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Global Optimal Solution for a System of Differential Equations via Measure of Noncompactness

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ABSTRACT: In this paper, we use a measure of noncompactness to give some best proximity point and best proximity pair results for cyclic (noncyclic) operators. As a consequence of our findings, we obtain an extension of Darbo's fixed point theorem. Furthermore, we investigate the existence of an optimal solution for a system of ordinary differential equations as an application of our results.

Key Words: Best proximity point, best proximity pair, strictly convex Banach spaces, ordinary differential equations.

Contents

1. Introduction and Preliminaries

Many optimization problems can be reformulated as best approximation problems. Due to this, optimization theory plays a key role in several areas, such as variational inequalities problems, fixed point problems, split feasibility problems, and so on. On the other hand, the study of nonself mappings is also fascinating because in this case best approximation exists by Ky Fan [6] technique. In this case we find a point x is an approximate solution such that the error $d(x, Tx)$ is minimum, that is, the point x is close proximity to Tx. However, when A is mapped into another subset B of X by T, the problem extends to determining a point that estimates the distance between these two subsets. These are referred to as best proximity points. If $T : A \to B$ is nonself mapping on a metric space (X, d), then a point $x \in A$ is known as the best proximity point (BPP) of a nonself mapping T, satisfying the condition

$$d(x, Tx) = d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$

where A and B are non-empty subsets of X such that $A \cap B = \emptyset$. Best approximation is an invariant approximation in the case of self mappings.
In this paper, we present some best proximity point and best proximity pair results for cyclic (noncyclic) operators by using a measure of noncompactness. As a consequence of our findings, we obtain an extension of Darbo's fixed point theorem. Also, we investigate the existence of an optimal solution for a system of ordinary differential equations as an application of our obtained results.

Definition 1.1 *Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic (respectively, noncyclic) map [5] if*

$$T(A) \subseteq B \text{ and } T(B) \subseteq A \text{ (respectively, } T(A) \subseteq A \text{ and } T(B) \subseteq B).$$

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Definition 1.2 *If the mapping T is noncyclic, a pair $(x, y) \in (A, B)$ is called a best proximity pair [5] if*

$$Tx = x,\ Ty = y \text{ and } d(x, y) = d(A, B).$$

Definition 1.3 *A self mapping T defined on a set $A \cup B$ is said to be relatively nonexpansive [5] if*

$$d(Tx, Ty) \leq d(x, y);\ \text{for all } x \in A \text{ and } y \in B,$$

where A and B be nonempty subsets of a metric space (X, d).

The set of all bounded subsets in a metric space (X, d) is represented by $\mathcal{B}(X)$.

Definition 1.4 *[2] The measure of noncompactness (MNC) is a function $\chi : \mathcal{B}(X) \to \mathbb{R}^+$ which fulfilled the following conditions:*

(i') $\chi(A) = 0$ if and only if A is relatively compact,

(i") $\chi(A) = \chi(\overline{A})$,

(i"') $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$,

If χ is an MNC on $\mathcal{B}(X)$, then the following properties will be concluded immediately:

(a') $\chi(A) = 0$ if and only if A is a finite set,

(b') $\chi(A \cap B) = \min\{\chi(A), \chi(B)\}$,

(c') If $\lim\limits_{n\to\infty}(A_n) = 0$ for a nonincreasing sequence $\{A_n\}$ of nonempty, closed and bounded subsets of X then $A_\infty = \cap_{n\geq 1} A_n$ is compact and nonempty,

(d') $A \subseteq B$ implies $\chi(A) \leq \chi(B)$,

Also, the following axioms hold if X is a Banach space

(e') $\chi(A) = \chi(\overline{con}A)$,

(f') $\chi(tA) = |t|\chi(A)$, for any number t and $A \in \mathcal{B}(X)$,

(g') $\chi(A + B) \leq \chi(A) + \chi(B)$, for all $A, B \in \mathcal{B}(X)$.

