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Application of contraction theory for the Synchronization of complex chaotic T-system

Rogayyeh Soltani, Saeed Nezhadhosein, Bashir Naderi* and Aghileh Heydari

ABSTRACT: In this paper, we investigate the synchronization of the complex chaotic T-system, with known and unknown parameters. In addition, the behavior of this system for new parameters is discussed. The controllers and estimation rule of unknown parameter are designed by using contraction theory (CT). The main feature of the proposed approach is that the control design and stability of the synchronization error are obtained without using Lyapunov functions. The numerical simulations confirm that the intended method is robust and applicable to the mentioned systems.

Key Words: Synchronization, contraction theory, chaos.

Contents

1. Introduction

Chaos, as an important behavior in nonlinear dynamical systems, has been studied over the last four decades [16,23,32,24,10,31,2]. Hyper (chaotic) systems are nonlinear deterministic systems, such that they have complex and unpredictable behavior without any random state variable or nondeterministic parameter. The sensitivity of chaotic systems dependents on initial conditions and the systems' parameter variations. These systems have many important applications in science and engineering, e.g., nonlinear circuits, laser physics, secure communications, neural networks, synchronization, control, and active wave propagation [10,25,30,21,12,3,1,6,8]. Also, there are many interesting cases involving complex variables that have been explored. For example the complex Lorenz equations used to describe and simulate the physics of a detuned laser and thermal convection of liquid flows [22,26,7]. The electric field amplitude and the atomic polarization amplitude are both complex, for details, see [26] and references therein. Complex Chen and Lü chaotic systems have also been introduced and studied recently in [17].

Synchronization of chaotic systems with real variables, described by Pecora and Carroll [24], has been received significant attention in the last few years. Chaos synchronization, as an essential subject in science, has been widely investigated in many fields, such as physics, chemistry, ecological science, and secure communications [9,4,31,40,42]. Recently, synchronization of chaotic complex systems was studied in [18,41]. The complete synchronization, of two identical hyper(chaotic) complex systems with certain and uncertain parameters is studied in [19,44,47]. The synchronization of different-order chaotic systems and externally perturbed chaotic oscillators are discussed in [46,45]. The antisynchronization and adaptive antisynchronization of two different chaotic complex systems are investigated in [15,14]. Phase and antiphase synchronization, adaptive synchronization, and optimal adaptive synchronization of two

* Corresponding author.
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identical hyperchaotic complex nonlinear systems are studied in [20,41,44]. Projective synchronization and modified projective synchronization are performed on the chaotic and complex systems in [17,11].

In 2005, G. Tigan [27] introduced a nonlinear dynamic chaotic system as follows:

$$\begin{cases} \dot{x} &= a(y-x) \\ \dot{y} &= (c-a)x - axz \\ \dot{z} &= xy - bz, \end{cases} \tag{1.1}$$

with $a \neq 0, b$ and c real parameters. The system (1.1) is called T-system, and some results about it are presented, in [28,29,13]. Recently, in [?], the dynamical behavior and synchronization of the complex chaotic T-system discussed for some value of parameters and complex conjugates of x and y. The corresponding state space equations are as follow:

$$\begin{cases} \dot{x} &= a(y-x) \\ \dot{y} &= (c-a)x - axz \\ \dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - bz, \end{cases} \tag{1.2}$$

where $x = x_1 + ix_2$, $y = y_1 + iy_2$ are complex variables, $i = \sqrt{-1}$ and z is a real variable. The over-bar $\bar{x}$ and $\bar{y}$ denote the complex conjugate of x and y, respectively.

Control and synchronization of complex chaotic T-system, given in (1.2), are discussed in previous works. For example, the authors in [44] used adaptive optimal control to synchronize this system. Also, the optimal adaptive sliding mode is used in [43] to synchronization.

