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# On Abu-Shady-Kaabar Fractional Chebyshev Differential Equation of the First Kind

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ABSTRACT: The newly proposed Abu-Shady-Kaabar (A-S-K) fractional derivative, initiated as generalized form of fractional derivative, is studied in this paper. New results are obtained via this newly proposed definition are investigated. The derivability and integrability of the sum function of fractional power series are studied in this context. Likewise, the homogeneous sequential linear A-S-K fractional differential equation of order $2\alpha$ solutions' existence around an ordinary point is analysed. The series solutions of the first kind A-S-K fractional Chebyshev differential equation, briefly named in this work as A-S-K Chebyshev-I for simplicity, are obtained. Some interesting properties of the A-S-K Chebyshev-I polynomials are also included. The novelty of this work lies in introducing the A-S-K derivative, which generalizes fractional calculus by integrating a new definition of the fractional derivative through an incremental ratio based on a power law. The A-S-K derivative provides a framework that is differentiable, integrable, and capable of handling complex fractional systems effectively.

Key Words: Fractional differential equation, fractional calculus, generalized fractional Chebyshev-I polynomials.

## Contents



## 1. Introduction

The theory of fractional calculus (FC) has been actively the current focus of several researchers in multidisciplinary fields of research due to its applicability in various systems in physics and engineering. FC can be categorized into two types: nonlocal and local definitions. The most common nonlocal definitions of FC are Riemann-Liouville and Caputo (sometimes known as Liouville-Caputo in some research works) (see [1,2,3,4,5] for more details about such definitions). One of the most commonly known local definitions of FC is called the conformable derivative definition [6], interpreted physically in [7] and geometrically in [8], which was proposed to overcome the obstacles associated with classical nonlocal FC definitions in solving differential equations, formulated in the context of FC, analytically without the need for modified or complex numerical methods. This definition has received some criticism in some research works [9], indicating that this definition has some disadvantages in comparison to the classical nonlocal definition of Caputo. In the cited paper it is shown how the use of the conformable operator produces a substantially larger error compared to the Caputo derivative when used to solve fractional models. As a result, in [10], Abu-Shady and Kaabar formulated the A-S-K fractional derivative, to create an alternative definition to all previously mentioned definitions that can efficiently solve differential equations of fractional orders analytically, while having the same results that agree with the nonlocal classical definitions of Riemann-Liouville and Liouville-Caputo results. A-S-K definition has been theoretically studied in [11]

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and applied in [17,18] for the diatomic molecules in the Deng-Fan model and heavy tetraquark masses spectra, respectively.
In this context, we can highlight the following topics as main objectives of this research work:

1. First, we extend the integral fractional ASK theory introduced in [10], studying important properties such as Barrow's rule, the mean value theorem or some properties of the modules of this integral.

2. The results on derivability and integrability of the sum function of fractional power series are established in this context.

3. The A-S-K fractional power series method is introduced.

4. The above method is employed to find the solutions of the A-S-K Chebyshev-I.

This study is constructed as: The A-S-K derivative with its essential properties are mentioned in Section 2. Novel results on the A-S-K $\alpha$-integrals are investigated in Section 3 to extend the obtained results in [10]. In Section 4, several essential results on the A-S-K fractional power series are discussed; in particular, the A-S-K fractional differentiability and the integrability of the sum of a power series are studied. In addition, the solutions' existence around an ordinary point of homogeneous sequential linear A-S-K fractional differential equation of order $2\alpha$ is studied in detail. In Section 5, the series solutions of A-S-K Chebyshev-I are obtained via the A-S-K fractional power series method. This research study is finalized by establishing interesting properties of the A-S-K Chebyshev-I polynomials. Section 6 concludes our work.

## 2. Preliminaries

In this section, basic concepts and definitions are stated to use them in the results of this work.
**Definition 1.** *For function $m : [0, \infty) \to R$, the A-S-K derivative of order $0 < \alpha \leq 1$, of $m$ at $t > 0$ is expressed as* [10]:

$$D^{A-S-K} \;\; m(t) = \lim_{\varepsilon \to 0} \frac{m\left(t + \left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\right)\varepsilon t^{1-\alpha}\right) - m(t)}{\varepsilon} \;,\; \beta > -1, \beta \in R^{+} \;. \tag{2.1}$$

*If $m$ is $\alpha$-differentiable function ($\alpha$-DF) in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} D^{A-S-K} \;\; m(t)$ exists, then it is written as:*

$$D^{A-S-K} \; m(0) = \lim_{t \to 0^+} D^{A-S-K} \; m(t) \;. \tag{2.2}$$

On one hand, in Eq. (2.1), the constraint $\beta > -1$ is mentioned so that the Gamma function in Definition 1 is well-defined positive and finite, since Gamma function is defined for all complex numbers except for the non-positive integers. On the other hand, if $\beta \leq -1$, the Gamma function will not hold, and this can cause it to be infinite, which will not be mathematically in agreement with the ASK derivative's definition.
**Theorem 1.** [10]. *Assume that $0 < \alpha \leq 1$, $\beta > -1, \beta \in R^{+}$ and suppose that $m, p$ be $\alpha$-DF at a point $t > 0$. Then, we get:*

1. $D^{A-S-K} \; [am + bp] = a \; D^{A-S-K} \; [m] + b \; D^{A-S-K} \; [p]$, $\forall a, b \in R$.

2. $D^{A-S-K} \; [t^s] = \frac{s\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} t^{s-\alpha}$, $\forall \; s \in R$.

3. $D^{A-S-K} \; [\mu] = 0$, $\forall$*constant functions* $m(t) = \mu$.

4. $D^{A-S-K} \; [mp] = m D^{A-S-K} \; [p] + p D^{A-S-K} \; [m]$.

5. $D^{A-S-K} \; \left[\left(\frac{m}{p}\right)\right] = \frac{p D^{A-S-K} \; [m] - m D^{A-S-K} \; [p]}{p^2}$.

6. *If, additionally, $m$ is DF, then we have:*

$$D^{A-S-K}m(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}\frac{dm}{dt}(t).$$

*The A-S-K $\alpha$-derivative of certain functions is:*

1. $D^{A-S-K}\ [1]=0,$
2. $D^{A-S-K}\ [sin\,(st)]=\frac{s\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}cos\,(st),$
3. $D^{A-S-K}\ [cos\,(st)]=-\frac{s\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}sin\,(st),$
4. $D^{A-S-K}\ [e^{st}]=\frac{s\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}e^{st}.$

*In addition, the A-S-K $\alpha$-derivative of the following functions are expressed as:*

1. $D^{A-S-K}\ \left[\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}t^{\alpha}\right]=1,$
2. $D^{A-S-K}\ \left[sin\left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}t^{\alpha}\right)\right]=cos\left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}t^{\alpha}\right),$
3. $D^{A-S-K}\ \left[cos\left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}t^{\alpha}\right)\right]=-sin\left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}t^{\alpha}\right),$
4. $D^{A-S-K}\ \left[e^{\left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}t^{\alpha}\right)}\right]=e^{\left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}t^{\alpha}\right)}.$

**Theorem 2** ***(Chain Rule).*** [11]. *Suppose that $0<\alpha\leq 1$, $\beta>-1,\beta\in R^{+}$, $p$ is A-S-K $\alpha$-DF at $t>0$ and $m$ is DF at $p(t)$ then*

$$D^{A-S-K}\ [m\ \ p](t)=m'(p(t))\,D^{A-S-K}\ p(t). \tag{2.3}$$

**Remark 1.** [11]. Since the differentiability indicated that the *A-S-K* $\alpha$-differentiability and supposing that $p(t)>0$, the Eq. (2.3) can be expressed as:

$$D^{A-S-K}[m\ \ p](t)=\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}p(t)^{\alpha-1}D^{A-S-K}\ m(p(t))\,D^{A-S-K}\ p(t). \tag{2.4}$$

**Theorem 3.** ***(Rolle´s Theorem for A-S-K $\alpha$-DFs).*** [7]. *Suppose that $a>0$, $\alpha\in(0,1]$ and $m:[a,b]\to R$ be a given function that satisfies:*

1. *$m$ is continuous on $[a,b]$.*
2. *$m$ is A-S-K $\alpha$-DF on $(a,b)$.*
3. *$m(a)=m(b)$.*

*Then, $\exists\ v\in(a,b)$, $\ni D^{A-S-K}\ m(v)=0$.*

**Theorem 4.** ***(Mean Value Theorem for A-S-K $\alpha$-DFs).*** [7]. *Assume that $a>0$, $\alpha\in(0,1]$ and $m:[a,b]\to R$ be a given function that satisfies:*

1. *$m$ is continuous on $[a,b]$.*
2. *$m$ is A-S-K $\alpha$-DF on $(a,b)$.*

*Then,* $\exists\ v \in (a,b)$, $\ni$

$$D^{A-S-K}\ m\,(v) = \frac{m\,(b) - m\,(a)}{k\,(b^{\alpha} - a^{\alpha})}. \tag{2.5}$$

*where* $k = \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}$.

