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On a safety distribution for a class of SEIRS epidemic model

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ABSTRACT: The present paper analyse conditions on a given a non-linear SEIRS epidemic model parameters, to provide a safety set for the epidemic containment. After having determined this set, we look for control actions constraining the epidemic to remain within the safety set with infection rates below an allowed threshold. This means that for any initial state in a certain safety set of the state space, there exists an appropriate control strategy maintaining the state of the system in the considered safety set. To ensure the solvability under feedback control of our problem, sufficient assumptions are derived in terms of linear inequalities on the input vectors at the vertices of a polytope.

Key Words: Epidemic process, discrete-time system, positivity, stability, feedback control.

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1. Introduction

Infectious disease surveillance plays a major role in analyzing the evolution of epidemic containment within populations, since it allows public health services to prepare recommendations and intervention strategies. For this, it is well known that the essential tool is the epidemic modelling. Indeed, based on mathematical models and surveillance data, it describe dynamics and spread of such epidemics [7, 17,25,26,16]. Kermack and McK-endrick, in 1927, proposed the first mathematical epidemiology model, namely, a basic SIR model (Susceptible-Infectious-Recovered), which assumes that a fixed population, can be divided into three compartments: susceptible people (not infected but could become infected), infectious people (have the disease and are able to infect others), and recovered people (those who were infected and are now immune). This model assumes that the total number of people remains constant as well as homogeneous mixing, i.e., each individual is equally likely to come in contact with any other, see [2,9,10,17,27] when the individual can recover from the disease with immunity, and [1,8,11] for SIRS models, i.e., when the recovered individual does not show immunity, and then it is again a susceptible individual. In the case of some infectious diseases such as tuberculosis, HIV or influenza-like illnesses, one needs to extend the standard SIR model and introduce a fourth compartment and state corresponding to the disease's latency period, when a person is infected but not yet able to infect others [12,17,18]. This extension is called the SEIR models (Susceptible-Exposed-Infected-Recovered) and SEIRS models if we assume that recovered individual does not show immunity, and then becomes again susceptible. Several estimators have been previously designed to track the states of SEIR models, and strong assumptions these model parameters allow the design of estimators converging towards the true state values [3,17,22,30]. However, the problem of observer design for SEIR models becomes very challenging in practice, when it is necessary to take into account the presence of uncertain parameters whose values are only known to belong to an interval or a polytope [15,16,19]. An interval/polytope estimation approach can address such problems. Indeed, using input and output measurements, an observer has to estimate the set of

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admissible values (interval or polytope) for the states at each instant of time. A major advantage of the interval/polytope estimation methodology is that it allows us to take into account many types of uncertainties in the system, see i.e., [15,13].

Here, we are looking for conditions to determine a safety distribution of population (or safety set of population) Susceptible, Exposed, Infected and Recovered, where the proportion of infected individuals can be reduced or maintained within the appropriate levels. This is a case of models representing the evolution of an epidemic where it is very important to design control actions so that the evolution of the disease remains within certain security levels and does not extend to the entire population [4,5,14, 20,21,29,32]. This paper deals with a particular class of non-linear discrete-time SEIRS epidemic models and aims to verify if it is possible to maintain the correspond state inside some safety set by choosing an adequate control actions, such as population recruitment, i.e., to ensure that the state of the epidemic system belongs in a certain region of the state space. For that, the spread of a disease is modelled by means of a non-autonomous SEIRS model with a bounded population, along the time.

Our adequate control approach is geometric, i.e., by considering a polytope that limits the proportion of infected individuals, as a safety set to guarantee an acceptable evolution of the disease. Moreover, a class of functions based on the vertices of the polytope is constructed. This kind of functions will allow us to derive the feedback control ensuring that the state trajectory of the closed-loop system stays in such a region. This control strategy allows us to increase the susceptible population and maintain the distribution of individuals within the appropriate levels established by such a safety set. Motivated by the recent paper [13], we focus here on the analysis of a general SEIRS epidemic model.

The rest of the paper is organised as follows. Section 2 gives a description of the epidemic model to be studied and the positivity property is also examined. In Section 3, an adequate control policy to keep the trajectory of the epidemic system in some safety set is chosen. In Section 4, some conditions are provided, to find control actions so that the epidemic remains within the security set with infection rates below an allowed amount. Finally, some conclusions are given, and an appendix to includes the notation and preliminaries results used throughout this work, is provided.

2. SEIRS Model Statement

Consider a SEIRS dynamic process for an epidemic disease where the population of individuals can be divided into four dynamic sub-populations : Susceptible $x_1(t)$, Exposed $x_2(t)$, Infected $x_3(t)$ and Immune or Recovered $x_4(t)$. The mathematical model of such model is given by a system (A.1) in term of the

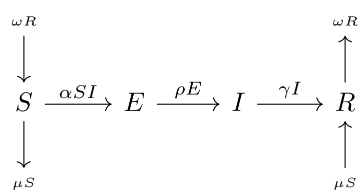

Figure 1: The Susceptible-Exposed-Infected-Recovered (SEIR) model.

variable $x = (x_i)_{1\le i\le 4} \in C^1([0,\infty),\mathbb{R}^4_+)$ and the non-negative input $u \in C^1([0,\infty),\mathbb{R}_+)$, that is

$$x(t+1) = Ax(t) + f(x(t)) + B(u(t)). \tag{2.1}$$

To provide A, f and B in the previous system, we first need to introduce different sub-populations rates

s, e : Survival rates of susceptible individuals and exposed individuals, respectively,

i, r : Survival rates of infected individuals and recovered individuals, respectively.

and theirs corresponding parameters of transmission rates, given as follows.

α	:	Exposition rate of susceptible individuals by contact with an infected individual.
ρ	:	Transition rate of exposed individuals to infected individuals.
γ	:	Transition rate of infected individuals to recovered individuals.
μ	:	Transition rate of susceptible individuals to recovered individuals (immune individuals).
ω	:	Transition rate of recovered individuals to susceptible individuals.

These parameters has an epidemiological meaning if it is assumed that they satisfy the constraints

$$\alpha > 0, \quad 0 \leq \mu < s < 1, \quad 0 < \rho < e < 1, \quad 0 < \gamma < i < 1 \;\text{ and }\; 0 \leq \omega < r < 1.$$

These constraints ensure non-negativity of the state x of System (A.1). According to these, we have

$$A = \begin{bmatrix} s-\mu & 0 & 0 & \omega \\ 0 & e-\rho & 0 & 0 \\ 0 & \rho & i-\gamma & 0 \\ \mu & 0 & \gamma & r-\omega \end{bmatrix}$$

$$f(x(t)) = \begin{bmatrix} -\alpha\, x_1(t)x_3(t) \\ \alpha\, x_1(t)x_3(t) \\ 0 \\ 0 \end{bmatrix}$$

Since new susceptible individuals can only be added by increasing the susceptible population x_1, then

$$B = e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

A newly added individual population is time-dependent and can be seen as a control action to achieve our goals, for example, model stability and efficiency, or maintaining adequate population size. In our case, this control action is used to find a safety distribution population, called a safety set. It follows from Proposition A.1 that System (2.1) is positive if and only if one has

$$\alpha\, x_1(t)x_3(t) \leq (s-\mu)\, x_1(t) + \omega\, x_4(t), \quad \forall\, t \geq 0,$$

or equivalently, for all $t > 0$, if and only if

$$0 \leq \big(s - \mu - \alpha\, x_3(t)\big)\, x_1(t) + \omega\, x_4(t).$$

Hence, to ensure the positivity of System (2.1), the following condition are sufficient

$$x_3(t)) \leq \frac{s-\mu}{\alpha}. \tag{2.2}$$

Let us assume that, for an initial condition $x(0) = (x_i(0))_i \in \mathbb{R}^4_+$, System (2.1) describes the evolution of an initial population $P_0 = x_1(0) + x_2(0) + x_3(0) + x_4(0) \in \mathbb{R}_+$, and hence it must obviously be a positive system, and if no controls are applied, due to the survival of individuals, the solution must tend to zero. Moreover, the state $x(t)$ of an autonomous system can be formulated as follows

$$x(t) = A^t\, x(0) + [I\; A\; \cdots\; A^{t-1}] \begin{bmatrix} f(x(t-1)) \\ \vdots \\ f(x(0)) \end{bmatrix}, \quad \forall\, t \geq 0. \tag{2.3}$$

Thus, if A is stable matrix and $f : t \mapsto f(t)$ is a bounded function, the population tends to extinct. Let's go back now to System (2.1), we determine the eigenvalues of A by resolving the following equation of an unknown λ.

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} s-\mu-\lambda & 0 & 0 & \omega \\ 0 & e-\rho-\lambda & 0 & 0 \\ 0 & \rho & i-\gamma-\lambda & 0 \\ \mu & 0 & \gamma & r-\omega-\lambda \end{vmatrix} = 0$$

or equivalently,

$$(e-\rho-\lambda)(i-\gamma-\lambda)\big[\lambda^2 - (s+r-\mu-\omega)\,\lambda + s(r-\omega) - r\mu\big] = 0.$$

Hence

$$\lambda = e-\rho \quad \text{ou} \quad \lambda = i-\gamma \quad \text{ou} \quad \lambda^2 - (s-\mu+r-\omega)\,\lambda + (sr - s\omega - r\mu) = 0.$$

Consequently, the eigenvalues of the matrix A are $e-\rho$, $i-\gamma$ and

$$\lambda_{1,2} = \frac{1}{2}\left((s+r-\mu-\omega) \pm \sqrt{(s-\mu+r-\omega)^2 - 4\,(sr - s\omega - r\mu)}\,\right). \tag{2.4}$$

Using Gershgorin circle Theorem A.1, it is deduced that the spectrum of A is contained into the line segment $]0,1[$. It is due to the fact that every eigenvalue of A lies on one of Gershgorin's discs, which are

$$D(s-\mu, s), \;\; D(e-\rho, e), \;\; D(i-\gamma, i) \;\; \text{and} \;\; D(r-\omega, r).$$

Hence $|\lambda_{3,4}| < 1$, that is, the matrix A is stable. Then, considering an initial population P_0 such that $P_0 \leq \dfrac{s-\mu}{\alpha}$ and zero controls, thus the autonomous system (2.1) is positive and its solution tends to 0.