Contraction theory (CT), was introduced by Slotine in 1998 [37], is used as a robust tool for analyzing the convergence behavior of nonlinear systems. In [34], CT was applied to control uncertain nonlinear systems and used to generalize of Krasovskii's theorem for the stability of these systems. In 2000, CT was used to design the controller of mechanical systems [33]. In 2009, this method was used to synchronize chaotic systems [37]. In 2013, CT has been used for the synchronization of the Lurie system [39]. In 2017, the system controller was based on the nonlinear feedback of the contractile metric [36]. An adaptive nonlinear control was proposed as a contraction metric in [35].

Motivated by the mentioned advantages of CT in control of dynamical systems, here, we used it for the synchronization of complex T-system with new unknown parameters. The dynamical behavior and synchronization of this system for new unknown parameters are presented. Moreover, we use this idea with a parameter estimation roll as an adaptive approach, for a complex system with unknown parameters. The main feature of the proposed controller is that it has less computational cost.

The paper is outlined as follows: In Section 2, the behavior of complex T-system and the concept of CT are discussed. The synchronization of this system, based on the CT approach and numerical simulations with known and unknown parameters, is given in section 3. Our concluding remarks are given in Section 4.

2. Preliminaries

In this section, we discuss the dynamics and behavior of complex T-system with new parameters. Also, a brief introduction to CT is given.

2.1. Complex T-system

As a extension of the chaotic T-system (1.1), which introduced by G. Tigan in 2005 [27], consider the following complex version of this system [41,44]:

$$\begin{cases} \dot{x_1} &= a(y_1 - x_1) \\ \dot{x_2} &= a(y_2 - x_2) \\ \dot{y_1} &= (c-a)x_1 - ax_1z_1 \\ \dot{y_2} &= (c-a)x_2 - ax_2z_1 \\ \dot{z_1} &= x_1y_1 + x_2y_2 - bz_1. \end{cases} \tag{2.1}$$

where a, b and c are given similar to (1.1) in [40]. In [40] the behavior of system (2.1) studied for $b = 0.6$, $a = 2.1$ and $0 < c < 30$. Recently, the dynamical behavior, synchronization, and application are discussed in [41,44,43].

In different approach, in this study, we focus on a complex T-system with new parameters $b = 0.6$, $c = 29$ and $0 < a < 30$. The goal is to investigate the dynamical behavior and synchronization of it. First of all, we calculate the Lyapunov exponents and bifurcation diagram as a chaotically proof of complex T-system (2.1). The graphical results of Lyaponuve exponents and bifurcation are given in Figures 1 and 2, respectively. These figures show that the complex T-system is chaotic. Attractors of system (2.1) are given in Figure 3.

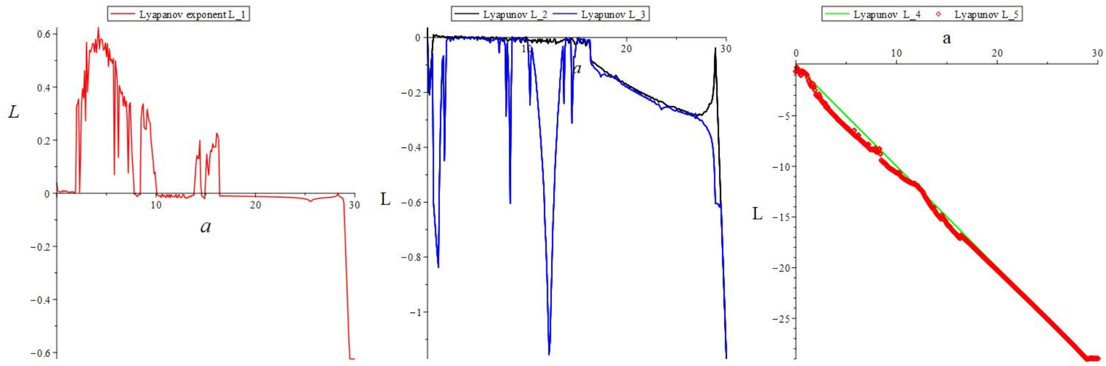

Figure 1: Lyapunov exponents of system (2.1), for $b = 0.6$, $c = 29$ and $0 < a < 30$, and the initial conditions $x_1(0) = 1$, $x_2(0) = -1$, $x_3(0) = 2$, $x_4(0) = -2$, and $x_5(0) = 0$