Definition 1.5 *A map $\kappa : \mathcal{B}(X) \to \mathbb{R}^+$ is said to be a Kuratowski MNC [11] if*

$$\kappa(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i, A_i \in \mathcal{B}(X),\ diam(A_i) < \epsilon, i \in \mathbb{N}\}.$$

Definition 1.6 *[8] Let T be a noncyclic (cyclic) self mapping defined on $A \cup B$. T is said to be compact whenever the pair $(T(A), T(B))$ is compact, that is, both $T|A$ and $T|B$ are compact, where (A, B) be a nonempty and bounded pair in a normed linear space X.*

Theorem 1.1 *[7] A relatively nonexpansive cyclic self mapping T which defined on $A \cup B$ has a best proximity point if (A, B) is a nonempty, bounded, closed, and convex pair in a Banach space X, A_0 is nonempty and T is compact.*

Theorem 1.2 *[7] A relatively nonexpansive cyclic self mapping T which defined on $A \cup B$ has a best proximity point if (A, B) is a nonempty, bounded, closed, and convex pair in a strictly convex Banach space X, A_0 is nonempty and T is compact.*

The following extensions of topological Schauder fixed point theorem and classical Banach fixed point theorem were proved by Darbo (respectively, Sadovskii) in 1955 (respectively, 1972).

Theorem 1.3 *[4] Let T be a continuous self mapping defined on a set A and satisfies the following*

$$\chi(T(A_1)) \leq k\chi(A_1);\ \ k \in [0,1),$$

for any $A_1 \subseteq A$, where χ is the Kuratowski MNC on Banach space X and A is a subset of X. Then T has a fixed point.

Theorem 1.4 *[12] Let T be a continuous self mapping defined on a set A and satisfies the following*

$$\chi(A_1) > 0 \text{ implies } \chi(T(A_1)) \leq k\chi(A_1);\ \ k \in [0,1),$$

for any $A_1 \subseteq A$, where χ is the Kuratowski MNC on Banach space X and A is a subset of X. Then T has a fixed point.

Throughout the paper, A and B are nonempty and convex subsets of a Banach space X, NBCC denotes nonempty, bounded, closed and convex subsets of a Banach space X, χ is MNC on X and $\overline{con}(A)$ is closed and convex hull of a set A. $B(x,r)$ denotes the closed ball with center x and radius r in a real Banach space $(X, ||.||)$. For two non-empty subsets A and B of a Banach space X. Define

$$A_0 = \{x \in A : \text{there exists some } y \in B \text{ such that } ||x-y|| = ||A-B||\},$$
$$B_0 = \{y \in B : \text{there exists some } x \in A \text{ such that } ||x-y|| = ||A-B||\}$$

If (A,B) is a pair of nonempty, convex and weakly compact subsets of X, then the respective pair (A_0, B_0) is of the same kind. For details, see [10]. If $A_0 = A$ and $B_0 = B$ then the pair (A,B) is said to be proximinal.

2. Main results

To prove the main results, we introduce the following definitions:

Definition 2.1 *A mapping $T : A\cup B \to A\cup B$ is said to be ς-condensing cyclic (respectively, noncyclic), provided that for any NBCC proximal and T invariant pair $(K_1,K_2) \subseteq (A,B)$ such that $||K_1 - K_2|| = ||A-B||$ with $T(A) \subseteq B$ and $T(B) \subseteq A$ (respectively, $T(A) \subseteq A$ and $T(B) \subseteq B$), we have*

$$\chi(T(K_1) \cup T(K_2)) \leq \varsigma(\chi(K_1 \cup K_2))\chi(K_1 \cup K_2),$$

where $\varsigma : [0,\infty) \to [0,1)$ is a function such that $\limsup_{s\to t^*} \varsigma(s) < 1$ for all $t \in [0,\infty)$.

Example 2.1 *Define $\varsigma : [0,\infty) \to [0,1)$ by*

$$\varsigma(t) = \frac{\sin(t)}{2}.$$

Then $\limsup_{s\to t^*} \varsigma(s) < 1$ for all $t \in [0,\infty)$.