**Theorem 5.** [11]. *Suppose that* $a > 0$, $\alpha \in (0,1]$ *and* $m : [a,b] \to R$ *be a given function that satisfies*

1. $m$ *is continuous on* $[a,b]$.
2. $m$ *is A-S-K* $\alpha$*-DF on* $(a,b)$.

*If* $D^{A-S-K}\ m\,(t) = 0$, $\forall t \in (a,b)$, *then* $m$ *is a constant on* $[a,b]$.

**Corollary 1.** [11]. *Assume that* $a > 0$, $\alpha \in (0,1]$, *and* $M, P : [a,b] \to R$ *be functions such that* $D^{A-S-K}\ M\,(t) = D^{A-S-K}\ P\,(t)\ \forall\ t \in (a,b)$. *Then,* $\exists$ *a real constant* $C$, *such that*

$$M\,(t) = P\,(t) + C. \tag{2.6}$$

The A-S-K $\alpha$-integral of a function $f$ starting from $a \geq 0$ is expressed as:

**Definition 2.** [10] $I^{a}_{\alpha}(m)(t) = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^t \frac{m(x)}{x^{1-\alpha}}dx$, *where this integral is the known Riemann improper integration, and* $\alpha \in (0\ ,1]$.

According to Definition 2, the following is obtained:

**Theorem 6.** [10]. $D^{A-S-K}\ I^{a}_{\alpha}\,(m)\,(t) = m(t)$, *for* $t \geq a$, *where* $m$ *is any continuous function in the domain of* $I_{\alpha}$.

To call the A-S-K derivative a fractional derivative, we need to prove that the fundamental property of every fractional order of derivative must be obey the semigroup property as follows:

Suppose that the function $m(t)$ in Definition 1 is $m\,(t) = t^{s}\ ,\ s \in R^{+}$, *where* $s > -1$, we show that $D^{A-S-K}_{\alpha} D^{A-S-K}_{\beta} t^{s} = D^{A-S-K}_{\alpha+\beta} t^{s}$ as follows:

By using the results in [10], we obtain: $D^{A-S-K}_{\beta} t^{s} = \frac{\Gamma(s+1)}{\Gamma(s-\beta+1)} t^{s-\beta}$.

From the left-hand side, we have:

$$\begin{aligned} D^{A-S-K}_{\alpha} D^{A-S-K}_{\beta} t^{s} =& \frac{\Gamma(s+1)}{\Gamma(s-\beta+1)} D^{A-S-K}_{\alpha} t^{s-\beta} = \frac{\Gamma(s+1)}{\Gamma(s-\beta+1)} \frac{\Gamma(s-\beta+1)}{\Gamma(s-\beta-\alpha+1)} t^{s-\beta-\alpha} \\ =& \frac{\Gamma(s+1)}{\Gamma(s-\beta-\alpha+1)} t^{s-\beta-\alpha} \end{aligned}$$

From the right-hand side, we get:

$$D^{A-S-K}_{\alpha+\beta} t^{s} = \frac{\Gamma(s+1)}{\Gamma(s-(\alpha+\beta)+1)} t^{s-(\alpha+\beta)} = \frac{\Gamma(s+1)}{\Gamma(s-\alpha-\beta+1)} t^{s-\alpha-\beta}$$

Both sides are equivalent, and hence the A-S-K derivative satisfies the semigroup property.

For A-S-K integral, from the results in [10], we suppose that the function $m(t)$ in Definition 2 is $m\,(t) = t^{s}\ ,\ s \in R^{+}$ *where* $s > -1$, and $a = 0$, we show that $I^{A-S-K}_{\alpha} I^{A-S-K}_{\beta} t^{s} = I^{A-S-K}_{\alpha+\beta} t^{s}$.

From the left-hand side, we obtain:

$$I^{A-S-K}_{\beta} t^{s} = \frac{\Gamma\,(\beta-\beta+1)}{\Gamma\,(\beta)} \int_0^t \frac{x^{s}}{x^{1-\beta}} dx = \frac{\Gamma\,(\beta-\beta+1)}{\Gamma\,(\beta)} \int_0^t x^{s-1+\beta} dx = \frac{\Gamma\,(\beta-\beta+1)}{\Gamma\,(\beta)} \frac{t^{s+\beta}}{s+\beta}$$

$$= \frac{\Gamma\,(1)}{\Gamma\,(\beta)} \frac{t^{s+\beta}}{s+\beta} = \frac{1}{\Gamma\,(\beta)} \frac{t^{s+\beta}}{s+\beta} = \frac{t^{s+\beta}}{\Gamma\,(\beta)\,(s+\beta)}$$

$$I^{A-S-K}_{\alpha}\left(\frac{t^{s+\beta}}{\Gamma\,(\beta)\,(s+\beta)}\right) = \frac{\Gamma\,(\alpha-\alpha+1)}{\Gamma\,(\alpha)\,\Gamma\,(\beta)\,(s+\beta)} \int_0^t \frac{x^{s+\beta}}{x^{1-\alpha}} dx = \frac{\Gamma\,(\alpha-\alpha+1)}{\Gamma\,(\alpha)\,\Gamma\,(\beta)\,(s+\beta)} \int_0^t x^{s+\beta-1+\alpha} dx$$

$$= \frac{\mathit{\Gamma}\left(\alpha - \alpha + 1\right)}{\mathit{\Gamma}\left(\alpha\right)\mathit{\Gamma}\left(\beta\right)\left(s+\beta\right)}\frac{t^{s+\beta+\alpha}}{s+\beta+\alpha} = \frac{\mathit{\Gamma}\left(1\right)}{\mathit{\Gamma}\left(\alpha\right)\mathit{\Gamma}\left(\beta\right)\left(s+\beta\right)}\frac{t^{s+\beta+\alpha}}{s+\beta+\alpha} = \frac{t^{s+\beta+\alpha}}{\mathit{\Gamma}\left(\alpha\right)\mathit{\Gamma}\left(\beta\right)\left(s+\beta\right)(s+\beta+\alpha)}$$

From the right-hand side, we obtain:

$$\begin{aligned} I_{\alpha+\beta}^{A-S-K} t^s =& \frac{\mathit{\Gamma}\left(\alpha+\beta-(\alpha+\beta)+1\right)}{\mathit{\Gamma}\left(\alpha+\beta\right)}\int_0^t \frac{x^s}{x^{1-(\alpha+\beta)}}dx = \frac{\mathit{\Gamma}\left(\alpha+\beta-(\alpha+\beta)+1\right)}{\mathit{\Gamma}\left(\alpha+\beta\right)}\int_0^t x^{s-1+\alpha+\beta}dx \\ =& \frac{\Gamma(1)}{\mathit{\Gamma}\left(\alpha+\beta\right)}\frac{t^{s+\alpha+\beta}}{s+\alpha+\beta} \end{aligned}$$

$$= \frac{1}{\mathit{\Gamma}\left(\alpha+\beta\right)}\frac{t^{s+\alpha+\beta}}{s+\alpha+\beta} = \frac{t^{s+\alpha+\beta}}{\mathit{\Gamma}\left(\alpha+\beta\right)\left(s+\alpha+\beta\right)}$$

From Gamma function identity and Beta function, we have:
$\Gamma\left(\alpha+\beta\right) = \frac{\Gamma(\alpha)\Gamma(\beta)}{B(\alpha,\beta)}$ where $B\left(\alpha,\beta\right) = \frac{1}{s+\beta}$ in this example.
Thus, $\Gamma\left(\alpha+\beta\right) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\frac{1}{s+\beta}} = (s+\beta)\Gamma\left(\alpha\right)\Gamma(\beta)$.
Then, by substitution the above in the result of right-hand side, we get:

$$I_{\alpha+\beta}^{A-S-K} t^s = \frac{t^{s+\alpha+\beta}}{(s+\beta)\Gamma\left(\alpha\right)\Gamma(\beta)(s+\alpha+\beta)}$$