3. Equilibrium points for a bounded non-autonomous system

We focus on general case of a non-autonomous system with a bounded population along the time, i.e.,

$$\sum_{i=1}^{4} x_i(t) \leq P, \quad \forall\, t \geq 0.$$

As this model describes an epidemic process, from an initial state $x(0) = \big(x_1(0)\, x_2(0)\, 0\, 0\big)^{\mathrm{T}}$, we are interested in analysing the dynamic process when some infected individuals are introduced. The main goal of this study is to provide a safety set and to look for the conditions so that if the state $x(t)$ of System (2.1) belongs to the provided set, then the trajectory of a system remains in that set forever. Once this set is determined, one could study under what conditions, the system trajectory can be leads to that set. However, we will show a case in which we can search for the set of reachable states that have the structure we are looking for, and how we can find a control sequence that leads us to that set. To do that, consider that the state $x(t)$ of System (2.1) can be written as follows.

$$x(t) = A^t\, x(0) + [B\; AB\; \cdots\; A^{t-1}B] \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix} + [I\; A\; \cdots\; A^{t-1}] \begin{bmatrix} f(x(t-1)) \\ \vdots \\ f(x(0)) \end{bmatrix}. \tag{3.1}$$

We introduce the concept of positively reachable (non-negative) state at t steps. A non-negative state $\overline{x}$ is positively reachable from $x(0) \geq 0$ if there exists a non-negative sequence $u(0), \ldots, u(t-1) \geq 0$ such that $x(i) \geq 0$ for all $i \in \{1, \ldots, t-1\}$ and $x(t) = \overline{x}$. For instance, if $\mu = 0$, one has

$$[B\; AB\; \cdots\; A^{t-1}B] \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix} = \begin{bmatrix} u(t-1) + \cdots + p^{t-1}u(0) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.2}$$

Then, the set of positively reachable states from $x(0)$ in t steps is given by

$$R_t(x(0)) = \Big\{x \in \mathbb{R}^4_+ : \ \exists\, \alpha_i \geq 0\,;\ x = A^t x(0) + \sum_{i=0}^{i-1} \big[p^{t-1-i}\alpha_i e_1 + A^{t-1-i} f(x(i))\big]\Big\}. \tag{3.3}$$

We from now would consider the control sequence that in t steps takes the path from the initial state $x(0)$ to a final state $x \in R_t(x(0))$ belonging to the safety set.

To construct the safety set, we must first study conditions on the asymptotic stability of System (2.1). For that, we consider that our equilibrium point is of the form $(x^*, u^*)^{\mathrm{T}}$. Note that the equilibrium points $(x^*, u^*)^{\mathrm{T}}$ are defined by the solution of the following algebraic system

$$x^* = Ax^* + f(x^*) + Bu^* \ \text{ under the condition } \ \sum_{i=1}^{4} (x^*)_i \leq P. \tag{3.4}$$

These solutions are the disease-free equilibrium point E_f and the endemic equilibrium points. Taking u^* and focused on the disease-free equilibrium point $E_f = (x^*, u^*)^{\mathrm{T}}$, that is, the point at which no disease in the population, so $x_3^* = 0$. Then, DFE point $E_f = (x^*, u^*)^{\mathrm{T}} = (x_1^*, x_2^*, 0, x_4^*, u^*)^{\mathrm{T}}$ is given by the constrained system (3.4), that is

$$\begin{bmatrix} x_1^* \\ x_2^* \\ 0 \\ x_4^* \end{bmatrix} = \begin{bmatrix} s-\mu & 0 & 0 & \omega \\ 0 & e-\rho & 0 & 0 \\ 0 & \rho & i-\gamma & 0 \\ \mu & 0 & \gamma & r-\omega \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ 0 \\ x_4^* \end{bmatrix} + \begin{bmatrix} -\,\alpha\, x_1^*(t)\, x_3^*(t) \\ \alpha\, x_1^*(t)\, x_3^*(t) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u^* \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the previous equation leads to the following system

$$\begin{cases} x_1^* = (s-\mu)\, x_1^* + \omega\, x_4^* + u^* \\ x_2^* = (e-\rho)\, x_2^* \\ 0 = \rho\, x_2^* \\ x_4^* = \mu\, x_1^* + (r-\omega)\, x_4^* \end{cases}$$

which implies that

$$x_2^* = 0\,, \quad x_4^* = \frac{\mu}{1-r+\omega}\, x_1^*\,, \quad \big(1 - s + \mu - \frac{\mu\omega}{1-r+\omega}\big)\, x_1^* = u^*.$$

Hence

$$\begin{cases} x_1^* = \dfrac{1-r+\omega}{(1-s+\mu)(1-r+\omega) - \mu\omega}\, u^* \\[2ex] x_2^* = 0 \\[2ex] x_4^* = \dfrac{\mu}{1-r+\omega}\, \dfrac{1-r+\omega}{(1-s+\mu)(1-r+\omega) - \mu\omega}\, u^*. \end{cases}$$

Finally, we conclude that the DFE point E_f is as follows

$$E_f = \big((x_1^*, x_2^*, 0, x_4^*), u^*\big) = \Big(\frac{(1-r+\omega)\, u^*}{(1-s+\mu)(1-r+\omega) - \mu\omega}\, \big(1, 0, 0, \frac{\mu}{1-r+\omega}\big), u^*\Big). \tag{3.5}$$

Moreover, we have $x_3^* = \dfrac{s-\mu}{\alpha} > 0$ (see (2.3)), hence, E_f is a positive equilibrium point of System (2.1).

Recalling that the population is assumed to be bounded, i.e.; $\sum\limits_{i=1}^{4} x_i(t) \leq P$, it comes from (3.5) that

$$\begin{aligned} \sum_{i=1}^{4} x_i^* = x_1^* + x_4^* &= \frac{(1-r+\omega)}{(1-s+\mu)(1-r+\omega) - \mu\omega}\, \big(1 + \frac{\mu}{1-r+\omega}\big)\, u^* \\ &= \frac{1-r+\omega+\mu}{(1-s+\mu)(1-r+\omega) - \mu\omega}\, u^* \leq P. \end{aligned} \tag{3.6}$$

Then, one has

$$u^* \le \frac{P}{1-r+\omega+\mu}\left[(1-s+\mu)(1-r+\omega)-\mu\omega\right]. \tag{3.7}$$

Now, we study the behaviour of disease-free equilibrium for System (2.1). To achieve this, the model is linearised around the DFE point and we obtain the eigenvalues of the coefficient matrix. The stability of the DFE point is then directly related to the spectral radius of this coefficient matrix. We notice that around E_f, we have $x_l(t) = x(t) - x^*$ and $u_l(t) = u(t) - u^*$, then by approximating $f(x(t))$, we get

$$f(x(t)) = \alpha \begin{bmatrix}-1\\1\\0\\0\end{bmatrix} x_1(t)x_3(t) \approx \alpha \begin{bmatrix}-1\\1\\0\\0\end{bmatrix} (x_1^*x_3(t) + x_1(t)x_3^*) = \alpha \begin{bmatrix}-1\\1\\0\\0\end{bmatrix} x_1^*x_3(t). \tag{3.8}$$

Thus, the new linearised systems is as follows

$$x_l(t+1) = (A+\tilde{A})\,x_l(t) + B\,u_l(t) \text{ where } \tilde{A} = \begin{bmatrix}0 & 0 & -\alpha\,x_1^* & 0\\ 0 & 0 & \alpha\,x_1^* & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{3.9}$$

Then, the approximation of the solution of a non-linear system is

$$x_{app}(t) = x_l(t) + x^*. \tag{3.10}$$

Note that the characteristic polynomial of $A+\tilde{A}$ is as follows

$$\det((A+\tilde{A}) - \lambda I) = \begin{vmatrix} s-\mu-\lambda & 0 & -\alpha\,x_1^* & \omega\\ 0 & e-\rho-\lambda & \alpha\,x_1^* & 0\\ 0 & \rho & i-\gamma-\lambda & 0\\ \mu & 0 & \gamma & r-\omega-\lambda \end{vmatrix}$$

$$= \left(\lambda^2 - \left(s-\mu+r-\omega\right)\lambda + s(r-\omega) - \mu r\right)\right)$$
$$\left(\lambda^2 - \left(e-\rho+i-\gamma\right)\lambda + (e-\rho)(i-\gamma) - \rho\alpha\,x_1^*\right).$$