Definition 2.2 *A mapping $T : A\cup B \to A\cup B$ is said to be ψ-condensing cyclic (respectively, noncyclic) if provided that for any NBCC proximal and T invariant pair $(K_1,K_2) \subseteq (A,B)$ such that $||K_1 - K_2|| = ||A-B||$, with $T(A) \subseteq B$ and $T(B) \subseteq A$ (respectively, $T(A) \subseteq A$ and $T(B) \subseteq B$), we have*

$$\chi(T(K_1) \cup T(K_2)) \leq \chi(K_1 \cup K_2) - \psi(\chi(K_1 \cup K_2)),$$

where $\psi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\psi(t) = 0$ if and only if $t = 0$; $t \in [0,\infty)$.

Example 2.2 *Define $\psi : [0,\infty) \to [0,\infty)$ by*

$$\psi(x) = \frac{x^2}{3}.$$

Then $\psi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\psi(x) = 0$ if and only if $x = 0$; $x \in [0,\infty)$.

Theorem 2.1 *Let $T : A \cup B \to A \cup B$ be a ς-condensing cyclic operator in the sense of Definition 2.1. Then T has a BPP if T is a relatively nonexpansive mapping and A_0 is nonempty.*

Proof: As A_0 is nonempty, (A_0, B_0) is nonempty. Also one can show that (A_0, B_0) is convex, closed, T -invariant and proximinal pair considering conditions on T. For $u \in A_0$, there is a $v \in B_0$ such that $||u - v|| = ||A - B||$. Since T is relatively nonexpansive

$$||Tu - Tv|| \leq ||u - v|| = ||A - B||,$$

which gives $Tu \in B_0$, that is, $T(A_0) \subseteq B_0$. Similarly, $T(B_0) \subseteq A_0$ and so T is cyclic on $A_0 \cup B_0$. Let us define a pair (G_n, H_n) as $G_n = \overline{con}(T(G_{n-1}))$ and $H_n = \overline{con}(T(H_{n-1}))$, $n \geq 1$ with $G_0 = A_0$ and $H_0 = B_0$. We claim that $G_{n+1} \subseteq H_n$ and $H_n \subseteq G_{n-1}$ for all $n \in \mathbb{N}$. We have $H_1 = \overline{con}(T(H_0)) = \overline{con}(T(B_0)) = \overline{con}(A_0) \subseteq A_0 = G_0$. Therefore, $T(H_1) \subseteq T(G_0)$. So $H_2 = \overline{con}(T(H_1)) \subseteq \overline{con}(T(G_0)) \subseteq G_1$. Repeated the steps, we have $H_n \subseteq G_{n-1}$ by induction. Using similar lines, we get $G_{n+1} \subseteq H_n$ for all $n \in \mathbb{N}$. Thus $G_{n+2} \subset H_{n+1} \subseteq G_n \subseteq H_{n-1}$ for all $n \in \mathbb{N}$. Hence, we have a nonincreasing sequence $\{(G_{2n}, H_{2n})\}$ of nonempty, closed and convex pairs in $A \times B$. Moreover, $T(H_{2n}) \subseteq T(G_{2n-1}) \subset \overline{con}(T(G_{2n-1})) = G_{2n}$ and $T(G_{2n}) \subseteq T(H_{2n-1}) \subseteq \overline{con}(T(H_{2n-1})) = H_{2n}$. This show that the pair (G_{2n}, H_{2n}) is T -invariant. Now if $(u, v) \in A \times B$ is a proximinal pair then

$$d(G_{2n}, H_{2n}) \leq ||T^{2n}u - T^{2n}v|| \leq ||u - v|| = ||A - B||.$$

By induction we prove that the pair (G_n, H_n) is proximinal. The pair (G_0, H_0) is proximinal. It is true for $n = 0$. Suppose that it is true for $n = k$. Next we prove that it is true for $n = k + 1$. Let x be an arbitrary member in $G_{k+1} = \overline{con}(T(G_k))$. It can be written as $x = \sum_{l=1}^{m'} \lambda_l T(x_l)$ with $x_l \in G_k$, $m' \in \mathbb{N}$, $\lambda_l \geq 0$ and $\sum_{l=1}^{m'} \lambda_l = 1$. Since the pair (G_k, H_k) is proximinal, there exists $y_l \in H_k$ for $1 \leq l \leq m'$ such that $x_l - y_l = ||G_k - H_k|| = ||A - B||$. Take $y = \sum_{l=1}^{m'} \lambda_l T(y_l)$. Then $y \in \overline{con}(T(H_k)) = H_{k+1}$ and

$$||x - y|| = \left|\left|\sum_{l=1}^{m'} \lambda_l T(x_l) - \sum_{l=1}^{m'} \lambda_l T(y_l)\right|\right| \leq \sum_{l=1}^{m'} \lambda_l ||x_l - y_l|| = ||A - B||.$$