Since both sides are equivalent, and the A-S-K integral satisfies the semigroup property.
Now, we suppose that the function $m(t)$ in Definitions 1 and 2 is $m\left(t\right) = t^s\ ,\ s \in R^+,\ where\ s > -1$, we show that $D_{\alpha}^{A-S-K}(I_{\beta}^{A-S-K} t^s) = D_{\alpha-\beta}^{A-S-K} t^s$ as follows:
By using all of the previously obtained results, we get the following:
From the left-hand side, we have:

$$\begin{aligned} I_{\beta}^{A-S-K} t^s =& \frac{t^{s+\beta}}{\mathit{\Gamma}\left(\beta\right)\left(s+\beta\right)} \\ D_{\alpha}^{A-S-K}\left(\frac{t^{s+\beta}}{\mathit{\Gamma}\left(\beta\right)\left(s+\beta\right)}\right) =& \frac{1}{\mathit{\Gamma}\left(\beta\right)\left(s+\beta\right)}\frac{\mathit{\Gamma}\left(s+\beta+1\right)}{\mathit{\Gamma}\left(s+\beta-\alpha+1\right)}t^{s+\beta-\alpha} \\ =& \frac{\mathit{\Gamma}\left(s+\beta+1\right)}{\mathit{\Gamma}\left(\beta\right)\left(s+\beta\right)\mathit{\Gamma}\left(s+\beta-\alpha+1\right)}t^{s+\beta-\alpha} \\ =& \frac{\left(s+\beta\right)\mathit{\Gamma}\left(s+\beta\right)}{\mathit{\Gamma}\left(\beta\right)\left(s+\beta\right)\mathit{\Gamma}\left(s+\beta-\alpha+1\right)}t^{s+\beta-\alpha} \\ =& \frac{\mathit{\Gamma}\left(s+\beta\right)}{\mathit{\Gamma}\left(\beta\right)\mathit{\Gamma}\left(s+\beta-\alpha+1\right)}t^{s+\beta-\alpha} \end{aligned}$$

By using the Beta function relationship with Gamma function, we get:

$$\begin{aligned} D_{\alpha}^{A-S-K}\left(\frac{t^{s+\beta}}{\mathit{\Gamma}\left(\beta\right)\left(s+\beta\right)}\right) =& \frac{\mathit{\Gamma}\left(s+\beta\right)}{\mathit{\Gamma}\left(\beta\right)\mathit{\Gamma}\left(s+\beta-\alpha+1\right)}t^{s+\beta-\alpha} \\ =& \frac{\mathit{\Gamma}\left(s+1\right)\Gamma(\beta)}{\mathit{\Gamma}\left(\beta\right)\mathit{\Gamma}\left(s+\beta-\alpha+1\right)}t^{s+\beta-\alpha} = \frac{\mathit{\Gamma}\left(s+1\right)}{\mathit{\Gamma}\left(s+\beta-\alpha+1\right)}t^{s+\beta-\alpha} \end{aligned}$$

From the right-hand side, we obtain:

$$D_{\alpha-\beta}^{A-S-K} t^s = \frac{\mathit{\Gamma}\left(s+1\right)}{\mathit{\Gamma}\left(s+\beta-\alpha+1\right)}t^{s+\beta-\alpha}$$

Since both sides are equal to each other, then the semigroup property is satisfied only if $\alpha > \beta$. As a result, the ASK derivative is a fractional derivative.

## 3. Novel results on A-S-K $\alpha$-Integrals

New results concerning the A-S-K $\alpha$-integral, extended from [10], are discussed in detail in this section.

**Theorem 7.** *Assume that $a > 0$, $\alpha \in (0,1]$ and $m$ be a continuous real-valued function (R-VF) on $[a,b]$. Let $P$ any R-VF with the property $D^{A-S-K}(P)(t) = m(t)$ for all $t \in [a,b]$. Then, we get:*

$$\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}\int_a^b \frac{m(t)}{t^{1-\alpha}}dt = P(b) - P(a). \tag{3.1}$$

**Proof.** We suppose that $M$ be a function on $[a,b]$ defined as $M(t) = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}\int_a^t \frac{m(x)}{x^{1-\alpha}}dx$, which can be named as the *A-S-K $\alpha-$ integral function of $m$.*
From Theorem 6, $D^{A-S-K}(M)(t) = m(t)$ for all $t \in [a,b]$.
Since $M$ and $P$ have the same A-S-K derivative, then by Corollary 1, $\exists$ a real constant $C$ such that $P(t) = M(t) + C \ \forall\, t \in [a,b]$.
Thus, $P(b) - P(a)$ is written as:

$$\begin{aligned} P(b) - P(a) &= (M(b) + C) - (M(a) + C) \\ &= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}\left(\int_a^b \frac{m(t)}{t^{1-\alpha}}dt - \int_a^a \frac{m(t)}{t^{1-\alpha}}dt\right) = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}\int_a^b \frac{m(t)}{t^{1-\alpha}}dt. \end{aligned} \tag{3.2}$$

Note that the term $\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}$ in the definite integral in Theorem 7 can be denoted as the coefficient $c(\alpha,\ \beta)$, which can be interpreted as the Mellin transform of the product $m(t)\cdot w(a,b)$, where the $w(a,b)$ is the window function such that

$$w(a,b) = \begin{cases} 1, & if\ a \le t \le b, \\ 0, & otherwise \end{cases}$$

where this Mellin transform exists for $a \ge 0$, and it is generally defined in the complex domain with $\gamma = \rho + i\eta$ where $\rho \ge 0$ . As a result, the Mellin transform can be generalized as follows:

$$\mathcal{M}[m(t)\cdot w(a,b)](\gamma) = \int_a^b \frac{m(t)}{t^{1-\gamma}}dt,$$

that extends naturally to the complex domain with $\gamma = \rho + i\eta$, and the A-S-K integral can be re-defined in terms of Mellin transform as follows:
$I_\beta^{A-S-K}[m(t)](t) = c(\alpha,\ \beta)\int_a^b \frac{m(x)}{x^{1-\gamma}}dx$ with an inverse Mellin transform, expressed as:

$$m(t)\cdot w(a,b) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\mathcal{M}[m(t)](\gamma)}{t^\gamma}d\gamma.$$

The above approach is just an outline of future extension of the current results to make them applicable in broader settings such as computational modeling in engineering sciences.

**Remark 2.** It is easy to propose the following conformable version of the classical Mean Value Theorem for A-S-K $\alpha$- integrals.

**Theorem 8.** *Assume that $a > 0$ and $m : [a,b] \to R$ be a given function that satisfies:*
*(i)* $\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}\int_a^t \frac{m(x)}{x^{1-\alpha}}dx \ \forall t \in [a,b]$, *exists for some* $\alpha \in (0,1]$.
*(ii)* $\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}h \le m(t) \le \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}H \ \forall t \in [a,b]$, *for certain real numbers $h$ and $H$.*
*Then, $\exists\ \mu \in [h,H]$ such that*

$$\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}\int_a^b \frac{m(t)}{t^{1-\alpha}}dt = \frac{\mu}{\alpha}(b^\alpha - a^\alpha). \tag{3.3}$$

**Proof.** By the monotonicity property of the usual definite integrals, [12], we have:

$$\frac{h}{\alpha}(b^\alpha - a^\alpha) = \int_a^b \frac{h}{t^{1-\alpha}}dt \le \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}\int_a^b \frac{m(t)}{t^{1-\alpha}}dt \le \int_a^b \frac{H}{t^{1-\alpha}}dt \le \frac{H}{\alpha}(b^\alpha - a^\alpha)$$

Multiplying by $\frac{\alpha}{b^{\alpha}-a^{\alpha}}$, then

$$h \le \mu \le H. \tag{3.4}$$

where $\mu = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \frac{\alpha}{b^{\alpha}-a^{\alpha}} \int_a^b \frac{m(t)}{t^{1-\alpha}} dt$. Hence, Eq. (3.3) is obtained, and the proof is completed.