Hence, the spectrum of matrix $A+\tilde{A}$ is given as follows

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{2}\left((s-\mu+r-\omega) \pm \sqrt{(s-\mu+r-\omega)^2 - 4\left(s(r-\omega)-\mu r\right)}\,\right),\\
\lambda_{3,4} &= \frac{1}{2}\left((e-\rho+i-\gamma) \pm \sqrt{(e-\rho+i-\gamma)^2 - 4\left((e-\rho)(i-\gamma)-\rho\alpha\,x_1^*\right)}\,\right)\\
&= \frac{1}{2}\left((e-\rho+i-\gamma) \pm \sqrt{\left((e-\rho)-(i-\gamma)\right)^2 + 4\,\rho\alpha\,x_1^*}\,\right) \in \mathbb{R}.
\end{aligned} \tag{3.11}$$

See that the eigenvalue $\lambda_{1,2}$ of $A+\tilde{A}$ is the same eigenvalue of A given in (2.4), which satisfies $|\lambda_{1,2}| < 1$. Then, to ensure the stability of the matrix $A+\tilde{A}$, the following condition is necessary and sufficient

$$4\,\rho\alpha\,x_1^* < \left(2 - (e-\rho+i-\gamma)\right)^2 - \left((e-\rho)-(i-\gamma)\right)^2.$$

That is

$$\rho\alpha\,x_1^* < (1+\gamma-i)(1+\rho-e).$$

Remembering $x_1^* = \dfrac{(1-r+\omega)\,u^*}{(1-s+\mu)(1-r+\omega)-\mu\omega}$, given by (3.5), the previous condition becomes

$$u^* < \frac{(1+\gamma-i)(1+\rho-e)}{\rho\alpha\,(1-r+\omega)}\left((1-s+\mu)(1-r+\omega)-\mu\omega\right). \tag{3.12}$$

Hence, the system stability is directly related to the control action that can be used in the process. This result can be interpreted as the maximum population recruitment to fulfil the stability of the model. In our case the control u^* must satisfy condition (3.12). Moreover, since the spectral radius of matrix $A+\tilde{A}$ is less than 1, the linearised system (3.9) is asymptotically stable $\lim_{t\infty} x_I(t) = 0$, which means that the disease-free equilibrium point is globally asymptotically stable.

The previous comments will now be summarized in the following theorems.

Theorem 3.1 *If the control input u^* satisfy (3.7) and (3.12), then $\sum_{i=1}^{4} x_i^* \leq P$ holds and System (2.1) is stable.*

Theorem 3.2 *If u^* satisfies condition (3.7) and not condition (3.12), that is*

$$\frac{(1+\gamma-i)(1+\rho-e)}{\rho\alpha\,(1-r+\omega)} \leq \frac{u^*}{(1-s+\mu)(1-r+\omega)-\mu\omega} \leq \frac{P}{1-r+\omega+\mu}, \tag{3.13}$$

then the contrainte $\sum_{i=1}^{4} x_i^* \leq P$ holds and System (2.1) is unstable.

We are looking for conditions to determine a distribution of population, Susceptible, Exposed, Infected and Recovered, where the proportion of infected individuals can be maintained within the appropriate levels. This means that there are still some infected individuals in the population while condition of Theorem 3.2 should hold and therefore, the population size satisfies

$$\big(1+\gamma-i\big)\Big(1+\frac{1-e}{\rho}\Big)\Big(1+\frac{\mu}{(1-r+\omega)}\Big) \leq \alpha P.$$

4. Safety population distribution

Here, we obtain some conditions to find control actions so that the epidemic remains within the safety set with infection rates below an allowed positive amount ε. First, let us assume that condition of Theorem 3.2 holds. Then, the population size P satisfies

$$\big(1+\gamma-i\big)\Big(1+\frac{1-e}{\rho}\Big)\Big(1+\frac{\mu}{(1-r+\omega)}\Big) \leq \alpha P. \tag{4.1}$$

In this case, the epidemic process is not stable around the DFE point. Despite having system instability, we analyse if it is possible to choose an adequate control strategy to keep the trajectory of an epidemic system inside some safety set, that is, to guarantee that the system state $x(t) = (x_i(t))_{1\leq i\leq 4}$ belongs in a certain subset of the admissible state space such that a percentage of infected population is less than a safety bound

$$\epsilon = \tau P > 0 \quad \text{with} \quad 0 < \tau < 1.$$

Recalling a non-negative condition (2.3) of System (2.1), the size of the infected population should satisfy

$$x_3(t) \leq \epsilon \leq \frac{s-\mu}{\alpha} \tag{4.2}$$

Moreover, we can assume that the susceptible individuals are a suitable percentage of the non-infected individuals $(1-\tau)P$. Thus, for any population size P and any $\tau > 0$ satisfying (4.1)-(4.2), we gather these conditions in an admissible set $\mathcal{X}$ of all $x = (x_i)_{1\leq i\leq 4} \in \mathbb{R}^4_+$ satisfying

$$\begin{cases} \sum_{i=1}^{4} x_i \leq P, \\ x_1 \leq k(1-\tau)P, \\ x_2 \leq \alpha\, k\tau(1-\tau)P, \\ x_3 \leq \tau P \end{cases} \quad \text{where} \quad 0 \leq \tau < k(1-\tau) < 1 \;\text{ and }\; 0 < k \leq 1. \tag{4.3}$$

Then, see that $\mathcal{X} = \mathcal{P}(\mathcal{V})$ is a polytope for the vertices set $\mathcal{V} = \{v_i\,;\ 0 \le i \le 15\}$ defined as follows

$$
\begin{array}{ll}
v_0 = (0,0,0,0) & v_1 = P(k(1-\tau),0,0,0) \\
v_2 = P(0,\alpha k\tau(1-\tau),0,0) & v_3 = P\,(0,0,\tau,0) \\
v_4 = P\,(0,0,0,1) & v_5 = P(k(1-\tau),\alpha k\tau(1-\tau),0,0) \\
v_6 = P(k(1-\tau),0,\tau,0) & v_7 = P(0,\alpha k\tau(1-\tau),\tau,0) \\
v_8 = P(k(1-\tau),0,0,1-k(1-\tau)) & v_9 = P(0,\alpha k\tau(1-\tau),0,1-\alpha k\tau(1-\tau)) \\
v_{10} = P(0,0,\tau,1-\tau) & v_{11} = P(0,\alpha k\tau(1-\tau),\tau,1-\tau-\alpha k\tau(1-\tau)) \\
v_{12} = P(k(1-\tau),0,\tau,1-\tau-k(1-\tau)) & \\
v_{13} = P(k(1-\tau),\alpha k\tau(1-\tau),0,1-k(1-\tau)-\alpha k\tau(1-\tau)) & \\
v_{14} = P(k(1-\tau),\alpha k\tau(1-\tau),\tau,0) & \\
v_{15} = P(k(1-\tau),\alpha k\tau(1-\tau),\tau,1-\tau-k(1-\tau)-\alpha k\tau(1-\tau)). &
\end{array}
\tag{4.4}
$$

Now, we propose to solve the following problem : *From an initial state $x(0)$ in a subset of $\mathcal{X}$, to construct a feedback control $u(t) = h(x(t))$ keeping the state trajectory $x(t)$ of the closed-loop system in this set, for all $t \ge 0$.* For that, we focus our attention on the following function class

$$
\Theta = \Big\{\theta : \mathcal{X} \to \mathbb{R}^4;\ \ \theta(x) = \sum_{i=1}^{4} c_i\,x_i + c_5\,x_1x_2 + c_6\,x_1x_3 + c_7\,x_2x_3 \ \text{ where } c_i \neq 0 \in \mathbb{R}\Big\} \tag{4.5}
$$

First, for each function $\theta \in \Theta$, it is straightforward to see that for any state $x \in \mathcal{X}$, the image $\theta(x)$ can be written as a linear combination of the images of the vertices set $\mathcal{V}$, defined in (4.4), of a polytope $\mathcal{X}$.

Theorem 4.1 *Let $\theta \in \Theta$ be given. Then, for all state $x \in \mathcal{X}$, one has $\theta(x) = \sum_{i=1}^{7} \alpha_i \theta(v_i)$ where*

$$
\begin{aligned}
&\alpha_1 = \frac{x_1}{k(1-\tau)P} - \frac{\tau\,x_1x_2}{(\alpha k^2\tau^2(1-\tau)^2)P^2} - \frac{\alpha k\tau(1-\tau)\,x_1x_3 - k(1-\tau)\,x_2x_3}{(\alpha k^2\tau^2(1-\tau)^2)P^2},\\
&\alpha_2 = \frac{x_2}{\alpha k\tau(1-\tau)P} - \frac{\tau\,x_1x_2 - k(1-\tau)\,x_2x_3}{(\alpha k^2\tau^2(1-\tau)^2)P^2}\ \ ,\quad \alpha_3 = \frac{x_3}{\tau P} - \frac{\alpha\tau\,x_1x_3}{(\alpha k\tau^2(1-\tau))P^2}\\
&\alpha_4 = \frac{x_4}{P}\ \ ,\quad \alpha_5 = \frac{x_1x_2}{(\alpha k^2\tau(1-\tau)^2)P^2}\ \ ,\quad \alpha_6 = \frac{\alpha\tau\,x_1x_3 - x_2x_3}{(\alpha k\tau^2(1-\tau))P^2}\ \ ,\quad \alpha_7 = \frac{x_2x_3}{(\alpha k\tau^2(1-\tau))P^2}.
\end{aligned}
\tag{4.6}
$$