This shows that the pair (G_{k+1}, H_{k+1}) is proximinal. By induction we can say that (G_n, H_n) is proximinal for all $n \in \mathbb{N}$. Now, it is understood that there arise two cases: namely either $\max\{\chi(G_{2j}), \chi(H_{2j})\} = 0$ for some $j \in \mathbb{N}$ or $\max\{\chi(G_{2n}), \chi(H_{2n})\} > 0$ for all $n \in \mathbb{N}$.

First, assume that $\max\{\chi(G_{2j}), \chi(H_{2j})\} = 0$ for some $j \in \mathbb{N}$, then $T : G_{2j} \cup H_{2j} \to G_{2j} \cup H_{2j}$ is compact. Apply the Theorem 1.1 we get the result.

Now, suppose that $\max\{\chi(G_{2n}), \chi(H_{2n})\} > 0$ for all $n \in \mathbb{N}$. As $G_{2n+1} \subseteq T(G_{2n})$ and $H_{2n+1} \subseteq T(H_{2n})$, we have

$$\begin{aligned}
\chi(G_{2n+1} \cup H_{2n+1}) &= \max\{\chi(G_{2n+1}), \chi(H_{2n+1})\} \\
&= \max\{(\overline{con}(T(G_{2n}))), \chi(\overline{con}(T(H_{2n})))\} \\
&= \max\{\chi(T(G_{2n})), \chi(T(H_{2n}))\} \\
&= \chi(T(G_{2n}) \cup T(H_{2n})) \\
&\leq \varsigma(\chi(G_{2n} \cup H_{2n}))\chi(G_{2n} \cup H_{2n}) \\
&\leq \chi(G_{2n} \cup H_{2n}).
\end{aligned}$$

$\chi_n = \chi(G_{2n} \cup H_{2n})$, is a nonincreasing sequence of positive real numbers. Then there exists $\alpha \geq 0$ such that $\chi_n \to \alpha$ as $n \to \infty$. Suppose $\alpha > 0$, for all $n \in \mathbb{N}$, we have

$$\frac{\chi(G_{2n+1} \cup H_{2n+1})}{\chi(G_{2n} \cup H_{2n})} \leq \varsigma(\chi(G_{2n} \cup H_{2n})).$$

By above inequality $\varsigma(\chi(G_{2n} \cup H_{2n})) \geq 1$, which is contradiction. So $\alpha = 0$ and $\chi_n = \chi(G_{2n} \cup H_{2n}) \to 0$ as $n \to \infty$. Now, let $G_\infty = \cap_{n=0}^{\infty} G_{2n}$ and $H_\infty = \cap_{n=0}^{\infty} H_{2n}$. By property (d') of MNC, the pair (G_∞, H_∞) is nonempty, convex, compact and T -invariant with $||G_\infty - H_\infty|| = ||A - B||$. Therefore, T has a BPP. □

Theorem 2.2 *Let $T : A \cup B \to A \cup B$ be a ψ-condensing cyclic operator in the sense of Definition 2.2. Then T has a BPP if T is a relatively nonexpansive mapping and A_0 is nonempty.*