**Corollary 2.** *If $m$ is a continuous function on the closed interval $[a, b]$, then $\exists\, v \in [a, b]$ such that*

$$\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^b \frac{m(t)}{t^{1-\alpha}} dt = \frac{\mathrm{m}(v)}{\alpha} \left(b^{\alpha} - a^{\alpha}\right). \tag{3.5}$$

**Proof.** By simply applying the classical Maximum and Minimum Value Theorem, [13], to Eq. (3.5).

**Theorem 9.** *Let $a > 0$ and $m : [a, b] \to R$ be continuous function. Then, for $\alpha \in (0, 1]$,*

$$\left| \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^b \frac{m(t)}{t^{1-\alpha}} dt \right| \le \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^b \frac{|m(t)|}{t^{1-\alpha}} dt. \tag{3.6}$$

**Proof.** The result follows directly since

$$\left| \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^b \frac{m(t)}{t^{1-\alpha}} dt \right| \le \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^b \left| \frac{m(t)}{t^{1-\alpha}} \right| dt \le \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^b \frac{|m(t)|}{t^{1-\alpha}} dt. \tag{3.7}$$

**Corollary 3.** *Let $m : [a, b] \to R$ be continuous function such that*

$$H = \max_{[a,b]} |m(t)|\,.$$

*Then, any $t \in [a, b]$, $\alpha \in (0, 1]$,*

$$\left| \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^b \frac{m(t)}{t^{1-\alpha}} dt \right| \le \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \frac{H}{\alpha} \left(b^{\alpha} - a^{\alpha}\right). \tag{3.8}$$

**Proof.** From Theorem 9, we have that for any $t \in [a, b]$, $\alpha \in (0, 1]$,

$$\begin{aligned} \left| \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^b \frac{m(t)}{t^{1-\alpha}} dt \right| &\le \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_a^b \frac{|m(t)|}{t^{1-\alpha}} dt \\ &\le \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} H \int_a^b t^{\alpha-1} dt = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \frac{H}{\alpha} \left(b^{\alpha} - a^{\alpha}\right). \end{aligned}$$

## 4. The A-S-K Power Series Representation

In [14], some essential results concerning the classical power series are generalized to the fractional power series. In this section, our goal is to establish various differentiability and integrability theorems for the sum of a fractional power series in the sense of the A-S-K derivative and integral definitions.
First, let us recall some elements about the fractional power series [14].

**Definition 3.** *A power series representation is expressed in the following form:*

$$\sum_{s=0}^{\infty} c_s (t - t_0)^{s\alpha} = c_0 + c_1 (t - t_0)^{\alpha} + c_2 (t - t_0)^{2\alpha} + \ldots. \tag{4.1}$$

*where $0 < \alpha \le 1$ and $t \ge t_0 \ge 0$ is called as the fractional power series about $t_0$, and $c_s$´s are constants which are called the coefficients of the series.*

In particular, when $t_0 = 0$, $\sum_{s=0}^{\infty} c_s t^{s\alpha}$ is known as a fractional Maclaurin series. For simplicity, we will only present the case when $t_0 = 0$ as follows:

**Theorem 10.** [14]. *We have the following two cases for the fractional power series $\sum_{s=0}^{\infty} c_s t^{s\alpha}$, $t \ge 0$:*

1. *If the fractional power series* $\sum_{s=0}^{\infty} c_s t^{s\alpha}$ *is convergent when* $t = b > 0$, *then it is convergent whenever* $0 \leq t < b$.

2. *If the fractional power series* $\sum_{s=0}^{\infty} c_s t^{s\alpha}$ *is divergent when* $t = d > 0$, *then it is divergent whenever* $t > d$.

**Theorem 11.** [14]. *For the fractional power series* $\sum_{s=0}^{\infty} c_s t^{s\alpha}$ , $t \geq 0$, *there are only three cases:*

1. $\sum c_s t^{s\alpha}$ *is convergent only when* $t = 0$,

2. $\sum c_s t^{s\alpha}$*is convergent for each* $t \geq 0$,

3. *There is a positive real number* $R$ *such that the series is convergent whenever* $0 \leq t < R$, *and it is divergent whenever* $t > R$.

**Remark 3.** From Theorem 11 case (3), $R$ is the radius of convergence of the fractional power series. By convention, the radius of convergence is $R = 0$ in case 1 and $R = \infty$ in case (2).
**Remark 4.** Assume that the following fractional power series $\sum_{s=0}^{\infty} c_s t^{s\alpha}$, $t \geq 0$ has $R \geq 0$. As in the classical case, [12], it is easy to prove that if $0 < \rho < R$, then the fractional power series converges uniformly on the interval $[0, \rho]$, and the sum of the series is continuous in $[0, R)$.
**Remark 5.** To investigate the convergence of the fractional power series, we test that by ratio test as follows [15]:

$$\lim_{s\to\infty} \left| \frac{c_s t^{s\alpha}}{c_{s+1} t^{(s+1)\alpha}} \right| = \lim_{s\to\infty} \left| \frac{c_s}{c_{s+1} t^{\alpha}} \right|$$

Now if we assume $l = \lim_{s\to\infty} \left| \frac{c_s}{c_{s+1}} \right|$ , then:

1. If $l = 0$, the series will diverge for all $t \neq 0$.

2. If $0 < l < \infty$, then $\sum c_s t^{s\alpha}$ converges if $0 \leq t < l^{\frac{1}{\alpha}}$ and diverges if $t > l^{\frac{1}{\alpha}}$.

3. If $l = \infty$, the series converges for all $t \in [0, \infty)$.

**Example 1.** Applying the above result, we study the convergence of some fractional power series.

1. The geometric series $\sum_{k=0}^{\infty} (-1)^k t^{k\alpha} = 1 + t^\alpha + t^{2\alpha} + t^{3\alpha} + \ldots$ has radius of convergence $R = 1$, since $l = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| = 1$. At $t = 1$, the series becomes $1 + 1 + 1 + 1 + \ldots$, which diverges. Thus, the interval of convergence of the power series is $[0, 1)$. The series converges uniformly on $[0, \rho]$ for every $0 < \rho < 1$, but it does not converge uniformly on $[0, 1)$.

2. Let $0 < \alpha < 1$. The power series $\sum_{k=0}^{\infty} \left( \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)} \right)^k \frac{t^{k\alpha}}{k!} = 1 + \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)} t^\alpha + \left( \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)} \right)^2 \frac{t^{2\alpha}}{2!} + \left( \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)} \right)^3 \frac{t^{3\alpha}}{3!} + \ldots$ has a radius of convergence $R = \infty$, since $l = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{\left( \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} \right)^{k+1} (k+1)!}{\left( \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} \right)^k k!} \right| = \lim_{k\to\infty} \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} (k+1) = \infty$ .

3. The power series $\sum_{k=0}^{\infty} ((2k+1)!) t^{k\alpha} = 1 + (3!) t^\alpha + (5!) t^{2\alpha} + (7!) t^{3\alpha} + \ldots$ has a radius of convergence $R = 0$, since $l = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{(2k+1)!}{(2k+3)!} \right| = \lim_{k\to\infty} \frac{1}{(2k+2)(2k+3)} = 0$ .

**Remark 6.** As in the classical case, we will show that the sum of a fractional power series $m(t) = c_0 + c_1 t^\alpha + c_2 t^{2\alpha} + c_3 t^{3\alpha} + \ldots$ is infinitely A-S-K $\alpha$-*DF inside* its interval of convergence, and it's A-S-K $\alpha-$derivative:
$D^{A-S-K}\ m(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} \alpha c_1 + \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} 2\alpha c_2 t^\alpha + \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} 3\alpha c_3 t^{2\alpha} + \ldots$ is given by the term-by-term differentiation. To show this, we first prove that the term-by-term A-S-K $\alpha-$derivative of a fractional power series has the same radius of convergence as the original fractional power series.

**Theorem 12.** *Assume that the fractional power series* $\sum_{s=0}^{\infty} c_s t^{s\alpha}$ *has radius of convergence* $R$. *Then, the fractional power series* $\sum_{s=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} s\alpha c_s t^{(s-1)\alpha}$ *also has radius of convergence* $R$.