Proof: First, consider the image of θ on the vertices set $\mathcal{V}$ characterising a polytope $\mathcal{X}$, that is

$$\theta(v_1) = c_1\,k(1-\tau)P \ \ ;\ \ \theta(v_2) = c_2\,\alpha k\tau(1-\tau)P \ \ ;\ \ \theta(v_3) = c_3\,\tau P \ \ ;\ \ \theta(v_4) = c_4 P,$$

$$\theta(v_5) = c_1\,k(1-\tau)P + c_2\,\alpha k\tau(1-\tau)P + c_5\,\alpha k^2\tau(1-\tau)^2P^2,$$

$$\theta(v_6) = c_1\,k(1-\tau)P + c_3\,\tau P + c_6\,k\tau(1-\tau)P^2,$$

$$\theta(v_7) = c_2\,\alpha k\tau(1-\tau)P + c_3\,\tau P + c_7\,\alpha k\tau^2(1-\tau)P^2,$$

$$\theta(v_8) = c_1\,k(1-\tau)P + c_4\,(1-k(1-\tau))P,$$

$$\theta(v_9) = c_2\,\alpha k\tau(1-\tau)P + c_4\,(1-\alpha k\tau(1-\tau))P,$$

$$\theta(v_{10}) = c_3\,\tau P + c_4\,(1-\tau)P,$$

$$\theta(v_{11}) = c_2\,\alpha k\tau(1-\tau)P + c_3\,\tau P + c_4\,(1-\tau-\alpha k\tau(1-\tau))P + c_7\,\alpha k\tau^2(1-\tau)\,P^2,$$

$$\theta(v_{12}) = c_1\, k(1-\tau)P + c_3\, \tau P + c_4\,(1-\tau-k(1-\tau))P + c_6\, k\tau(1-\tau)P^2,$$

$$\theta(v_{13}) = c_1\, k(1-\tau)P + c_2\, \alpha k\tau(1-\tau)P + c_4\,(1-k(1-\tau)-\alpha k\tau(1-\tau))P \\ + c_5\, \alpha k^2\tau(1-\tau)^2P^2,$$

$$\theta(v_{14}) = c_1\, k(1-\tau)P + c_2\, \alpha k\tau(1-\tau)P + c_3\, \tau P + c_5\, \alpha k^2\tau(1-\tau)^2P^2 + c_6\, k\tau(1-\tau)P^2 \\ + c_7\, \alpha k\tau^2(1-\tau)P^2,$$

$$\theta(v_{15}) = c_1\, k(1-\tau)P + c_2\, \alpha k\tau(1-\tau)P + c_3\, \tau P + c_4\,(1-\tau-k(1-\tau)-\alpha k\tau(1-\tau))P \\ + c_5\, \alpha k^2\tau(1-\tau)^2P^2 + c_6\, k\tau(1-\tau)P^2 + c_7\, \alpha k\tau^2(1-\tau)P^2.$$

Then, equation $\theta(x) = \sum_{i=1}^{15} \alpha_i \theta(v_i)$ which can be written as follows

$$\sum_{i=1}^{4} c_i\, x_i + c_5\, x_1x_2 + c_6\, x_1x_3 + c_7\, x_2x_3 = \sum_{i=1}^{15} \alpha_i\, \theta(v_i)$$

implies that

$$\begin{aligned}
x_1 &= k(1-\tau)P\,(\alpha_1+\alpha_5+\alpha_6+\alpha_8+\alpha_{12}+\alpha_{13}+\alpha_{14}+\alpha_{15})\\
x_2 &= \alpha k\tau(1-\tau)P\,(\alpha_2+\alpha_5+\alpha_7+\alpha_9+\alpha_{11}+\alpha_{13}+\alpha_{14}+\alpha_{15})\\
x_3 &= \tau P\,(\alpha_3+\alpha_6+\alpha_7+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{14}+\alpha_{15})\\
x_4 &= P\,(\alpha_4 + (1-k(1-\tau))\,\alpha_8 + (1-\alpha k\tau(1-\tau))\,\alpha_9 + (1-\tau)\,\alpha_{10}\\
&\quad + (1-\tau-\alpha k\tau(1-\tau))\,\alpha_{11} + (1-\tau-k(1-\tau))\,\alpha_{12}\\
&\quad + (1-k(1-\tau)-\alpha k\tau(1-\tau))\,\alpha_{13} + (1-\tau-k(1-\tau)-\alpha k\tau(1-\tau))\,\alpha_{15})),\\
x_1x_2 &= (\alpha k^2\tau(1-\tau)^2)P^2\,(\alpha_5+\alpha_{13}+\alpha_{15}),\\
x_1x_3 &= (k\tau(1-\tau))P^2\,(\alpha_6+\alpha_7+\alpha_{14}+\alpha_{15}),\\
x_2x_3 &= (\alpha k\tau^2(1-\tau))P^2\,(\alpha_7+\alpha_{11}+\alpha_{14}).
\end{aligned}$$

Now, by considering the solution corresponding to $\alpha_8 = \alpha_9, \dots, \alpha_{15} = 0$, one deduce

$$\begin{aligned}
&x_1 = k(1-\tau)P\,(\alpha_1+\alpha_5+\alpha_6) \quad , \quad x_2 = \alpha k\tau(1-\tau)P\,(\alpha_2+\alpha_5+\alpha_7)\\
&x_3 = \tau P\,(\alpha_3+\alpha_6+\alpha_7) \quad , \quad x_4 = P\,\alpha_4 \quad , \quad x_1x_2 = (\alpha k^2\tau(1-\tau)^2)P^2\,\alpha_5,\\
&x_1x_3 = (k\tau(1-\tau))P^2\,(\alpha_6+\alpha_7) \quad , \quad x_2x_3 = (\alpha k\tau^2(1-\tau))P^2\,\alpha_7,
\end{aligned}$$

that is

$$\begin{aligned}
\alpha_4 &= \frac{x_4}{P} \quad , \quad \alpha_5 = \frac{x_1x_2}{(\alpha k^2\tau(1-\tau)^2)P^2} \quad , \quad \alpha_7 = \frac{x_2x_3}{(\alpha k\tau^2(1-\tau))P^2},\\
\alpha_6 &= \frac{x_1x_3}{(k\tau(1-\tau))P^2} - \alpha_7 = \frac{x_1x_3}{(k\tau(1-\tau))P^2} - \frac{x_2x_3}{(\alpha k\tau^2(1-\tau))P^2} = \frac{\alpha\tau\, x_1x_3 - x_2x_3}{(\alpha k\tau^2(1-\tau))P^2},\\
\alpha_3 &= \frac{x_3}{\tau P} - \alpha_6 - \alpha_7 = \frac{x_3}{\tau P} - \frac{\alpha\tau\, x_1x_3}{(\alpha k\tau^2(1-\tau))P^2},\\
\alpha_2 &= \frac{x_2}{\alpha k\tau(1-\tau)P} - \frac{x_1x_2}{(\alpha k^2\tau(1-\tau)^2)P^2} - \frac{x_2x_3}{(\alpha k\tau^2(1-\tau))P^2},\\
&= \frac{x_2}{\alpha k\tau(1-\tau)P} - \frac{\tau\, x_1x_2 - k(1-\tau)\, x_2x_3}{(\alpha k^2\tau^2(1-\tau)^2)P^2},
\end{aligned}$$

$$\begin{aligned}\alpha_1 &= \frac{x_1}{k(1-\tau)P} - \frac{x_1x_2}{(\alpha k^2\tau(1-\tau)^2)P^2} - \frac{\alpha\tau\, x_1x_3 - x_2x_3}{(\alpha k\tau^2(1-\tau))P^2} \\ &= \frac{x_1}{k(1-\tau)P} - \frac{\tau\, x_1x_2}{(\alpha k^2\tau^2(1-\tau)^2)P^2} - \frac{\alpha k\tau(1-\tau)\, x_1x_3 - k(1-\tau)\, x_2x_3}{(\alpha k^2\tau^2(1-\tau)^2)P^2}.\end{aligned}$$

□

Our goal is to find a subset $\widetilde{\mathcal{X}}$ of the state space $\mathcal{X}$, in which for any given $\theta \in \Theta$ satisfying

$$\theta(v_i) \in \widetilde{\mathcal{X}}, \quad \forall\, v_i \in \widetilde{V} = \{v_1, v_2, \ldots, v_7\},$$

we can ensure that the image of any state of that region is also in it. First, we look for an admissible subset $\widetilde{\mathcal{X}} \subseteq \mathcal{X}$, so that the linear combination $\theta(x) = \sum_{i=1}^{7} \alpha_i \theta(v_i)$ is a convex linear combination, i.e., their coefficients α_i are non-negative and their coefficients sum $\sum_{i=1}^{7} \alpha_i$ less than 1. For this, we can state the following theorem.