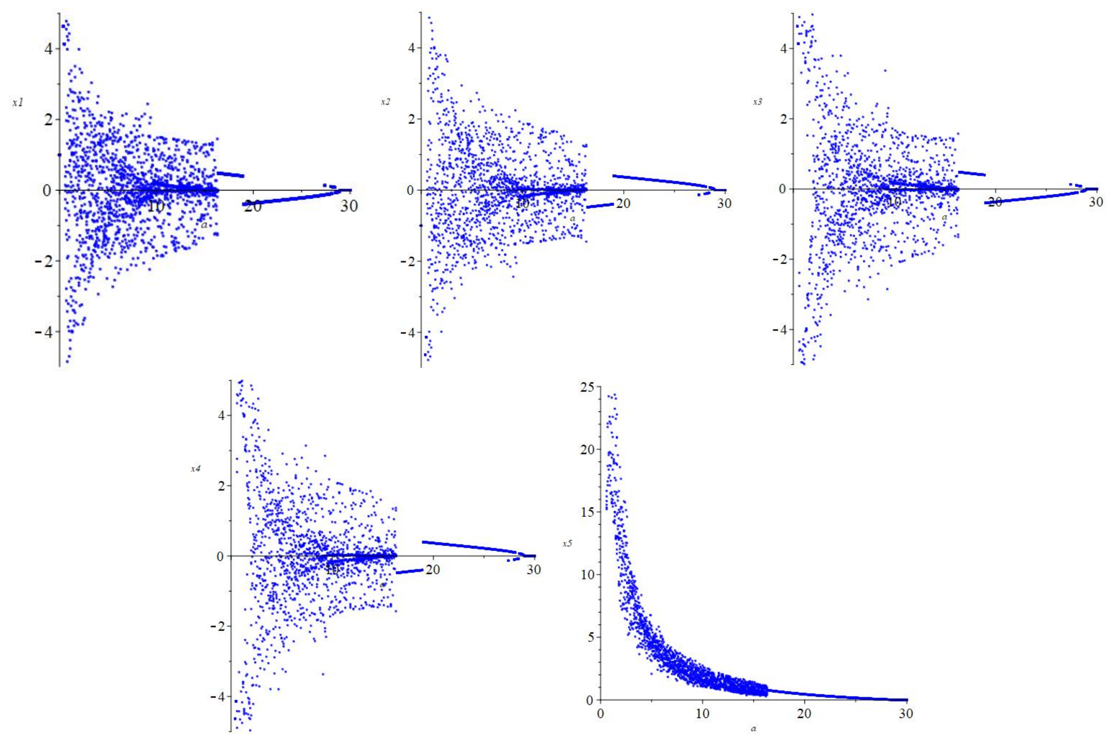

Figure 2: Bifurcation diagram of system (2.1), for $b = 0.6$, $c = 29$ and $0 < a < 30$, and the initial conditions $x_1(0) = 1$, $x_2(0) = -1$, $x_3(0) = 2$, $x_4(0) = -2$, and $x_5(0) = 0$.