**Proof.** See the proof of Theorem 3.3 in [14], and note that $\sum_{s=0}^{\infty} c_s t^{s\alpha}$ and $\sum_{s=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} s\alpha c_s t^{(s-1)\alpha}$ have the same behaviour.

**Theorem 13.** *Assume that the fractional power series* $m(t) = \sum_{s=0}^{\infty} c_s t^{s\alpha}$ *for all* $t \in [0, R)$, *has the radius of convergence* $R > 0$ *and sum* $m$. *Then,* $m$ *is A-S-K* $\alpha$*-DF in* $[0, R)$, *and*

$$D^{A-S-K}\ m(t) = \sum_{s=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} s\alpha c_s t^{(s-1)\alpha}.$$

**Proof.** According to the term-by-term differentiation of the fractional power series, it is convergent in $[0, R)$ by Theorem 12. Suppose that its sum is $p(t) = \sum_{s=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} s\alpha c_s t^{(s-1)\alpha}$. Assume that $0 < \rho < R$. Then, by Remark 4, the fractional power series for $m$ and $p$ both converge uniformly in $[0, \rho)$. By applying the classical result on the differentiability of the sequences of functions [9] to their succession of partial sums, we conclude that $m$ is differentiable in $[0, \rho)$ and $D^{A-S-K}\ m(t) = p(t)$. Since this holds for every $0 < \rho < R$, it follows that $m$ is differentiable in $[0, R)$ and $D^{A-S-K}\ m(t) = p(t)$, which the result is proven.

Repeated application of theorem 13 implies that the sum of a fractional power series is infinitely A-S-K $\alpha$-*DF* inside its interval of convergence and its derivatives are given by the term-by-term differentiation of the fractional power series. In addition, we can get an expression for the coefficients $c_s$ in terms of the function $m$, named as as the fractional Taylor coefficients of $m$ at $t = 0$.

**Theorem 14.** *If the fractional power series* $m(t) = \sum_{s=0}^{\infty} c_s t^{s\alpha}$ *has a radius of convergence* $R > 0$, *then* $m$ *is infinitely A-S-K* $\alpha$*-DF in* $[0, R)$, *and* $c_s = \left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\right)^s \frac{{}^{(s)}D^{A-S-K}\ m(0)}{s!}$, *where* ${}^{(s)}D^{A-S-K}\ m = D^{A-S-K}\ D^{A-S-K} \ldots D^{A-S-K}\ m$, $s$ *times.*

**Proof.** By applying Theorem 13 to the power series $m(t) = c_0 + c_1 t^{\alpha} + c_2 t^{2\alpha} + c_3 t^{3\alpha} + \cdots + c_s t^{s\alpha} + \ldots$, $n$ times, we find that $m$ has $\alpha-$derivatives of every order in $[0, R)$, and

$$\begin{aligned}
{}^{(1)}D^{A-S-K}\ m(t) =& \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha c_1 + 2\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha c_2 t^{\alpha} + 3\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha c_3 t^{2\alpha} + \ldots \\
&+ s\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha c_s t^{(s-1)\alpha} + \ldots \\
{}^{(2)}D^{A-S-K}\ m(t) =& 2\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 c_2 + (3 \cdot 2)\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 c_3 t^{\alpha} + \ldots \\
&+ s(s-1)\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 c_s t^{(s-2)\alpha} + \ldots \\
{}^{(3)}D^{A-S-K}\ m(t) =& (3 \cdot 2 \cdot 1)\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^3 c_3 + \ldots \\
&+ s(s-1)(s-2)\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^3 c_s t^{(s-3)\alpha} + \ldots \\
{}^{(n)}D^{A-S-K}\ m(t) =& (n!)\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^n c_n + \ldots \\
&+ s(s-1)(s-2)\ldots(s-n+1)\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^n c_s t^{(s-n)\alpha} + \ldots
\end{aligned}$$

where all of these power series have the same radius of convergence $R$. Finally, we obtain: $c_0 = m(0)$, $c_1 = \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)} {}^{(1)}D^{A-S-K}\ m(0)$, $\ldots$, $c_n = \left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\right)^n \frac{{}^{(n)}D^{A-S-K}\ m(0)}{n!}$, which proves the result.

**Remark 7.** If the power series is centred at point $t_0 > 0$, that is,
$m(t) = \sum_{s=0}^{\infty} c_s (t - t_0)^{s\alpha}$, for all $t \in [t_0, t_0 + R)$.

from the above theorem, it is easy to obtain:
$c_s = \left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\right)^s \frac{{}^{(s)}D^{A-S-K}\ m(t_0)}{s!}$, for every $s = 0, 1, 2, \ldots$
Finally, we will show that the fractional power series can be integrated term by term on every closed interval that contains in its interval of convergence.

**Theorem 15.** *Assume that the fractional power series $\sum_{s=0}^{\infty} c_s t^{s\alpha}$ has a radius of convergence $R > 0$. Then:*

1. *The A-S-K $\alpha-$integrated power series $\sum_{s=0}^{\infty} \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)(s+1)} c_s t^{(s+1)\alpha}$ converges for all $t \in [0, R)$.*
2. *If $t \in [0, R)$, it is verified that*

$$\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_0^t \left(\sum_{s=0}^{\infty} c_s x^{s\alpha}\right) \frac{dx}{x^{1-\alpha}} = \sum_{s=0}^{\infty} \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_0^t c_s x^{s\alpha} \frac{dx}{x^{1-\alpha}}$$
$$= \sum_{s=0}^{\infty} \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)(s+1)} c_s t^{(s+1)\alpha}.$$

**Proof.**

1. From a certain $s$, $\frac{\Gamma(\beta-\alpha+1)t}{\alpha\Gamma(\beta)(s+1)} \leq 1,\ \forall t \in [0, R)$. Then, we have:

$$\left|\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)(s+1)} c_s t^{(s+1)\alpha}\right| \leq \frac{\Gamma(\beta-\alpha+1)\,t}{\alpha\Gamma(\beta)(s+1)} |c_s|\, t^{s\alpha} \leq |c_s|\, t^{s\alpha}.$$

According to the comparison test, the integrated series converges.

1. Since $\sum c_s x^{s\alpha}$ is a series of continuous functions that converge uniformly in the closed interval $[0, t]$, we can integrate term by term. The above equality is a consequence of the convergence of $\sum \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)(s+1)} c_s t^{(s+1)\alpha}$ at both 0 and $t$.

**Example 2.** Let $0 < \alpha < 1$. The power series $\sum_{K=o}^{\infty} (k+1) \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} t^{k\alpha}$ has a radius of convergence $R = 1$ since we have: $l = \lim_{k\to\infty} \left|\frac{c_k}{c_{k+1}}\right| = \lim_{k\to\infty} \frac{\frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}(k+1)}{\frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}(k+2)} = 1.$
By taking $f(t) = \sum (k+1) \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} t^{k\alpha}$ for all $t \in [0, 1)$, we have:

$$\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_0^t f(t) \frac{dt}{t^{1-\alpha}} = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_0^t \left(\sum_{k=o}^{\infty} (k+1) \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} t^{k\alpha}\right) \frac{dt}{t^{1-\alpha}}$$
$$= \sum_{k=0}^{\infty} \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \int_0^t (k+1) \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} t^{k\alpha} \frac{dt}{t^{1-\alpha}}$$
$$= \sum_{k=0}^{\infty} t^{(k+1)\alpha} = \frac{t^{\alpha}}{1-t^{\alpha}}.$$

By differentiating both sides of the above equality, we obtain:

$$D^{A-S-K} f(t) = D^{A-S-K} \left(\frac{t^{\alpha}}{1-t^{\alpha}}\right) = \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} \frac{1}{(1-t^{\alpha})^2}.$$

Now, we consider the most general sequential linear homogeneous A-S-K fractional differential equation, expressed as:

$${}^{(n)}D^{A-S-K}\ y(t) + a_{n-1}(t)\,{}^{(n-1)}D^{A-S-K}\ y(t) + \cdots + a_1(t)\,{}^{(1)}D^{A-S-K}\ y(t) + a_0(t)\,y(t) = 0. \quad (4.2)$$