Theorem 4.2 *The coefficients α_i in (4.6) satisfy $0 \le \alpha_i \le 1$ and $\sum_{i=1}^{7} \alpha_i \le 1$ if and only if*

$$x = (x_i)_{i=1}^{4} \in \widetilde{\mathcal{X}} \subseteq \mathcal{X},$$

where

$$\widetilde{\mathcal{X}} = \Big\{ x \in \mathcal{X}\, ;\;\; x_4 \le P\,\big(1 - \frac{x_1}{k(1-\tau)P}\big)\big(1 - \frac{x_2}{\alpha k\tau(1-\tau)P} - \frac{x_3}{\tau P}\big) - 2\,\frac{x_2x_3}{(\alpha k\tau^2(1-\tau))P} \Big\}. \tag{4.7}$$

Proof: The fact that $(\alpha_i)_{1\le i\le 7}$ belong on $[0,1)$ follows directly from conditions (4.3). Moreover, we have

$$\begin{aligned}\sum_{i=1}^{7} \alpha_i &= \frac{x_1}{k(1-\tau)P} - \frac{\tau\, x_1x_2}{(\alpha k^2\tau^2(1-\tau)^2)P^2} - \frac{\alpha k\tau(1-\tau)\, x_1x_3 - k(1-\tau)\, x_2x_3}{(\alpha k^2\tau^2(1-\tau)^2)P^2} \\ &\quad + \frac{x_2}{\alpha k\tau(1-\tau)P} - \frac{\tau\, x_1x_2 - k(1-\tau)\, x_2x_3}{(\alpha k^2\tau^2(1-\tau)^2)P^2} + \frac{x_3}{\tau P} - \frac{\alpha\tau\, x_1x_3}{(\alpha k\tau^2(1-\tau))P^2} \\ &\quad + \frac{x_4}{P} + \frac{x_1x_2}{(\alpha k^2\tau(1-\tau)^2)P^2} + \frac{\alpha\tau\, x_1x_3 - x_2x_3}{(\alpha k\tau^2(1-\tau))P^2} + \frac{x_2x_3}{(\alpha k\tau^2(1-\tau))P^2}. \\ &= \frac{x_1}{k(1-\tau)P} + \frac{x_2}{\alpha k\tau(1-\tau)P} + \frac{x_3}{\tau P} + \frac{x_4}{P} - \frac{x_1x_2}{(\alpha k^2\tau(1-\tau)^2)P^2} \\ &\quad - \frac{x_1x_3}{(k\tau(1-\tau))P^2} + 2\,\frac{x_2x_3}{(\alpha k\tau^2(1-\tau))P^2}.\end{aligned}$$

Then, the coefficients sum satisfies $\sum\limits_{i=1}^{7} \alpha_i \le 1$ if and only if

$$\frac{x_4}{P} \le \Big(1 - \frac{x_1}{k(1-\tau)P}\Big)\Big(1 - \frac{x_2}{\alpha k\tau(1-\tau)P} - \frac{x_3}{\tau P}\Big) - 2\,\frac{x_2x_3}{(\alpha k\tau^2(1-\tau))P^2}.$$

or equivalently,

$$x_4 \le P\,\Big(1 - \frac{x_1}{k(1-\tau)P}\Big)\Big(1 - \frac{x_2}{\alpha k\tau(1-\tau)P} - \frac{x_3}{\tau P}\Big) - 2\,\frac{x_2x_3}{(\alpha k\tau^2(1-\tau))P}.$$

From $x \in \mathcal{X}$, the condition $\sum\limits_{i=1}^{7} \alpha_i \le 1$ holds if and only if $x \in \widetilde{\mathcal{X}} \subseteq \mathcal{X}$ with $\tilde{\mathcal{X}}$ defined in (4.7).

□

Remark 4.1 *It is straightforward to observe that* $\mathcal{X}$ *is a polytope and* $\widetilde{\mathcal{X}} \subseteq \mathcal{X}$ *is not it. This remark motivates the need to reduce the set of states by considering a polytope contained in* $\widetilde{\mathcal{X}}$*. For that, it is sufficient to consider the region limited by some tangent hyperplan to the geometrical form* S *defined by*

$$S: \quad \frac{x_4}{P} = \Big(1 - \frac{x_1}{k(1-\tau)P}\Big)\Big(1 - \frac{x_2}{\alpha k\tau(1-\tau)P} - \frac{x_3}{\tau P}\Big) - 2\,\frac{x_2 x_3}{(\alpha k\tau^2(1-\tau))P^2}.$$

at a point $N = (x_1^0\ x_2^0\ x_3^0\ x_4^0)^T \in S$*. Then, since this tangent hyperplan is wrtten as follows*

$$\xi_1\,(x_1 - x_1^0) + \xi_2\,(x_2 - x_2^0) + \xi_3\,(x_3 - x_3^0) - (x_4 - x_4^0) = 0. \tag{4.8}$$

where

$$\begin{aligned}
\xi_1 = \frac{\partial x_4}{\partial x_1}(N) &= -\frac{1}{k(1-\tau)}\Big(1 - \frac{x_2^0}{\alpha k\tau(1-\tau)P} - \frac{x_3^0}{\tau P}\Big)\\
&= -\frac{1}{k(1-\tau)}\Big(\frac{\alpha k\tau(1-\tau)P - x_2^0 - \alpha k(1-\tau)\,x_3^0}{\alpha k\tau(1-\tau)P}\Big),\\
\xi_2 = \frac{\partial x_4}{\partial x_2}(N) &= -\frac{1}{\alpha k\tau(1-\tau)}\Big(1 - \frac{x_1^0}{k(1-\tau)P} + \frac{2\,x_3^0}{\tau P}\Big)\\
&= -\frac{1}{\alpha k\tau(1-\tau)}\Big(\frac{k\tau(1-\tau)P - \tau\,x_1^0 + 2k(1-\tau)\,x_3^0}{k\tau(1-\tau)P}\Big),\\
\xi_3 = \frac{\partial x_4}{\partial x_3}(N) &= -\frac{1}{\tau}\Big(1 - \frac{x_1^0}{k(1-\tau)P} + \frac{2\,x_2^0}{(\alpha k\tau(1-\tau))P}\Big)\\
&= -\frac{1}{\tau}\Big(\frac{\alpha k\tau(1-\tau)P - \alpha\tau\,x_1^0 + 2\,x_2^0}{\alpha k\tau(1-\tau)P}\Big).
\end{aligned}$$

Next, by multiplying by $-\alpha k^2\tau^2(1-\tau)^2P$*, equation* (4.8) *can be rewrite as follows*

$$\xi_1'\,(x_1 - x_1^0) + \xi_2'\,(x_2 - x_2^0) + \xi_3'\,(x_3 - x_3^0) + \xi_4'\,(x_4 - x_4^0) = 0. \tag{4.9}$$

where

$$\begin{aligned}
\xi_1' &= (\alpha k\tau^2(1-\tau)P - \tau\,x_2^0 - \alpha k\tau(1-\tau)\,x_3^0),\\
\xi_2' &= (k\tau(1-\tau)P - \tau\,x_1^0 + 2k(1-\tau)\,x_3^0),\\
\xi_3' &= (\alpha k^2\tau(1-\tau)^2P - \alpha k\tau(1-\tau)\,x_1^0 + 2k(1-\tau)\,x_2^0),\\
\xi_4' &= \alpha k^2\tau^2(1-\tau)^2P.
\end{aligned} \tag{4.10}$$

According to the above remark, equation (4.8) can be reformulated in the following way

$$\xi_1'\,x_1 + \xi_2'\,x_2 + \xi_3'\,x_3 + \xi_4'\,x_4 = \mathcal{M} \quad \text{with} \quad \mathcal{M} = \sum_{i=1}^{4} \xi_i'\,x_i^0. \tag{4.11}$$

Now, by using equation (4.11), we are able to consider a polytope included in $\widetilde{\mathcal{X}}$, that is

$$\mathcal{P}_N = \big\{x = (x_i)_{i=1}^4 \in \widetilde{\mathcal{X}};\ \ \xi_1'\,x_1 + \xi_2'\,x_2 + \xi_3'\,x_3 + \xi_4'\,x_4 \leq \mathcal{M}\big\} \tag{4.12}$$

where coefficients ξ_1' and a constant $\mathcal{M}$ are defined in equations (4.10) and (4.11).

Theorem 4.3 *Consider* $\mathcal{P}_N$ *and* $\widetilde{V} = \{v_1, .., v_7\}$ *previously defined. If* $\theta \in \Theta$ *satisfies* $\theta(v_i) \in \mathcal{P}_N$ *for any* $v_i \in \widetilde{V}$*, then, one has* $\theta(x) \in \mathcal{P}_N$ *for all* $x \in \mathcal{P}_N$*.*

Proof: For each $v_i \in \widetilde{V}$, let $\theta(v_i) = \big(\theta(v_i)_1\ \theta(v_i)_2\ \theta(v_i)_3\ \theta(v_i)_4\big)^T$. Since $\theta(v_i) \in \mathcal{P}_N \subseteq \widetilde{\mathcal{X}} \subseteq \mathcal{X}$, we have

$$\sum_{j=1}^{4} \theta(v_i)_j \leq P\,, \quad \theta(v_i)_1 \leq k(1-\tau)P\,, \quad \theta(v_i)_2 \leq \alpha\, k\tau(1-\tau)P,$$

and

$$\theta(v_i)_3 \leq \tau P\,, \quad \sum_{j=1}^{4} \xi'_j\, \theta(v_i)_j \leq \mathcal{M}.$$

Given $x \in \mathcal{P}_N$, it follows from Theorem 4.1 and Theorem 4.2 that for all $v_i \in \widetilde{V}$, we can write

$$\theta(x) = \sum_{i=1}^{7} \alpha_i\, \theta(v_i) \ \text{ with } \ 0 \leq \alpha_i \leq 1 \ \text{ and } \ \sum_{i=1}^{7} \alpha_i \leq 1.$$

Then, we deduce

$$\theta(x)_1 = \sum_{i=1}^{7} \alpha_i\, \theta(v_i)_1 \leq \sum_{i=1}^{7} \alpha_i\, k(1-\tau)P \leq k(1-\tau)P,$$

$$\theta(x)_2 = \sum_{i=1}^{7} \alpha_i\, \theta(v_i)_2 \leq \sum_{i=1}^{7} \alpha_i\, \alpha k\tau(1-\tau)P \leq \alpha\, k\tau(1-\tau)P,$$

$$\theta(x)_3 = \sum_{i=1}^{7} \alpha_i\, \theta(v_i)_3 \leq \sum_{i=1}^{7} \alpha_i\, \tau P \leq \tau P.$$

Moreover, we have

$$\sum_{j=1}^{4} \theta(x)_j = \sum_{j=1}^{4}\sum_{i=1}^{7} \alpha_i\, \theta(v_i)_j = \sum_{i=1}^{7} \alpha_i \sum_{j=1}^{4} \theta(v_i)_j \leq \sum_{i=1}^{7} \alpha_i P \leq P.$$

and

$$\sum_{j=1}^{4} \xi'_j\, \theta(x)_j = \sum_{j=1}^{4} \xi'_j\, \big(\sum_{i=1}^{7} \alpha_i\, \theta(v_i)_j\big) = \sum_{i=1}^{7} \alpha_i\, \big(\sum_{j=1}^{4} \xi'_j\, \theta(v_i)_j\big) \leq \sum_{i=1}^{7} \alpha_i\, \mathcal{M} \leq \mathcal{M}.$$

Finally, we conclude that $\theta(x) \in \mathcal{P}_N$, and that finishes the proof.