Proof: Using the similar lines of proof Theorem 2.1, we obtain (G_n, H_n) is proximinal for all $n \in \mathbb{N}$. There are arise two facts: namely either $\max\{\chi(G_{2j}), \chi(H_{2j})\} = 0$ for some $j \in \mathbb{N}$ or $\max\{\chi(G_{2n}), \chi(H_{2n})\} > 0$ for all $n \in \mathbb{N}$. If we take $\max\{\chi(G_{2j}), \chi(H_{2j})\} = 0$ for some $j \in \mathbb{N}$, then $T : G_{2j} \cup H_{2j} \to G_{2j} \cup H_{2j}$ is compact, by Theorem 1.1 we get the results. Next, consider $\max\{\chi(G_{2n}), \chi(H_{2n})\} > 0$ for all $n \in \mathbb{N}$. As $G_{2n+1} \subseteq T(G_{2n})$ and $H_{2n+1} \subseteq T(H_{2n})$, we have

$$\begin{aligned}
\chi(G_{2n+1} \cup H_{2n+1}) &= \max\{\chi(G_{2n+1}), \chi(H_{2n+1})\} \\
&= \max\{(\overline{con}(T(G_{2n}))), \chi(\overline{con}(T(H_{2n})))\} \\
&= \max\{\chi(T(G_{2n})), \chi(T(H_{2n}))\} \\
&= \chi(T(G_{2n}) \cup T(H_{2n})) \\
&\leq \chi(G_{2n} \cup H_{2n}) - \psi(\chi(G_{2n} \cup H_{2n})) \\
&\leq \chi(G_{2n} \cup H_{2n}).
\end{aligned}$$

So, $\chi_n = \chi(G_{2n} \cup H_{2n})$ is a nonincreasing sequence of positive real numbers. Thus there exists $\alpha \geq 0$ such that $\chi_n \to \alpha$ as $n \to \infty$. Suppose $\alpha > 0$, for all $n \in \mathbb{N}$, we have

$$\chi(G_{2n+1} \cup H_{2n+1}) - \chi(G_{2n} \cup H_{2n}) \leq -\psi(\chi(G_{2n} \cup H_{2n})).$$

Taking $n \to \infty$ in above inequality we get $\psi(\alpha) = 0$, for $\alpha > 0$, which is contradiction. So $\alpha = 0$ and $\chi_n = \chi(G_{2n} \cup H_{2n}) \to 0$ as $n \to \infty$. Now, let $G_\infty = \cap_{n=0}^{\infty} G_{2n}$ and $H_\infty = \cap_{n=0}^{\infty} H_{2n}$. By property (d') of MNC, the pair (G_∞, H_∞) is T -invariant, nonempty, compact and convex with $||G_\infty - H_\infty|| = ||A - B||$. Therefore, T has a BPP. □

Theorem 2.3 *Suppose that a pair (A, B) is NBCC subset of a strictly convex Banach space X and $T : A \cup B \to A \cup B$ is a ς condensing noncyclic operator in the sense of Definition 2.1. Then T has a best proximity pair if T is a relatively nonexpansive mapping and A_0 is nonempty.*

Proof: As A_0 is nonempty, (A_0, B_0) is nonempty. Also one can show that (A_0, B_0) is T -invariant, proximinal pair, closed and convex considering conditions on T. For $u \in A_0$, there is a $v \in B_0$ such that $||u - v|| = ||A - B||$. Since T is relatively nonexpansive

$$||Tu - Tv|| \leq ||u - v|| = ||A - B||,$$

which gives $Tu \in A_0$, that is, $T(A_0) \subseteq A_0$. Similarly, $T(B_0) \subseteq B_0$ and so T is noncyclic on $A_0 \cup B_0$. Using the similiar lines of proof Theorem 2.1, define a pair (G_n, H_n) as $G_n = \overline{con}(T(G_{(n-1)}))$, $H_n = \overline{con}(T(H_{n-1}))$, $n \geq 1$ with $G_0 = A_0$ and $H_0 = B_0$ and $\{(G_n, H_n)\}$ is a nonincreasing sequence of pairs which are NBCC in $A_0 \times B_0$. Also, $T(H_n) \subseteq T(H_{n-1}) \subseteq \overline{con}(T(H_{n-1})) = H_n$ and $T(G_n) \subseteq T(G_{n-1}) \subseteq \overline{con}(T(G_{n-1})) = G_n$. Hence, the pair (G_n, H_n) is T -invariant for all $n \in \mathbb{N}$. It follows from Theorem 2.1, we have (G_n, H_n) is proximinal for all $n \in \mathbb{N}$ with $||G_n - H_n|| = ||A - B||$. If $\max\{\chi(G_j), \chi(H_j)\} = 0$ for some $j \in \mathbb{N}$, then $T : G_j \cup H_j \to G_{2j} \cup H_{2j}$ is compact, by Theorem 1.2 we get the result. Consider $\max\{\chi(G_n), \chi(H_n)\} > 0$ for all $n \in \mathbb{N}$. Since $G_{n+1} \subseteq T(G_n)$ and $H_{n+1} \subseteq T(H_n)$, we have