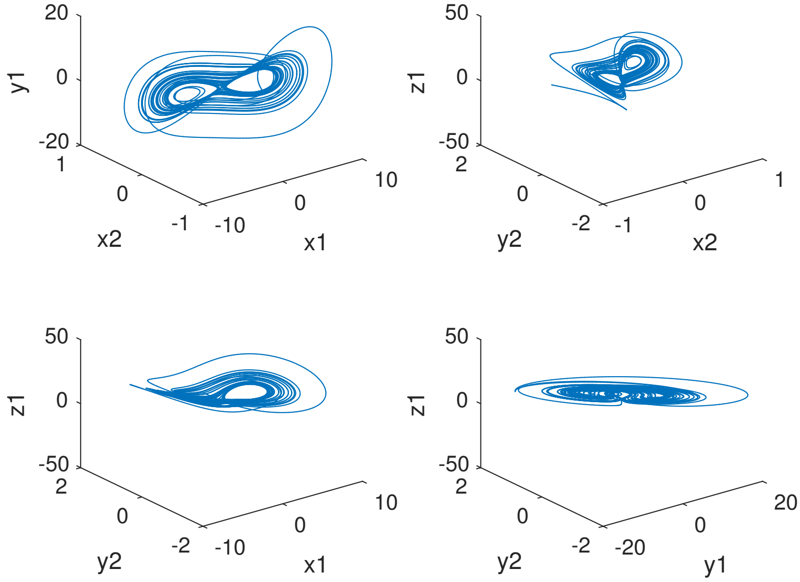

Figure 3: Attractor of system (2.1).

2.2. Contraction theory (CT)

In this subsection, some basic preliminary of CT are discussed. Based on the concept of virtual displacement between two different neighboring trajectories, TC scheme was introduced by Lohmiller and Slotine in 1998 [38]. Let the nonlinear, non-autonomous dynamic system as follows:

$$\dot{x} = f(X, t), \tag{2.2}$$

where x is the $n \times 1$ dimensional state vector and f is the $n \times 1$ nonlinear vector function, with $J = \frac{\partial f}{\partial X}$ is the Jacobian matrices of system (2.2). Now, consider the following inequality:

$$\frac{d}{dt}(\delta^T X \delta X) \leq 2\lambda_{max}(\delta^T X \delta X),$$

where $\delta X = X(t + \delta t) - X(t)$ is the virtual displacement between two different neighboring trajectories and λ_{max} is the largest eigenvalue of the symmetric part of the Jacobian matrix $J_s = \frac{1}{2}(J + J^t)$.

It can be proved that:

$$\|\delta x\| \leq \|\delta x_0\| e^{\int_0^t \lambda_{max}(x,\tau)d\tau}. \tag{2.3}$$

From the inequality (2.3), it is evident that if $\lambda_{max}(x, t)$ is uniformly strictly negative, then $\|\delta x\|$ will converge exponentially to zero. So the solution of system (2.2) will converge to a single trajectory, exponentially [38].

Definition 2.1 *[39] Consider the dynamic system $\dot{X} = f(X, Y)$. A region of the state space is called a contraction region if the Jacobian matrix $J = \dfrac{\partial f}{\partial X}$ is uniformly negative definite in that region, that is:*

$$\exists \beta > 0,\ \forall x, \forall t \geq 0, \qquad J_s = \frac{1}{2}\left(\frac{\partial f}{\partial X} + \frac{\partial^T f}{\partial X}\right) \leq -\beta I < 0. \tag{2.4}$$

The CT, is used in Section 3 for synchronization of complex T-system. For more details about the CT, such as generalized Jacobian matrix and partial CT, see [34,36,37]. In previous works [41,44], for investigating the stability of synchronization of CT, the Lyapunov function should be defined, but in the proposed method, it is not required. Also, designing the controllers is simpler.

3. Synchronization of complex T-system

To use CT for synchronization of system (2.1), consider two identical systems, drive and slave system. The goal of designing a controller is make the response system follow the drive system, until they ultimately become the same. Let the drive system as follows:

$$\begin{cases} \dot{x}_{1d} &= a(y_{1d} - x_{1d}) \\ \dot{x}_{2d} &= a(y_{2d} - x_{2d}) \\ \dot{y}_{1d} &= (c-a)x_{1d} - ax_{1d}z_{1d} \\ \dot{y}_{2d} &= (c-a)x_{2d} - ax_{2d}z_{1d} \\ \dot{z}_{1d} &= x_{1d}y_{1d} + x_{2d}y_{2d} - bz_{1d}. \end{cases} \tag{3.1}$$

Where, subscript id is used to determine the drive system. Now, we discuss the synchronization of complex T-system in two following cases:

- Synchronization with known parameter
- Synchronization with unknown parameter

3.1. Synchronization with known parameter

Let system (3.1) as a drive system, and define the slave system with known parameters as follows, respectively:

$$\begin{cases} \dot{x}_{1s} &= a(y_{1s} - x_{1s}) + k_1(x_{1s} - x_{1d}) \\ \dot{x}_{2s} &= a(y_{2s} - x_{2s}) + k_2(x_{2s} - x_{2d}) \\ \dot{y}_{1s} &= (c-a)x_{1s} - ax_{1s}z_{1s} + k_3(y_{1s} - y_{1d}) \\ \dot{y}_{2s} &= (c-a)x_{2s} - ax_{2s}z_{1s} + k_4(y_{2s} - y_{2d}) \\ \dot{z}_{1s} &= x_{1s}y_{1s} + x_{2s}y_{2s} - bz_{1s} + k_5(z_{1s} - z_{1d}), \end{cases} \tag{3.2}$$

Now, similar to [5], we define the virtual system for (3.1) and (3.2) as follow:

$$\begin{cases} \dot{x_{1v}} &= a(y_{1v} - x_{1v}) + k_1(x_{1v} - x_{1d}) \\ \dot{x_{2v}} &= a(y_{2v} - x_{2v}) + k_2(x_{2v} - x_{2d}) \\ \dot{y_{1v}} &= (c-a)x_{1v} - ax_{1v}z_{1v} + k_3(y_{1v} - y_{1d}) \\ \dot{y_{2v}} &= (c-a)x_{2v} - ax_{2v}z_{1v} + k_4(y_{2v} - y_{2d}) \\ \dot{z_{1v}} &= x_{1v}y_{1v} + x_{2v}y_{2v} - bz_{1v} + k_5(z_{1v} - z_{1d}), \end{cases} \tag{3.3}$$

Let J be the Jacobian matrix of (3.3) and J_s is the symmetric part of J, which are follows:

$$J = \begin{bmatrix} -a+k_1 & 0 & a & 0 & 0 \\ 0 & -a+k_2 & 0 & a & 0 \\ (c-a)-az_{1v} & 0 & k_3 & 0 & -ax_{1v} \\ 0 & (c-a)-az_{1v} & 0 & k_4 & -ax_{2v} \\ y_{1v} & y_{2v} & x_{1v} & x_{2v} & -b+k_5 \end{bmatrix} \tag{3.4}$$

$$-J_s = -\frac{J+J^T}{2} = \begin{bmatrix} a-k_1 & 0 & \frac{c-aM}{2}-a & 0 & \frac{M}{2} \\ 0 & -a-k_2 & 0 & \frac{c-aM}{2} & \frac{M}{2} \\ \frac{c-aM}{2}-a & 0 & k_3 & 0 & \frac{M(1-a)}{2} \\ 0 & \frac{c-aM}{2} & 0 & -k_4 & \frac{M(1-a)}{2} \\ \frac{M}{2} & \frac{M}{2} & \frac{M(1-a)}{2} & \frac{M(1-a)}{2} & -b-k_5 \end{bmatrix} \tag{3.5}$$

where, M is the maximum of state variables. For simplicity in calculating the determinant of matrix and sub-matrix (3.5), we let

$$J_{ps} = -J_s = \begin{bmatrix} H & 0 & A & 0 & C \\ 0 & J & 0 & K & C \\ A & 0 & L & 0 & B \\ 0 & K & 0 & N & B \\ C & C & B & B & O \end{bmatrix} \tag{3.6}$$

Based on CT, for synchronization, we need the J_{ps} to be a positive definite matrix. So, the following conditions be satisfied:

- $H > 0$
- $HJ > 0$
- $-A^2J + HJL > 0$
- $(HL - A^2)(HJ - K^2) > 0$
- $(-JB^2+L(OJ-C^2))H^2+((-JL+K^2)B^2+2C(AJ+KL)B+(A^2-JL)C^2-O(A^2J+K^2L))H+A^2B^2J-2ACK(A+K)B+K^2(OA^2+C^2L) > 0$