□

Now, in order to simplify the explicit expression of the control that maintains the solution trajectory within the polytope, we consider the tangent hyperplan to S at $\overline{N} = v_4 = (0\ 0\ 0\ P)^T$. Hence, keeping in mind (4.10)-(4.12), the polytope corresponding to this point is given by

$$\mathcal{P}_{\overline{N}} = \Big\{x = (x_i)_{i=1}^4 \in \widetilde{\mathcal{X}}\,;\ \frac{1}{k(1-\tau)}\, x_1 + \frac{1}{\alpha\, k\tau(1-\tau)}\, x_2 + \frac{1}{\tau}\, x_3 + x_4 \leq P\Big\} \tag{4.13}$$

As the coefficients $(\alpha_i)_{1\leq i\leq 7}$ satisfy conditions in Theorem 4.2, then for $x \in \mathcal{P}_{\overline{N}}$ and $\theta \in \Theta$ given, on has

$$\theta(v_i) \in \mathcal{P}_{\overline{N}},\ \forall\, v_i \in \widetilde{\mathcal{V}} \implies \theta(x) \in \mathcal{P}_{\overline{N}}.$$

From here, let us assume that the initial state is in $\mathcal{P}_{\overline{N}}$, and that, for all $\theta \in \Theta$, $\theta(x)$ is a convex combination of the images of the vertices in $\widetilde{\mathcal{V}}$. Remembering System (2.1), we have

$$x(t+1) = g(x(t)) + Bu(t) \quad \text{with} \quad g(x) = Ax + f(x).$$

Keeping in mind (4.5), the function g belongs to Θ, since it can be written as follows

$$g(x) = \begin{bmatrix} s-\mu & 0 & 0 & \omega \\ 0 & e-\rho & 0 & 0 \\ 0 & \rho & i-\gamma & 0 \\ \mu & 0 & \gamma & r-\omega \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} -\alpha\, x_1 x_3 \\ \alpha\, x_1 x_3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (s-\mu)\, x_1 + \omega\, x_4 \\ (e-\rho)\, x_2 \\ \rho\, x_2 + (i-\gamma)\, x_3 \\ \mu\, x_1 + \gamma\, x_3 + (r-\omega)\, x_4 \end{bmatrix} + \begin{bmatrix} -\alpha\, x_1 x_3 \\ \alpha\, x_1 x_3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} s-\mu \\ 0 \\ 0 \\ \mu \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ e-\rho \\ \rho \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ i-\gamma \\ \gamma \end{bmatrix} x_3 + \begin{bmatrix} \omega \\ 0 \\ 0 \\ r-\omega \end{bmatrix} x_4 + \begin{bmatrix} -\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} x_1 x_3.$$

We are now going to look for a feedback control $u(t) = h(x(t))$ such that the new function

$$g_c : x \mapsto g(x) + \tilde{h}(x) \;\text{ where }\; \tilde{h} = Bh \in \Theta \tag{4.14}$$

of the closed-loop system

$$x(t+1) = g(x(t)) + \tilde{h}(x(t)),\; x(0) \in \mathcal{X},\;\; \forall\, t \in [0, \infty), \tag{4.15}$$

belonging also in Θ and that $\mathcal{P}_{\overline{N}}$ is invariant by function g_c, that is $g_c(\mathcal{P}_{\overline{N}}) \subseteq \mathcal{P}_{\overline{N}}$. To this end, using Theorem 4.3, it suffices to analyse conditions in order to get

$$g_c(v_i) = g(v_i) + \tilde{h}(v_i) \in \mathcal{P}_{\overline{N}}\,,\;\; \forall\, v_i \in \widetilde{\mathcal{V}}.$$

Theorem 4.4 *Let $P > 0$, $\varepsilon = \tau P$ with $\tau \in (0,1)$, satisfying (4.1)-(4.2). Consider $k \in (0,1]$ such that*

$$\alpha\rho\, k(1-\tau) \le 1 - (i-\gamma). \tag{4.16}$$

Assume a function $h : \mathcal{P}_{\overline{N}} \to \mathbb{R}$ satisfies $Bh(x) \in \Theta$ and the following conditions hold.

$$\begin{aligned} &h(v_1) \le Pk(1-\tau)\,\mathbf{a}\;,\;\; h(v_2) \le Pk(1-\tau)\,\mathbf{b}\;,\;\; h(v_3) \le Pk(1-\tau)\,\mathbf{c} \\ &h(v_4) \le Pk(1-\tau)\,\mathbf{d} - Pw\;,\;\; h(v_5) \le Pk(1-\tau)\,\mathbf{e}, \\ &h(v_6) \le P\,\mathbf{f}\;,\;\; h(v_7) \le P\,\mathbf{g}\;,\;\; v_i \in \widetilde{\mathcal{V}}\;\; \textit{for}\;\; i = 1,2,\dots,7. \end{aligned} \tag{4.17}$$

where

$$\mathbf{a} = 1 - s + \mu - \mu k(1-\tau)\;,\;\; \mathbf{b} = 1 - e + \rho - \alpha\rho\, k(1-\tau),$$

$$\mathbf{c} = 1 - i + \gamma(1-\tau)\;,\;\; \mathbf{d} = 1 - r + w\;,\;\; \mathbf{e} = \mathbf{a} + \mathbf{b} - 1,$$

$$\mathbf{f} = \mathbf{a} + \mathbf{c} - 1 + (\alpha\tau - 1)P\;,\;\;\; \mathbf{g} = \mathbf{b} + \mathbf{c} - 1.$$

Then, under the control feedback $u(t) = h(x(t))$ ($t \in [1, \infty)$, the trajectory of closed-loop system (4.15) remains in a polytope $\mathcal{P}_{\overline{N}}$ for all initial state $x(0) \in \mathcal{P}_{\overline{N}}$.

Remark 4.2 *The condition (4.16) and it direct consequence $\alpha\rho\, k(1-\tau) \le 1$ well be used to prove the theorem above.*

Proof: Since $Bh(x) \in \Theta$, Theorem 4.1 implies that $\widetilde{h}(x) = Bh(x) = B \sum_{i=1}^{7} \alpha_i h(v_i)$. Taking the control feedback $u(t) = h(x(t))$, we will prove that the function of the closed loop system

$$g_c(x) = Ax + f(x) + \widetilde{h}(x)$$

satisfies

$$g_c(v_i) \in P_{\overline{N}}, \quad \forall\, v_i \in \widetilde{\mathcal{V}}.$$

Note that from (4.13), a state x belong to $P_{\overline{N}}$ if and only if

$$\sum_{i=1}^{4} x_i \leq P, \tag{4.18}$$

$$x_1 \leq k(1-\tau)P, \tag{4.19}$$

$$x_2 \leq \alpha\, k\tau(1-\tau)P, \tag{4.20}$$

$$x_3 \leq \tau P, \tag{4.21}$$

$$\frac{1}{k(1-\tau)}\, x_1 + \frac{1}{\alpha\, k\tau(1-\tau)}\, x_2 + \frac{1}{\tau}\, x_3 + x_4 \leq P. \tag{4.22}$$

Since all three positive terms $k(1-\tau)$, $\alpha\, k\tau(1-\tau)$ and τ are less than 1, one has

$$\sum_{i=1}^{4} x_i \leq \frac{1}{k(1-\tau)}\, x_1 + \frac{1}{\alpha\, k\tau(1-\tau)}\, x_2 + \frac{1}{\tau}\, x_3 + x_4.$$

Hence the condition (4.22) implies the condition (4.18), and then, we have to prove that for each $v_i \in \widetilde{\mathcal{V}}$, the entries of the vectors $g_c(v_i)$ satisfy the conditions (4.19)-(4.22). For that, we note that

1. We have $g_c(v_1) = g(v_1) + Bh(v_1) = \big((s-\mu)Pk(1-\tau) + h(v_1), 0, 0, \mu Pk(1-\tau)\big)^T$, then

$$g_c\,(v_1) \in \mathcal{P}_{\overline{N}} \iff \begin{cases} (s-\mu)Pk(1-\tau) + h\,(v_1) \leq k(1-\tau)P \\ (s-\mu)P + \dfrac{h(v_1)}{k(1-\tau)} + \mu Pk(1-\tau) \leq P \end{cases}$$

$$\iff \begin{cases} h(v_1) \leq Pk(1-\tau)(1-s+\mu) \\ (s-\mu)P + \dfrac{h(v_1)}{k(1-\tau)} + \mu Pk(1-\tau) \leq P \end{cases}.$$