$$\begin{aligned}
\chi(G_{n+1} \cup H_{n+1}) &= \max\{\chi(G_{n+1}), \chi(H_{n+1})\} \\
&= \max\{\chi(\overline{con}(T(G_n))), \chi(\overline{con}(T(H_n)))\} \\
&= \max\{\chi(T(G_n)), \chi(T(H_n))\} \\
&= \chi(T(G_n) \cup T(H_n)) \\
&\leq \varsigma(\chi(G_n \cup H_n))\chi(G_n \cup H_n) \\
&\leq \chi(G_n \cup H_n).
\end{aligned}$$

$\chi_n = \chi(G_n \cup H_n)$, is a nonincreasing sequence of positive real numbers. Thus there exists $\alpha \geq 0$ such that $\chi_n \to \alpha$ as $n \to \infty$. Suppose $\alpha > 0$, for all $n \in \mathbb{N}$, we have

$$\frac{\chi(G_{n+1} \cup H_{n+1})}{\chi(G_n \cup H_n)} \leq \varsigma(\chi(G_n \cup H_n)).$$

By above inequality $\varsigma(\chi(G_n \cup H_n)) \geq 1$, which is contradiction. So $\alpha = 0$ and $\chi_n = \chi(G_n \cup H_n) \to 0$ as $n \to \infty$. Now, let $G_\infty = \cap_{n=0}^{\infty} G_n$ and $H_\infty = \cap_{n=0}^{\infty} H_n$. By property (d') of MNC, the pair (G_∞, H_∞) is T -invariant, nonempty, compact and convex with $||G_\infty - H_\infty|| = ||A - B||$. Therefore, T has a best proximity pair. □

Theorem 2.4 *Suppose that a pair (A, B) is NBCC subset of a strictly convex Banach space X and $T : A \cup B \to A \cup B$ is a ψ condensing noncyclic operator in the sense of Definition 2.2. Then T has a best proximity pair if T is a relatively nonexpansive mapping and A_0 is nonempty.*

Proof: Using the similar lines of proof Theorem 2.1, we have (G_n, H_n) is proximinal for all $n \in \mathbb{N}$ with $||G_n - H_n|| = ||A - B||$. If $\max\{\chi(G_j), \chi(H_j)\} = 0$ for some $j \in \mathbb{N}$, then $T : G_j \cup H_j \to G_{2j} \cup H_{2j}$ is compact, by Theorem 1.2 we get the result. consider $\max\{\chi(G_n), \chi(H_n)\} > 0$ for all $n \in \mathbb{N}$. Since $G_{n+1} \subseteq T(G_n)$ and $H_{n+1} \subseteq T(H_n)$, we have

$$\begin{aligned}
\chi(G_{n+1} \cup H_{n+1}) &= \max\{\chi(G_{n+1}), \chi(H_{n+1})\} \\
&= \max\{\chi(\overline{con}(T(G_n))), \chi(\overline{con}(T(H_n)))\} \\
&= \max\{\chi(T(G_n)), \chi(T(H_n))\} \\
&= \chi(T(G_n) \cup T(H_n)) \\
&\leq \chi(G_n \cup H_n) - \psi(\chi(G_n \cup H_n)) \\
&\leq \chi(G_n \cup H_n).
\end{aligned}$$

$\chi_n = \chi(G_n \cup H_n)$, is a nonincreasing sequence of positive real numbers. Thus there exists $\alpha \geq 0$ such that $\chi_n \to \alpha$ as $n \to \infty$. Suppose $\alpha > 0$, for all $n \in \mathbb{N}$, we have

$$\chi(G_{n+1} \cup H_{n+1}) - \chi(G_n \cup H_n) \leq -\psi(\chi(G_n \cup H_n)).$$