**Definition 4.** *Let* $\alpha \in (0,1]$ *and* $m(t)$ *be a real function defined on* $[0,a]$. *In this case,* $m(t)$ *is said to be a A-S-K* $\alpha$*- analytic at* 0 *if* $m(t)$ *can be written as:*

$$\sum_{s=0}^{\infty} c_s t^{s\alpha} \qquad (c_s \in R)$$

$\forall\, t \in [0,R)$. $R$ *is the radius of convergence of this fractional power series.*

**Definition 5.** *Let* $\alpha \in (0,1]$ *and the functions* $a_s(t)$ *be a A-S-K* $\alpha$*- analytic at* $t=0$, *for* $s = 0,1,2,\ldots,\ n-1$. *In this case, the point* $t=0$ *is said to be a A-S-K* $\alpha$*-ordinary point of Eq. (4.2).*

**Theorem 16.** *Let* $\alpha \in (0,1]$ *and* $c_0, c_1 \in R$. *If* $t=0$ *is a A-S-K* $\alpha$*- ordinary point of the fractional differential equation*

$$^{(2)}D^{A-S-K}\ y(t) + a(t)\, ^{(1)}D^{A-S-K}\ y(t) + b(t)\, y(t) = 0, \tag{4.3}$$

*then there is a solution of the above equation as:*

$$y(t) = \sum_{s=0}^{\infty} c_s t^{s\alpha}. \tag{4.4}$$

*for* $t \in [0,R)$ *with* $R = min\{R_1, R_2\}$ *and initial conditions* $y(0) = c_0$ *and* $^{(1)}D^{A-S-K}\ y(0) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha c_1$, *where* $R_1$ *and* $R_2$ *are the radii of convergence of* $a(t)$ *and* $b(t)$, *respectively.*

**Proof.** Since $t=0$ a A-S-K $\alpha$-analytic point of Eq. (4.3), by Definitions 4 and 5, we can write:

$$a(t) = \sum_{s=0}^{\infty} a_s t^{s\alpha},\ \forall t \in [0,R_1). \tag{4.5}$$

and

$$b(t) = \sum_{s=0}^{\infty} b_s t^{s\alpha},\ \forall t \in [0,R_1). \tag{4.6}$$

Now, we look for a solution in the form Eq. (4.4) of Eq. (4.3). By substituting Eq. (4.4) and it is A-S-K $\alpha$-derivative in Eq. (4.3), then we obtain:

$$\sum_{s=0}^{\infty}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 (s+2)(s+1)\, c_{s+2}\, t^{s\alpha} + \left(\sum_{s=0}^{\infty} a_s t^{s\alpha}\right)\left(\sum_{s=0}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha(s+1)\, c_{s+1}\, t^{s\alpha}\right)$$
$$+\left(\sum_{s=0}^{\infty} b_s t^{s\alpha}\right)\left(\sum_{s=0}^{\infty} c_s t^{s\alpha}\right) = 0.$$

Then, it is easy to see that the coefficients $c_s$ satisfy the following recurrence formula:

$$\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 (s+2)(s+1)\, c_{s+2} = -\sum_{n=0}^{s}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha(n+1)\, a_{s-n} c_{n+1} + b_{s-n} c_n\right). \tag{4.7}$$

Next, we are going to show that if the coefficients $c_s$ defined by the above formula, then the fractional series $y(t) = \sum_{s=0}^{\infty} c_s t^{s\alpha}$ is convergent for $t \in [0,R)$. Let us fix $\rho$, such that $0 < \rho < R$. Since the fractional series in Eq. (4.5) and Eq. (4.6) are convergent for $t \in [0,\rho)$, there is a constant $H > 0$ such that

$$|a_{s-n}| \le \frac{H}{\rho^{(s-n)\alpha}},\ |b_{s-n}| \le \frac{H}{\rho^{(s-n)\alpha}},\ (s \in N \cup \{0\},\ 0 \le n \le s). \tag{4.8}$$

Using Eq. (4.8) in Eq. (4.7), we obtain:

$$\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 (s+2)(s+1)c_{s+2} \leq \frac{H}{\rho^{s\alpha}}\sum_{n=0}^{s}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha(n+1)\left|c_{n+1}\right|+\left|c_n\right|\right)\rho^{n\alpha}$$
$$\leq \frac{H}{\rho^{s\alpha}}\sum_{n=0}^{s}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha(n+1)\left|c_{n+1}\right|+\left|c_n\right|\right)\rho^{n\alpha}+H\left|c_{n+1}\right|\rho^{\alpha}.$$

Now, we define:

$$C_0=\left|c_0\right|,\ C_1=\left|c_1\right|$$

and

$$\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 (s+2)(s+1)C_{s+2} = \frac{H}{\rho^{s\alpha}}\sum_{n=0}^{s}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha(n+1)C_{n+1}+C_n\right)\rho^{n\alpha}+HC_{s+1}\rho^{\alpha}. \tag{4.9}$$

for $s \geq 2$.
Using mathematical induction, it is easy to prove that $|c_n| \leq C_n$, $C_n \geq 0$ for $n=0,1,2,\ldots$
Now, we are going to study so that $t$ the series:

$$\sum_{s=0}^{\infty} C_s t^{s\alpha}. \tag{4.10}$$

is convergent.
Using Eq. (4.10), we obtain:

$$\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 s(s+1)C_{s+1} = \frac{H}{\rho^{(s-1)\alpha}}\sum_{n=0}^{s-1}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha(n+1)C_{n+1}+C_n\right)\rho^{n\alpha}+HC_s\rho^{\alpha}. \tag{4.11}$$

$$\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 s(s-1)C_{s} = \frac{H}{\rho^{(s-2)\alpha}}\sum_{n=0}^{s-2}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha(n+1)C_{n+1}+C_n\right)\rho^{n\alpha}+HC_{s-1}\rho^{\alpha}. \tag{4.12}$$

From Eq. (4.11) and Eq. (4.12), we have:

$$\rho^{\alpha}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 s(s+1)C_{s+1} = \left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 s(s-1)C_{s} + \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha Hs\rho^{\alpha}C_s+HC_s\rho^{2\alpha}. \tag{4.13}$$

Hence,

$$\frac{C_s}{C_{s+1}} = \frac{\rho^{\alpha}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 s(s+1)}{\left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 s(s-1)+\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha Hs\rho^{\alpha}+H\rho^{2\alpha}}.$$

is obtained. By the help of ratio test, we have that

$$\lim_{s\to\infty}\left|\frac{c_s}{c_{s+1}}\right| = \rho^{\alpha}.$$

Thus, the series Eq. (4.10) converges for $t \in [0,\rho)$. This implies that the series Eq. (4.4) converges for $t \in [0,\rho)$. Since $\rho$ was any number satisfying $0<\rho<R$, the series Eq. (4.4) converges for $t \in [0,R)$.

## 5. The A-S-K Chebyshev-I and its solutions

Consider the following A-S-K Chebyshev-I:

$$\left(1-t^{2\alpha}\right){}^{(2)}D^{A-S-K}\ y\left(t\right)-\frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta-\alpha+1\right)}\alpha t^{\alpha}\ {}^{(1)}D^{A-S-K}\ y\left(t\right)+\left(\frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta-\alpha+1\right)}\alpha\right)^{2}w^{2}y\left(t\right)=0. \tag{5.1}$$

where $\alpha\in[0,1)$, $t>0$ and $w$ is a real constant. If $\alpha=\beta=1$, then Eq. (5.1) becomes the classical Chebyshev's differential equation [13]. $t=0$ is an ordinary point of Eq. (5.1). Now, by using the A-S-K fractional power series method and proposing the following solution:

$$y\left(t\right)=\sum_{s=0}^{\infty}c_{s}t^{s\alpha}. \tag{5.2}$$

By substituting Eq. (5.2) and the A-S-K fractional derivatives in Eq. (5.1), we have:

$$2c_{2}+3\cdot 2\ c_{3}t^{\alpha}-c_{1}t^{\alpha}+w^{2}c_{0}+w^{2}c_{1}t^{\alpha}+\sum_{s=2}^{\infty}\left[\left(s+2\right)\left(s+1\right)c_{s+2}-\left(s^{2}-w^{2}\right)c_{s}\right]t^{s\alpha}=0.$$

The above equation implies:

$$2c_{2}+w^{2}c_{0}=0.$$
$$3\cdot 2\ c_{3}-\left(1-w^{2}\right)c_{1}=0.$$
$$\left(s+2\right)\left(s+1\right)c_{s+2}-\left(s^{2}-w^{2}\right)c_{s}\ ,s\geq 2\ .$$