For numerical experiments of the mentioned theoretical method, consider complex T-system with parameter in Table 1. It is clear that for the following values of parameters and controller gain in Table 1, the positive definite condition of matrix J_{ps} is satisfied. So the master system (3.1) and slave system (3.2) are synchronized for every initial conditions. The numerical simulation of results for arbitrary initial conditions, such as $(1, -.5, 1, 1, -1.5)$ and $(-4, 5, -4, -2, 2)$ are given in Figures 4 and 5. These Figures show that the synchronization and convergence occur after a bit of time.

Table 1: The parameters and controller gains.

$a = 4$	$k_1 = 0$
$b = 0.6$	$k_2 = 0$
$c = 29$	$k_3 = -170$
$M = 50$	$k_4 = -170$
$--$	$k_5 = -2413$

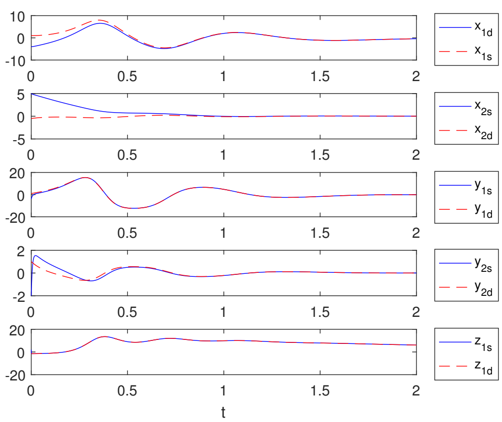

Figure 4: Synchronization of systems (3.1) and (3.2).

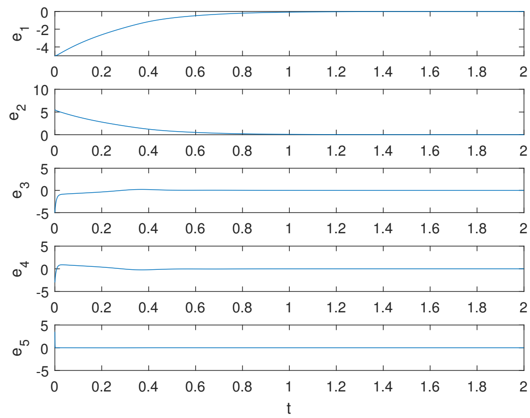

Figure 5: Error of synchronization systems (3.1) and (3.2).

3.2. Synchronization with unknown parameter

In this subsection, we define the slave system with unknown parameter b as follows:

$$\left\{\begin{array}{lcl} \dot{x}_{1s} & = & a(y_{1s}-x_{1s})+k_1(x_{1s}-x_{1d}) \\ \dot{x}_{2s} & = & a(y_{2s}-x_{2s})+k_2(x_{2s}-x_{2d}) \\ \dot{y}_{1s} & = & (c-a)x_{1s}-ax_{1s}z_{1s}+k_3(y_{1s}-y_{1d}) \\ \dot{y}_{2s} & = & (c-a)x_{2s}-ax_{2s}z_{1s}+k_4(y_{2s}-y_{2d}) \\ \dot{z}_{1s} & = & x_{1s}y_{1s}+x_{2s}y_{2s}-\hat{b}z_{1s}+k_5(z_{1s}-z_{1d}), \end{array}\right. \tag{3.7}$$