Taking $n \to \infty$ in above inequality we get $\psi(\alpha) = 0$, for $\alpha > 0$, which is contradiction. So $\alpha = 0$ and $\chi_n = \chi(G_n \cup H_n) \to 0$ as $n \to \infty$. Now, let $G_\infty = \cap_{n=0}^{\infty} G_n$ and $H_\infty = \cap_{n=0}^{\infty} H_n$. By property (d') of MNC, the pair (G_∞, H_∞) is T -invariant, nonempty, compact and convex with $||G_\infty - H_\infty|| = ||A - B||$. Therefore, T has a best proximity pair. □

3. Consequences

If we take $\varsigma(s) = k$; $k \in (0, 1)$ in Theorem 2.1, then we get the following result:

Corollary 3.1 *Let (K_1, K_2) be a nonempty, closed, convex, proximinal, and T-invariant pair with $||K_1 - K_2|| = ||A - B||$, and $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping such that*

$$\chi(T(K_1) \cup T(K_2)) \leq k\chi(K_1 \cup K_2).$$

Then T has a BPP provided A_0 is nonempty.

Remark 3.1 If we take $A = B$ in Corollary 3.1, we retrieve Darbo's fixed point theorem [4].
If we take $\varsigma(s) = k$; $k \in (0, 1)$ in Theorem 2.3, then we get the following result:

Corollary 3.2 *Let (K_1, K_2) be a nonempty, closed, convex, proximinal, and T-invariant pair with $|K_1 - K_2|| = ||A - B||$, and $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping such that*

$$\chi(T(K_1) \cup T(K_2)) \leq k\chi(K_1 \cup K_2); \; k \in (0, 1).$$

Then T has a best proximity pair provided A_0 is nonempty.

Corollary 3.3 *[3] Let $T : A \to A$ be a continuous operator satisfied with the following condition:*

$$\chi(T(A_1)) \leq \chi(A_1) - \psi(\chi(A_1)),$$

for any $A_1 \subseteq A$. Then T has at least one fixed point in A, where $\psi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\psi(t) = 0$ if and only if $t = 0$; $t \in [0, \infty)$ and A is a nonempty, bounded, closed and convex subset of a Banach space X.

4. Application

We investigate the existence of a global optimal solution for a system of differential equations as an application of our findings in this section.

First, we recall the following lemma:

Lemma 4.1 *[9] Let J be a real interval, X be a Banach space, and $f : J \to X$ be a differentiable mapping. Let $a, b \in J$ with $a < b$. Then*

$$f(b) - f(a) \in (b - a)\overline{con}(\{f'(t) : t \in [a, b]\}).$$

Definition 4.1 *Let $V_1 = B(x_0, b)$, $V_2 = B(x_1, b)$ be closed balls in a Banach space X, where $a, b \in \mathbb{R}^+$, $t_0 \in \mathbb{R}$ and $x_0, x_1 \in X$. Assume that $f : I \times V_1 \to X$ and $g : I \times V_2 \to X$ are continuous mappings. Consider the following system of differential equations:*

$$\begin{cases} x'(t) = & f(t, x(t)), \quad x(t_0) = x_0 \\ y'(t) = & f(t, y(t)), \quad y(t_0) = x_1, \end{cases} \tag{1}$$

defined on a closed real interval $J = [t_0 - h, t_0 + h]$ for some $h \in \mathbb{R}^+$. Let $C(J, X)$ consist of all continuous mappings from J into X with the supremum norm, and let

$$\begin{aligned} C(J, V_1) &= \{x \in C(J, X) : x(t_0) = x_0\}, \\ C(J, V_2) &= \{y \in C(J, X) : y(t_0) = x_1\}, \end{aligned}$$

$$||x - y||_\infty = \sup_{t \in J} ||x(t) - y(t)|| \geq ||x_0 - x_1||, \text{ for all } (x, y) \in C(J, V_1) \times C(J, V_2),$$

and so, $||C(J, V_1) - C(J, V_2)|| = ||x_0 - x_1||$. Let

$$T : C(J, V_1) \cup C(J, V_1) \to C(J, X)$$

be an operator defined as

$$\begin{aligned} Tx(t) =& x_1 + \int_{t_0}^{t} g(s, x(s))ds, \ x \in C(J, V_1) \\ Ty(t) =& x_0 + \int_{t_0}^{t} f(s, y(s))ds, \ y \in C(J, V_2). \end{aligned}$$

We say that $z \in C(J, V_1) \cup C(J, V_2)$ is an optimal solution for the system of differential equations given in (1), if $||z - Tz|| = ||A - B||$.