Since $sk(1-\tau) > 0$, we have $\mathbf{a} < 1-s+\mu$, then combined with hypothesis (4.17), we get

$$h(v_1) \leq Pk(1-\tau)\,\mathbf{a} \leq Pk(1-\tau)(1-s+\mu).$$

Thus, the first condition holds. For the second condition, it is straightforward to see that

$$\begin{aligned}(s-\mu)P + \frac{h(v_1)}{k(1-\tau)} + \mu Pk(1-\tau) &\leq \big((s-\mu) + \mathbf{a} + \mu k(1-\tau)\big)P \\ &\leq \big(s-\mu+1-s+\mu-\mu k(1-\tau)+\mu k(1-\tau)\big)P = P.\end{aligned}$$

2. We have $g_c(v_2) = \big(h(v_2), \alpha Pk(1-\tau)\tau(e-\rho), \alpha\rho Pk(1-\tau)\tau, 0\big)^T \in \mathcal{P}_{\overline{N}}$ if and only if

$$\begin{cases} h(v_2) \leq Pk(1-\tau), \\ \alpha Pk(1-\tau)\tau(e-\rho) \leq \alpha Pk(1-\tau)\tau, \\ \alpha\rho\, Pk(1-\tau)\tau \leq \tau P, \\ \dfrac{h(v_2)}{k(1-\tau)} + P(e-\rho) + \alpha\rho\, Pk(1-\tau) \leq P. \end{cases}$$

Keeping in mind the condition (4.16), the previous conditions are equivalent to

$$\begin{cases} h(v_2) \leq Pk(1-\tau), \\ e-\rho \leq 1, \\ h(v_2) \leq Pk(1-\tau)\big(1-e+\rho-\alpha\rho\, k(1-\tau)\big). \end{cases}$$

Recalling that $0 < \tau < k(1-\tau) \leq 1$, $0 < \rho < e < 1$ and $\alpha > 0$, we find $0 < e - \rho < 1$, and then $0 < \mathbf{b} = 1 - e + \rho - \alpha\rho\, k(1-\tau) < 1$. Thus, the second condition holds, the third condition follows from assumptions (4.17) and the first conditions comes from the third one since $h(v_2) \leq Pk(1-\tau)\mathbf{b}$ and as $\mathbf{b} \in (0,1)$. Finally, all the tree conditions holds.

3. We have $g_c(v_3) = \big(h(v_3), 0, (i-\gamma)P\tau, P\gamma\tau\big)^T$, then

$$g_c(v_3) \in \mathcal{P}_{\overline{N}} \iff \begin{cases} h(v_3) \leq Pk(1-\tau), \\ (i-\gamma)P\tau \leq P\tau, \\ \dfrac{h(v_3)}{k(1-\tau)} + (i-\gamma)P + P\gamma\tau \leq P. \end{cases}$$

$$\iff \begin{cases} h(v_3) \leq Pk(1-\tau), \\ i-\gamma \leq 1, \\ h(v_3) \leq Pk(1-\tau)\big(1 - i + \gamma(1-\tau)\big). \end{cases}$$

From $0 < \gamma(1-\tau) < \gamma < i < 1$, we deduce $i - \gamma < 1$ and $0 < \mathbf{c} = 1 - i + \gamma(1-\tau) < 1$. Hence the second condition holds, the third condition follows from assumption (4.17) and the first condition holds also since $\mathbf{c} \in (0,1)$.

4. We have $g_c(v_4) = (h(v_4) + wP, 0, 0, (r-w)P)^T$, then

$$g_c(v_4) \in \mathcal{P}_{\overline{N}} \iff \begin{cases} h(v_4) + wP \leq Pk(1-\tau) \\ \dfrac{h(v_4) + wP}{k(1-\tau)} + (r-w)P \leq P \end{cases}$$

$$\iff \begin{cases} h(v_4) + wP \leq Pk(1-\tau) \\ h(v_4) \leq Pk(1-\tau)(1-r+w) - wP \end{cases}$$

Using $0 < w < r < 1$, we get $0 < r - w < 1$ and then $0 < 1-(r-w) < 1$. Thus, the second condition implies the first one, since $h(v_4) + wP \leq Pk(1-\tau)(1-r+w) = Pk(1-\tau)(1-(r-w)) \leq Pk(1-\tau)$. Further, keeping in mind hypothesis (4.17), we get that all diserd conditions hold.

5. We have $g_c(v_5) = ((s-\mu)Pk(1-\tau) + h(v_5), \alpha(e-\rho)Pk\tau(1-\tau), \alpha\rho Pk\tau(1-\tau), \mu Pk(1-\tau))^T$, then

$$g_c(v_5) \in \mathcal{P}_{\overline{N}} \iff \begin{cases} (s-\mu)Pk(1-\tau) + h(v_5) \leq Pk(1-\tau), \\ \alpha(e-\rho)Pk\tau(1-\tau) \leq \alpha Pk\tau(1-\tau), \\ \alpha\rho Pk\tau(1-\tau) \leq \tau P, \\ (s-\mu)P + \dfrac{h(v_5)}{k(1-\tau)} + (e-\rho)P + \alpha\rho Pk(1-\tau) + \mu Pk(1-\tau) \leq P. \end{cases}$$

Recalling the condition (4.16), these conditions are equivalent to

$$\begin{cases} h\,(v_5) \leq Pk(1-\tau)(1-s+\mu), \\ e-\rho \leq 1, \\ h(v_5) \leq Pk(1-\tau)\big(\mathbf{a} + \mathbf{b} - 1\big). \end{cases}$$

Since $0 < \rho < e < 1$, we have $e - \rho < 1$, and since $\mu k(1-\tau) > 0$ and $\alpha\rho\, k(1-\tau) > 0$, we also have

$$\mathbf{e} = \mathbf{a} + \mathbf{b} - 1 < 1 - s + \mu - e + \rho < 1 - s + \mu.$$

Thus, the second condition holds, the fist condition follows from the third condition and this last one comes from assumptions (4.17). Finally, all the previous conditions hold.

6. First, we have $g_c(v_6) = (x_1, x_2, x_3, x_4)^T$ where $x_1 = (s-\mu)Pk(1-\tau) - \alpha P^2 k\tau(1-\tau) + h(v_6)$, $x_2 = \alpha P^2 k\tau(1-\tau)$, $x_3 = (i-\gamma)P\tau$ and $x_4 = \mu Pk(1-\tau) + \gamma P\tau$. Then, $g_c(v_6) \in \mathcal{P}_{\overline{N}}$ if and only if

$$\begin{cases} (s-\mu)Pk(1-\tau) - \alpha P^2 k\tau(1-\tau) + h\left(v_6\right) \leq Pk(1-\tau), \\ (i-\gamma)P\tau \leq \tau P, \\ (s-\mu)P - \alpha P^2\tau + \dfrac{h(v_6)}{k(1-\tau)} + P^2 + (i-\gamma)P + \mu Pk(1-\tau) + \gamma P\tau \leq P. \end{cases}$$

i.e.;

$$\begin{cases} h\left(v_6\right) \leq Pk(1-\tau)(1-s+\mu+\alpha P\tau), \\ i-\gamma \leq 1, \\ \dfrac{h(v_6)}{k(1-\tau)} \leq P\big(\mathbf{a}+\mathbf{c}-1+\alpha P\tau - P\big) = P\,\mathbf{f}. \end{cases}$$

Using $< 0 < \gamma < i \leq 1$, we get $i - \gamma \leq 1$, and then the second condition holds. Further, as seen previously, we have $\mathbf{a} < 1 - s + \mu$ and $c - 1 < 0$, then $\mathbf{f} \leq (1 - s + \mu + \alpha P\tau)$ and hence the first condition comes from the third condition and this last one follows from (4.17). Finally, all the desired conditions hold.

7. We have $g_c(v_7) = (h(v_7), \alpha(e-\rho)Pk\tau(1-\tau), \alpha\rho Pk\tau(1-\tau) + (i-\gamma)P\tau, \gamma P\tau)^T \in \mathcal{P}_{\overline{N}}$ if and only if

$$\begin{cases} h(v_7) \leq Pk(1-\tau), \\ \alpha(e-\rho)Pk\tau(1-\tau) \leq \alpha Pk\tau(1-\tau) \\ \alpha\rho Pk\tau(1-\tau) + (i-\gamma)P\tau \leq \tau P \\ \dfrac{h(v_7)}{k(1-\tau)} + (e-\rho)P + \alpha\rho Pk(1-\tau) + (i-\gamma)P + \gamma P\tau \leq P. \end{cases}$$

Taking into account the condition (4.16), these conditions are equivalent to

$$\begin{cases} h(v_7) \leq Pk(1-\tau), \\ e-\rho \leq 1, \\ \dfrac{h(v_7)}{k(1-\tau)} \leq P\left(\mathbf{b}+\mathbf{c}-1\right) = P\,\mathbf{g}. \end{cases}$$

As done for the previous cases, we have $0 < \rho < e < 1$, then $e - \rho < 1$. Furthermore, we have $\mathbf{b} < 1$ and $\mathbf{c} < 1$, then the fist condition follows from the third condition and this last one comes from assumption (4.17). Thus, all the desired conditions hold.

