
For $y \in X$,

$$\begin{aligned}
\limsup_{p\to\infty} m(y_p, y) &= \limsup_{p\to\infty} (y_p + y) \\
&= y \\
&\leq s \limsup_{p\to\infty} m(x_p, y) \\
&= s \limsup_{p\to\infty} (x_p + y) \\
&= s(y).
\end{aligned}$$

From here, we can say that (X, m) is a super metric space.
Now let us consider $F : X \to X$ be mapping defined as

$$F(x) = \begin{cases} 1, & x \neq 2, \\ \frac{1}{6}, & x = 2. \end{cases}$$

And it satisfy

$$m(F(x), F(y)) \geq \alpha m(x, y) + \beta[m(x, F(x)) + m(y, F(y))] + \gamma[m(x, F(y)) + m(y, F(x))] + \delta \left[\frac{m(x, F(x))m(y, F(y))}{m(x, y) + m(y, F(y))} \right] \quad \forall\, x, y \in X,\, x \neq y.$$

And $\alpha, \beta, \gamma, \delta \in [0, 1)$ and such that $\alpha + 2\beta + 2\gamma + \delta < 1$ and $\alpha + 2\gamma < 1$.
Then F has a fixed point 1 in X.

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