Now, similar to [5], we choose the parameter estimation rule as follows:

$$\dot{\hat{b}} = -z_{1s}(z_{1s}-z_{1d}) \tag{3.8}$$

Based on CT and similar to [5], consider t the virtual system for systems (3.8), (3.1) and (3.7) as follow:

$$\left\{\begin{array}{lcl} \dot{x}_{1v} & = & a(y_{1v}-x_{1v})+k_1(x_{1v}-x_{1d}) \\ \dot{x}_{2v} & = & a(y_{2v}-x_{2v})+k_2(x_{2v}-x_{2d}) \\ \dot{y}_{1v} & = & (c-a)x_{1v}-ax_{1v}z_{1v}+k_3(y_{1v}-y_{1d}) \\ \dot{y}_{2v} & = & (c-a)x_{2v}-ax_{2v}z_{1v}+k_4(y_{2v}-y_{2d}) \\ \dot{z}_{1v} & = & x_{1v}y_{1v}+x_{2v}y_{2v}-\hat{b}z_{1v}+k_5(z_{1v}-z_{1d}), \\ \dot{\hat{b}} & = & -z_{1v}(z_{1v}-z_{1d}) \end{array}\right. \tag{3.9}$$

Now, by calculating the Jacobian matrix J of (3.9) and J_s the symmetric part of it J, we have:

$$J = \begin{bmatrix} -a+k_1 & 0 & a & 0 & 0 & 0 \\ 0 & -a+k_2 & 0 & a & 0 & 0 \\ (c-a)-az_{1v} & 0 & k_3 & 0 & -ax_{1v} & 0 \\ 0 & (c-a)-az_{1v} & 0 & k_4 & -ax_{2v} & 0 \\ y_{1v} & y_{2v} & x_{1v} & x_{2v} & -\hat{b}+k_5 & -z_{1v} \\ 0 & 0 & 0 & 0 & -2z_{1v} & 0 \end{bmatrix} \tag{3.10}$$

we let the upper bound M $M1$ for state and parameter estimate variables.

$$J_{ps} = -J_s = -\frac{J+J^T}{2} = \begin{bmatrix} a-k_1 & 0 & \frac{c-aM}{2}-a & 0 & \frac{M}{2} & 0 \\ 0 & -a-k_2 & 0 & \frac{c-aM}{2} & \frac{M}{2} & 0 \\ \frac{c-aM}{2}-a & 0 & k_3 & 0 & \frac{M(1-a)}{2} & 0 \\ 0 & \frac{c-aM}{2} & 0 & -k_4 & \frac{M(1-a)}{2} & \\ \frac{M}{2} & \frac{M}{2} & \frac{M(1-a)}{2} & \frac{M(1-a)}{2} & -b-k_5 & \frac{3z_{1v}}{2} \\ 0 & 0 & 0 & 0 & \frac{3z_{1v}}{2} & 0 \end{bmatrix} \tag{3.11}$$

where, we let, M is the maximum of state variables. For simplicity in calculating the determinant of the matrix and sub-matrix, we let

$$J_s = \begin{bmatrix} H & 0 & A & 0 & C & 0 \\ 0 & J & 0 & K & C & 0 \\ A & 0 & L & 0 & B & 0 \\ 0 & K & 0 & N & B & 0 \\ C & C & B & B & O & P \\ 0 & 0 & 0 & 0 & P & 0 \end{bmatrix} \tag{3.12}$$

Based on CT, for synchronization, we need the J_s to be a negative definite matrix. So, the following conditions be satisfied:

- $H > 0$
- $H\,J^{\circ\ \circ} > 0$
- $-A^2 J + HJL > 0$
- $A^2\,K^2 - 1\,J\,N\,A^2 - 1\,H\,L\,K^2 + H\,J\,L\,N > 0$
- $A^2\,B^2\,J + A^2\,C^2\,N + B^2\,H\,K^2 + C^2\,K^2\,L + A^2\,K^2\,O - 2\,A\,B\,C\,K^2 - 2\,A^2\,B\,C\,K - B^2\,H\,J\,L - B^2\,H\,J\,N - C^2\,H\,L\,N - C^2\,J\,L\,N - A^2\,J\,N\,O - H\,K^2\,L\,O + 2\,A\,B\,C\,J\,N + 2\,B\,C\,H\,K\,L + H\,J\,L\,N\,O > 0$
- $-P\,\left(P\,A^2\,K^2 - 1\,J\,N\,P\,A^2 - 1\,H\,L\,P\,K^2 + H\,J\,L\,N\,P\right) > 0$

For numerical experiments, and evidence of the theoretical proposed method, consider the systems with new parameters 2. It is clear that the positive definite condition in J_{ps} is satisfied. So the master system (3.1) and slave system (3.7) are synchronized for estimation rule (3.8) every initial condition. The numerical simulation of results for initial conditions $(1, -.5, 1, 1, -1.5)$ and $(-4, 5, -4, -2, 2)$ are given in Figures 6, 7 and 8.