Note that the recurrence formula above can be written as:

$$c_{s+2}=\frac{s^{2}-w^{2}}{\left(s+2\right)\left(s+1\right)}c_{s}\ ,s\geq 2\ .$$

Hence, we obtain:

$$c_{2}=-\frac{w^{2}}{2}c_{0}.$$
$$c_{3}=\frac{1-w^{2}}{3!}c_{1}.$$
$$c_{4}=\frac{4-w^{2}}{4\cdot 3}c_{2}=-\frac{w^{2}\left(4-w^{2}\right)}{4!}c_{0}.$$
$$c_{5}=\frac{9-w^{2}}{5\cdot 4}c_{3}=\frac{\left(1-w^{2}\right)\left(9-w^{2}\right)}{5!}c_{1}.$$

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In general, we can write:

$$c_{2s}=\frac{\left(-w^{2}\right)\left(4-w^{2}\right)\left(16-w^{2}\right)\ldots\left(\left(2s-2\right)^{2}-w^{2}\right)}{\left(2s\right)!}c_{0}.$$

$$c_{2s+1}=\frac{\left(1-w^{2}\right)\left(9-w^{2}\right)\left(25-w^{2}\right)\ldots\left(\left(2s-1\right)^{2}-w^{2}\right)}{\left(2s+1\right)!}c_{1}.$$

for $s\geq 1$.

As a result, we have the following general solution of Eq. (5.1):

$$y\left(t\right)=c_{0}\left(1+\sum_{s=1}^{\infty}\frac{\left(-w^{2}\right)\left(4-w^{2}\right)\left(16-w^{2}\right)\ldots\left(\left(2s-2\right)^{2}-w^{2}\right)}{\left(2s\right)!}t^{2s\alpha}\right)$$

$$+c_1\left(t^\alpha+\sum_{s=1}^{\infty}\frac{\left(1-w^2\right)\left(9-w^2\right)\left(25-w^2\right)\ldots\left((2s-1)^2-w^2\right)}{(2s+1)!}t^{(2s+1)\alpha}\right).$$

**Remark 7.** Note that the above equation can be written in the following form:

$$y(t)=c_0y_1(t)+c_1y_2(t)$$

where

$$y_1(t)=1+\sum_{s=1}^{\infty}\frac{\left(-w^2\right)\left(4-w^2\right)\left(16-w^2\right)\ldots\left((2s-2)^2-w^2\right)}{(2s)!}t^{2w\alpha}. \tag{5.3}$$

$$y_2(t)=t^\alpha+\sum_{s=1}^{\infty}\frac{\left(1-w^2\right)\left(9-w^2\right)\left(25-w^2\right)\ldots\left((2s-1)^2-w^2\right)}{(2s+1)!}t^{(2s+1)\alpha}. \tag{5.4}$$

To find the radius of convergence of the series Eq. (5.3) and Eq. (5.4), we will use the ratio test as follows:

$$\lim_{s\to\infty}\left|\frac{(2s+2)!\left(-w^2\right)\left(4-w^2\right)\left(16-w^2\right)\ldots\left((2s-2)^2-w^2\right)}{(2s)!\left(-w^2\right)\left(4-w^2\right)\left(16-w^2\right)\ldots\left((2s)^2-w^2\right)}\right|=\lim_{s\to\infty}\frac{(2s+2)(2s+1)}{\left((2s)^2-w^2\right)}=1$$

$$\lim_{s\to\infty}\left|\frac{(2s+3)!\left(1-w^2\right)\left(9-w^2\right)\left(25-w^2\right)\ldots\left((2s-1)^2-w^2\right)}{(2s+1)!\left(1-w^2\right)\left(9-w^2\right)\left(25-w^2\right)\ldots\left((2s+1)^2-w^2\right)}\right|=\lim_{s\to\infty}\frac{(2s+3)(2s+2)}{\left((2s+1)^2-w^2\right)}=1$$

Hence, from Remark 5 it follows that the radius of convergence is $R=1$ in both cases, and in that case $y(t)$ converges for $t\in[0,1)$.

**Remark 8.** Note that if $w\in N$, one of the above series Eq. (5.3) or Eq. (5.4) is truncated at $s=\frac{w+2}{2}$ if $w$ is even, or at $s=\frac{w+1}{2}$ if $w$ is odd. In the first case, a polynomial of degree $w$ is obtained that has only even powers of $t^\alpha$, and in the second case, a polynomial that has only odd powers of $t^\alpha$, also of degree $w$, In short, is that if $w\in N$ polynomial solution of A-S-K Chebyshev-I is obtained.
For example, if $w=2,4,6,8$ the corresponding polynomial solutions are

$$W(t)=1-2t^{2\alpha}.$$

$$W(t)=1-8t^{2\alpha}+8t^{4\alpha}.$$

$$W(t)=1-18t^{2\alpha}+48t^{4\alpha}-32t^{6\alpha}.$$

$$W(t)=1-32t^{2\alpha}-160t^{4\alpha}+256t^{6\alpha}-128t^{8\alpha}.$$

and if $w=1,3,5,7$ the corresponding polynomial solutions are

$$W(t)=t^\alpha.$$

$$W(t)=t^\alpha-\frac{4}{3}t^{3\alpha}.$$

$$W(t)=t^\alpha-4t^{3\alpha}+\frac{16}{5}t^{5\alpha}.$$

$$W(t)=t^\alpha-8t^{3\alpha}+16t^{5\alpha}-\frac{64}{7}t^{7\alpha}.$$

These polynomials, when multiplied by a suitable constant, are known as A-S-K Chebyshev-I polynomials, as shown below.

**Remark 9.** Based on the definition of A-S-K of order $\alpha$ and its properties, the A-S-K Chebyshev-I polynomials can be expressed by the following Rodrigues' type formula:

$$T_w^\alpha(t) = \frac{(-2)^w w!}{(2w)!}\left(\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\right)^w \sqrt{1-t^{2\alpha}}^{(w)} D^{A-S-K}\left(1-t^{2\alpha}\right)^{w-\frac{1}{2}}. \tag{5.5}$$

for $w = 0, 1, 2, \ldots$ and $\alpha \in (0, 1]$. If $\alpha = \beta = 1$, then Eq. (5.5) becomes the classical Rodrigues' formula for Chebyshev-I polynomials [13]. Thus, from Eq. (5.5) the A-S-K Chebyshev-I polynomials can be obtained by putting $w = 0, 1, 2, \ldots$ as:

$$T_0^\alpha(t) = 1.$$

$$T_1^\alpha(t) = t^\alpha.$$

$$T_2^\alpha(t) = 2t^{2\alpha} - 1.$$

$$T_3^\alpha(t) = 4t^{3\alpha} - 3t^\alpha.$$

$$T_4^\alpha(t) = 8t^{4\alpha} - 8t^{2\alpha} + 1.$$

$$T_5^\alpha(t) = 16t^{5\alpha} - 20t^{3\alpha} + 5t^\alpha.$$

$$T_6^\alpha(t) = 32t^{6\alpha} - 48t^{4\alpha} + 18t^{2\alpha} - 1.$$

$$T_7^\alpha(t) = 64t^{7\alpha} - 112t^{5\alpha} + 56t^{3\alpha} - 7t^\alpha.$$

$$T_8^\alpha(t) = 128t^{8\alpha} - 256t^{6\alpha} + 160t^{4\alpha} - 32t^{2\alpha} + 1.$$

and so on.