□

Remark 4.3 *What is intended with the result of the previous theorem is to determine conditions on the control actions that can be taken to ensure that if the initial condition is within the security set then applying those control actions the trajectory of the model will remain within that set.*

If we take the function $h(x) = \sum_{i=1}^{7} \alpha_i h(v_i)$ where coefficients α_i given in (4.6) and quantities $h(v_i)$ are equal to the upper bounds given in (4.17), we get

$$\begin{aligned}
h(x) =& \alpha_1 h(v_1) + \alpha_2 h(v_2) + \alpha_3 h(v_3) + \alpha_4 h(v_4) + \alpha_5 (v_5) + \alpha_6 h(v_6) + \alpha_7 h(v_7) \\
=& \big(x_1 - \frac{x_1 x_2}{(\alpha k\tau(1-\tau))P} - \frac{\alpha\tau\, x_1 x_3 - x_2 x_3}{(\alpha\tau^2)P}\big)\,\mathbf{a} + \big(\frac{x_2}{\alpha\tau} - \frac{\tau\, x_1 x_2 - k(1-\tau)\, x_2 x_3}{(\alpha k\tau^2(1-\tau))P}\big)\,\mathbf{b} \\
&+ \big(\frac{x_3}{\tau} - \frac{x_1 x_3}{(k\tau(1-\tau))P}\big) k(1-\tau)\,\mathbf{c} + x_4\, k(1-\tau)\,\mathbf{d} + \frac{x_1 x_2}{(\alpha k\tau(1-\tau))P}\,\mathbf{e} \\
&+ \frac{\alpha\tau\, x_1 x_3 - x_2 x_3}{\alpha\tau^2 P}\,\mathbf{f} + \frac{x_2 x_3}{\alpha\tau^2 P}\,\mathbf{g} \\
=& \big(x_1 - \frac{x_1 x_2}{(\alpha k\tau(1-\tau))P} - \frac{\alpha\tau\, x_1 x_3 - x_2 x_3}{(\alpha\tau^2)P}\big)\,\mathbf{a} + \big(\frac{x_2}{\alpha\tau} - \frac{\tau\, x_1 x_2 - k(1-\tau)\, x_2 x_3}{(\alpha k\tau^2(1-\tau))P}\big)\,\mathbf{b} \\
&+ \big(\frac{x_3}{\tau} - \frac{x_1 x_3}{(k\tau(1-\tau))P}\big) k(1-\tau)\,\mathbf{c} + x_4\, k(1-\tau)\,\mathbf{d} + \frac{x_1 x_2}{(\alpha k\tau(1-\tau))P}\,(\mathbf{a}+\mathbf{b}-1) \\
&+ \frac{\alpha\tau\, x_1 x_3 - x_2 x_3}{\alpha\tau^2 P}\,(\mathbf{a}+\mathbf{c}-1+(\alpha\tau-1)P) + \frac{x_2 x_3}{\alpha\tau^2 P}\,(\mathbf{b}+\mathbf{c}-1) \\
=& \,\mathbf{a}\, x_1 + \frac{\mathbf{b}}{\alpha\tau}\, x_2 + \frac{\mathbf{c}}{\tau}\, x_3 + k(1-\tau)\,\mathbf{d}\, x_4 - \frac{1}{\alpha k\tau(1-\tau)P}\, x_1 x_2 \\
&+ \big(\frac{2\,\mathbf{b}}{\alpha\tau^2 P} + \frac{1-\alpha\tau}{\alpha\tau^2}\big)\, x_2 x_3 - \big(\frac{1}{\tau P} + \frac{1-\alpha\tau}{\tau}\big)\, x_1 x_3.
\end{aligned}$$

Hence, the function $\tilde{h} = Bh$ can be written as function of class Θ defined by (4.5), that is

$$\tilde{h}(x) = \sum_{i=1}^{4} \tilde{c}_i x_i + \tilde{c}_5 x_1 x_3$$

where the entries of the vectors $\tilde{c}_i$ are given as follows

$$(\tilde{c}_1)_1 = \mathbf{a}\ ,\ (\tilde{c}_2)_1 = \frac{\mathbf{b}}{\alpha\tau}\ ,\ (\tilde{c}_3)_1 = \frac{\mathbf{c}}{\tau}\ ,\ (\tilde{c}_4)_1 = k(1-\tau)\,\mathbf{d}$$
$$(\tilde{c}_5)_1 = \frac{-1-(1-\alpha\tau)P}{\tau P}\ ,\ (\tilde{c}_i)_j = 0 \ \text{ for } \ i = 1,..,5,\ j = 2,..,4.$$

Finally, for this specific control feedback $u(x(t)) = h(x(t))$, the closed-loop system is

$$x(t+1) = Ax(t) + f(x(t)) + \tilde{h}(x(t)) = \tilde{A}x(t) + F(x(t)) \tag{4.23}$$

where

$$\tilde{A} = \begin{bmatrix} s-\mu+(\tilde{c}_1)_1 & (\tilde{c}_2)_1 & (\tilde{c}_3)_1 & \omega+(\tilde{c}_4)_1 \\ 0 & e-\rho & 0 & 0 \\ 0 & \rho & i-\gamma & 0 \\ \mu & 0 & \gamma & r-\omega \end{bmatrix}$$

and

$$F(x(t)) = \begin{bmatrix} -\alpha+(\tilde{c}_5)_1 \\ \alpha \\ 0 \\ 0 \end{bmatrix} x_1(t)x_3(t).$$

Note that system (4.23) will be stable if $F(x(t))$ is bounded and the spectral radius of $\tilde{A}$ is smaller than the unit.

Conclusion

We have considered here a dynamic SEIRS process for the spread of a disease. We have obtained conditions to provide a distribution of the population (safety distribution) into Susceptible, Exposed, Infected and Recovered subgroups, where the proportion of infected individuals is assumed to be kept below a safety threshold. This has allowed us to define the security set and search for its structure as a polytope, determined by its vertices. Moreover, conditions are obtained so that if a state of the system lies in the safety set, the system trajectory remains in that safety set, forever. We note that contrary to dimension 3 where we can trace an exemple of a polytope of security and trajectories of the system studied (see for example [13]), it is more delicate in dimension 4 to illustrate obtained results by an example. Finally, we recall that ensuring solvability under feedback control requires certain assumptions, notably the boundedness of the studied population. Therefore, we plan to explore the problem without this restrictive hypothesis in future work.

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A. Appendix : Preliminaries

We recall here some useful notations, definitions and lemmas which will be used in the current paper

Definition A.1 *Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be real matrix. Then*

1. *A is said to be non-negative (positive), denoted $A \geq 0$ ($A > 0$), if*

$$a_{ij} \geq 0 \quad (a_{ij} > 0) \quad \text{for all} \quad 1 \leq i, j \leq n.$$

2. *The spectrum of A is the set of its eigenvalues, that is the set of reals λ such that*

$$\det(A - \lambda I_n) \neq 0.$$

3. *the spectral radius of $\sigma(A)$ is the maximum of the absolute values of its eigenvalues.*

Lemma A.1 (see [6]) *The matrix A is said to be stable if its spectral radius is less than 1, i.e.;*

$$A \text{ is stable} \iff \rho(A) < 1.$$

An autonomous discrete-time linear system $x(t+1) = Ax(t)$ is asymptotically stable to 0 if and only if

$$A \text{ is a stable matrix, that is } \rho(A) < 1.$$

Let $m \geq n+1$, we consider a set of points $V = \{v_1, \ldots, v_m\}$ in the space $\mathbb{R}^n$ such that there exist no hyperplanes of $\mathbb{R}^n$ containing all these m points. A full-dimensional polytope $P(V)$ is defined as the convex hull of V, see [19] and references theirein. If a point v_i, $i \in \{1, \ldots, m\}$ cannot be written as a convex combination of $V \setminus \{v_i\}$, the point v_i is called a polytope's vertex. We recall that, a full-dimensional polytope is characterised by its set of vertices.

Consider the discrete-time non-linear system given by

$$x(t+1) = Ax(t) + f(x(t)) + Bu(t), \quad \forall\, t \geq 0 \tag{A.1}$$

with $x(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}$, $A \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathbb{R}^n$ and $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ a bound differentiable function satisfying $f(0) = 0$ and $f(\lambda\, e_i) = 0$ for all $\lambda > 0$, where $\mathcal{B} = (e_i)_{i=1,..,n}$ the canonical basis of $\mathbb{R}^n$.

In general, positive systems are those systems whose trajectory from any initial non-negative condition remains in the positive orthant for all future time, that is, a system is positive if from any non-negative initial state and any non-negative input sequence the solution trajectory is non-negative. The notion of local positiveness of a non-linear time-varying system and necessary and sufficient conditions for positivity of a class of non-linear systems are introduced and established in [23,24].

Denoting $f(x) = (f_i(x))_{i=1,..,n}$ and $Ax = ((Ax)_i)_{i=1,..,n}$, we have the following characterization.

Proposition A.1 *System (A.1) is positive if and only if $A \geq 0$, $B \geq 0$ and for every $x \in \mathbb{R}^n$, one has*

$$|f_i(x)| \leq (Ax)_i \quad \text{for any } i \text{ satisfying } f_i(x) \leq 0.$$

Definition A.2 *Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{C}$. For each $i \in \{1, \cdots, n\}$, we define*

$$R_i := \sum_{j \neq i} |a_{i,j}|.$$

The sets $D_i := \{z \in \mathbb{C}\,;\ |z - a_{i,i}| \leq R_i\}$ with $i \in \{1, \cdots, n\}$, are called the Gershgorin disks of A.

Theorem A.1 (Gershgorin circle Theorem) *Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{C}$. The eigenvalues of A belong to the union of its Gershgorin disks.*

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