Table 2: parameters and controller gain

$a = 4$	$k_1 = -9$
$b = 0.6$	$k_2 = -10$
$c = 29$	$k_3 = -6$
$M = 50$	$k_4 = -26$
$--$	$k_5 = -9$

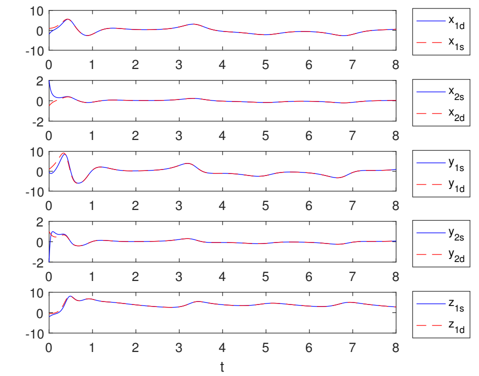

Figure 6: Synchronization of systems (3.1) and (3.7).

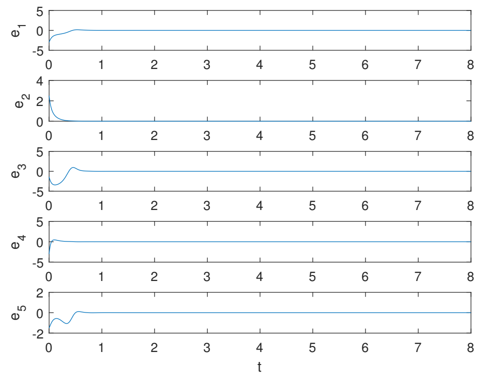

Figure 7: Error of synchronization systems (3.1) and (3.7).

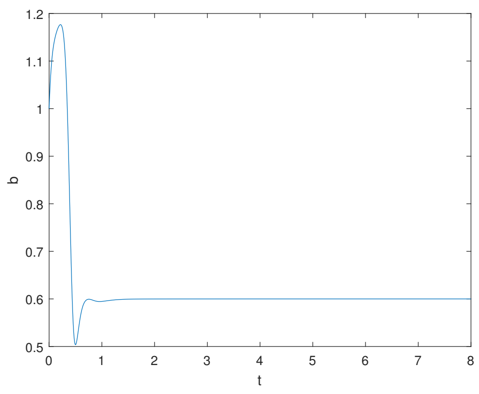

Figure 8: Estimation of unknown parameter in (3.7).

4. Conclusion

In this paper, we study the behavior of a chaotic complex T-system for some new parameters. Also, the synchronization of this system is discussed in terms of known and unknown parameters. The controllers and parameter estimation rules in synchronization are obtained by CT. In the proposed approach, the methods stability is satisfied without using Lyapunov functions. Numerical simulations show the efficiency of the method used.

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Rogayyeh Soltani,

Department of Mathematics,

Payame Noor University (PNU), Tehran, Iran.

E-mail address: rogayeh.soltani.50@student.pnu.ac.ir

and

Saeed Nezhadhosein,

Department of Mathematics,

Payame Noor University
Iran.
E-mail address: s_nejhadhosein@pnu.ac.ir

and

Bashir Naderi (Corresponding author),
Department of Mathematics,
Payame Noor University
Iran.
E-mail address: b_naderi@pnu.ac.ir

and

Aghileh Heydari ,
Department of Mathematics,
Payame Noor University
Iran.
E-mail address: a_heidari@pnu.ac.ir