**Remark 10.** As in the classical case, also the A-S-K Chebyshev-I polynomials can be defined by the condition:

$$T_w^\alpha(t) = cos\left(w\cos^{-1}(t^\alpha)\right),\ w \in N \cup \{0\},\ t \in [0, 1). \tag{5.6}$$

It is easy to show that these polynomials satisfy the following recurrence formula, which involves the A-S-K derivative of $T_w^\alpha(t)$:

$$\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\left(1-t^{2\alpha}\right) D^{A-S-K}\, \underset{w}{\overset{\alpha}{T}}(t) = -wt^\alpha T_w^\alpha(t) + wT_{w-1}^\alpha(t). \tag{5.7}$$

Indeed, if we take the following change of variable:

$$\theta = \cos^{-1}(t^\alpha).$$

Therefore, from Theorem 2 we have:

$$t^\alpha = cos\theta \Rightarrow 1 = -\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)} sin\theta D^{A-S-K}\,\theta \Rightarrow D^{A-S-K}\,\theta = -\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\,\frac{1}{sin\theta}.$$

Then, according to the equation above, we have that

$$D^{A-S-K}\, T_w^\alpha(t) = D^{A-S-K}\,(cos(w\theta)) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha w\frac{sin(w\theta)}{sin\theta}. \tag{5.8}$$

Finally, using Eq. (5.8) we are going to construct Eq. (5.7) from its right-hand side:

$$\begin{aligned} -wt^\alpha T_w^\alpha(t) + wT_{w-1}^\alpha(t) &= -\, wt^\alpha cos(w\theta) + wcos((w-1)\theta) \\ &= -\, wt^\alpha cos(w\theta) + w\left(cos(w\theta)\, cos\theta + sin(w\theta)\, sin\theta\right) \end{aligned}$$

$$= -wt^\alpha cos(w\theta) + wt^\alpha cos(w\theta) + wsin(w\theta)\, sin\theta = wsin(w\theta)\, sin\theta$$

$$= \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)} sin^2\theta \left( \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} \alpha w \frac{sin(w\theta)}{sin\theta} \right) = \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)} \left(1-t^{2\alpha}\right) D^{A-S-K} \underset{w}{\overset{\alpha}{T}}(t) \ .$$

**Remark 11.** When the first two A-S-K Chebyshev-I polynomials $T_0^\alpha(t)$ and $T_1^\alpha(t)$ are known, all the remaining polynomials $T_w^\alpha(t)$ for $w \geq 2$ can be obtained from the recurrence formula

$$T_{w+1}^\alpha(t) = 2t^\alpha T_w^\alpha(t) - T_{w-1}^\alpha(t) \ , \ t \in [0,1) \ . \tag{5.9}$$

In fact, if we take in Eq. (5.8) the change of variable $\theta = \cos^{-1}(t^\alpha)$ , one can write

$$\begin{aligned} T_{w+1}^\alpha(t) =& cos((w+1)\theta) = cos(w\theta)cos\theta + sin(w\theta)\, sin\ \theta \\ T_{w-1}^\alpha(t) =& cos((w-1)\theta) = cos(w\theta)cos\theta - sin(w\theta)\, sin\theta. \end{aligned}$$

Finally, by a*dding the above equations our result follows directly.

**Remark 12.** We end this section by including another form of the A-S-K Chebyshev-I in Eq. (5.1). Therefore, this equation can be converted to a simpler form using the change of variable $\theta = \cos^{-1}(t^\alpha)$ . Indeed, since $D^{A-S-K}\ \theta = -\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\frac{1}{sin\theta}$ using Theorem 2 we have:

$$D^{A-S-K}\ y = -\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\frac{1}{sin\theta}\frac{dy}{d\theta}.$$

$$^{(2)}D^{A-S-K}\ y = \left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\alpha\right)^2 \frac{1}{sin^2\theta}\left(\frac{d^2y}{d\theta^2} - \frac{cos\theta}{sin\theta}\frac{dy}{d\theta}\right).$$

Substituting the expressions of the A-S-K derivatives into the Eq. (5.1) gives:

$$\frac{d^2y}{d\theta^2} + w^2 y = 0.$$

The general solution of the above equation is given by the formula:

$$y(\theta) = C_1 cos(w\theta) + C_2 sin(w\theta),$$

which can be also written as:

$$y(\theta) = C cos(w\theta + \gamma).$$

Here, $C_1$, $C_2$, and $C$, $\gamma$ are arbitrary real numbers. For simplicity, we can set $\gamma = 0$. Then the general solution of the original A-S-K Chebyshev-I will be given by the formula:

$$y(t) = C cos\left(w\cos^{-1}(t^\alpha)\right).$$

In this expression, $w$ may be any real number. However, if $w \in N \cup \{0\}$, this given function represents, as known by the A-S-K Chebyshev-I polynomials.

## 6. Conclusion

Essential properties of the A-S-K fractional integral have been achieved, such as Barrow's rule or the mean value theorem. Likewise, the A-S-K fractional power series technique has been established, which has been successfully applied to find the solutions of the A-S-K Chebyshev-I. Also, some interesting properties of the A-S-K Chebyshev-I polynomials are derived. The research carried out highlights the fact that these recently obtained results are considered as a natural extension of classical differential calculus. The potential of this new definition of fractional derivative, both from a theoretical point of view and due to its applications, is evident through the developments and remarks included in the previous section. In short, this research can open a path for future works in which the results of classical mathematical analysis are extended in the sense of A-S-K derivative. Likewise, focusing on the field of applications, it may be interesting to extend the results obtained to important classical differential equations, such as those of Legendre, Hermite or Airy. In addition, a full numerical study based on our obtained theoretical results in this paper will be conducted in our future research studies.

## References


1. Podlubny, I., *Fractional Differential Equations*, Mathematics in Science and Engineering, Academic Press, Inc.: San Diego, CA, USA, (1999).

2. Miller, K. S., *An Introduction to Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Hoboken, NJ, USA, (1993).

3. Ahmed, H. M., Ahmed, A. M. S., Ragusa M. A., *On some non-instantaneous impulsive differential equations with fractional brownian motion and Poisson jumps.* TWMS Journal of Pure and Applied Mathematics 14(1), 125-140, (2023).

4. Hilal K., Kajouni A., Lmou H., *Existence and stability results for a coupled system of Hilfer fractional Langevin equation with non local integral boundary value conditions.* Filomat 37(4), 1241-1259, (2023).

5. Yang L., Li P. T., *Quantitative Weighted Bounds for Littlewood-Paley Functions Generated by Fractional Heat Semigroups Related with Schrödinger Operators.* Journal of Function Spaces 2023, Article ID 8001131, 1-16, (2023).

6. Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., *A new definition of fractional derivative.* Journal of Computational and Applied Mathematics 264, 65-70, (2014).

7. Zhao, D., Luo, M., *A new definition of fractional derivative.* Calcolo 54(3),903-917, (2017).

8. Khalil, R., Al Horani, M., Abu Hammad, M., *Geometric meaning of conformable derivative via fractional cords.* Journal of Mathematics and Computer Science 19(4),241-245, (2019).

9. Abdelhakim, A. A., *The flaw in the conformable calculus: it is conformable because it is not fractional.* Fractional Calculus and Applied Analysis 22(2),242-254, (2019).

10. Abu-Shady, M., Kaabar, M. K. A., *A Generalized Definition of the Fractional Derivative with Applications.* Mathematical Problems in Engineering 2021, Article ID 9444803, 1-9, (2021).

11. Martínez, F., Kaabar, M. K. A., *Novel Investigation of the Abu-Shady–Kaabar Fractional Derivative as a Modeling Tool for Science and Engineering.* Computational and Mathematical Methods in Medicine 2022, Article ID 119082, 1-8, (2022).

12. Apostol, T. M., *Calculus*, Volume 1 (2nd edition), John Wiley & Sons, USA, (1991).

13. Spivak, M., *Calculus*, Second Edition, Publish or Perisn. Inc, Houston, USA, (1980).

14. Martínez, F., Martínez, I., Kaabar, M. K. A., Paredes, S., *Some New Results on Conformable Fractional Power Series.* Asia Pacific Journal of Mathematics 7(31), 1-14, (2020)

15. Al Masalmeh, M., *Series Method to solve conformable fractional Riccati Differential equations.* International Journal of Applied Mathematics Research 6(1), 30-33, (2017)

16. Derrick, W. R., Grossman, S. I., *Elementary Differential Equations with Applications*, Addison-Wesley Publishing Company, Inc., de Reading, (1981).

17. Abu-Shady, M., Khokha, E. M., Abdel-Karim, T. A., *The generalized fractional NU method for the diatomic molecules in the Deng–Fan model.* The European Physical Journal D 76(9), , 159, (2022).

18. Abu-Shady, M., Ahmed, M. M. A., Gerish, N. H., *Generalized Fractional of the Extended Nikiforov-Uvaro Method for Heavy Tetraquark Masses Spectra.* Modern Physics Letters A 38(4), 2350028, (2023).



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