Now we are ready to state and prove the following theorem:

Theorem 4.1 *Under the assumptions of Definition 4.1 , we suppose that*

(i) $\chi f(I \times W_2) \cup g(I \times W_1) \leq \varsigma(\chi(W_1 \cup W_2))\chi(W_1 \cup W_2)$;

(ii) $||f(t, x) - g(t, y)|| \leq \frac{1}{h}(||x(t) - y(t)|| - ||x_1 - x_0||)$,

where $\varsigma = \frac{1}{2b}$ *and for any* $(W_1, W_2) \subseteq (V_1, V_2)$ *and* $h \leq \min\{a, b/M_1, b/M_2, 1/2b\}$*, where*

$$\begin{aligned} M_1 &= \sup\{f(t,x) : (t,x) \in I \times V_1\}, \\ M_2 &= \sup\{g(t,y) : (t,y) \in I \times V_2\}. \end{aligned}$$

Then system (1) has an optimum solution.

Proof: Notice that $(C(J,V_1), C(J,V_2))$ is a bounded, closed, and convex pair in $C(J,X)$ and that T is cyclic on $(C(J,V_1) \cup C(J,V_2))$. We assert that $T(C(J,V_1))$ is a bounded and equicontinuous subset of $C(J,V_2)$. Let $t, t_0 \in J$ and $x \in C(J,V_1)$. Then

$$\begin{aligned} ||Tx(t)|| = ||x_1 + \int_{t_0}^{t} g(s,x(s))ds|| &\leq ||x_1|| + \int_{t_0}^{t} ||g(s,x(s))||ds \\ &\leq ||x||_1 + M_2 h \leq ||x_1|| + b, \end{aligned}$$

and this leads to boundedness of $T(C(J,V_1))$. On the other hand,

$$\begin{aligned} ||Tx(t) - Tx(t')|| &= \left|\left| \int_{t_0}^{t} g(s,x(s))ds - \int_{t_0}^{t'} g(s,x(s))ds \right|\right| \\ &\leq \int_{t}^{t'} ||g(s,x(s))||ds \leq M_2|t - t'|, \end{aligned}$$

this imples, $T(C(J,V_1))$ is equicontinuous. Using similar lines we can prove that $T(C(J,V_2))$ is also bounded and equicontinuous. Therefore, by Arzela–Ascoli's theorem we conclude that the pair $(C(J,V_1), C(J,V_2))$ is relatively compact. Now we show that T is a ς cyclic operator. To this end, suppose that $(K_1, K_2) \subset (C(J,V_1), C(J,V_2))$ is a T invariant, closed, proximinal pair and convex and

$$||K_1 - K_2|| = ||C(J,V_1) - C(J,V_2)|| = ||x_0 - x_1||.$$

Using Theorem 2.11 of [1], we deduce that

$$\begin{aligned} \chi(T(K_1) \cup T(K_2)) &= \max\{\chi(T(K_1), \chi T(K_2)\} \\ &= \max\left\{\sup_{t\in J}\{\chi(\{Tx(t) : x \in K_1\})\}, \sup_{t\in J}\{\chi(\{Tx(t) : x \in K_2\})\}\right\} \\ &= \max\left\{\sup_{t\in J}\left\{\chi\left(\left\{x_1 + \int_{t_0}^{t} g(s,x(s))ds : x \in K_1\right\}\right)\right\}\right\}, \\ &\quad \sup_{t\in J}\left\{\chi\left(\left\{x_0 + \int_{t_0}^{t} f(s,x(s))ds : x \in K_2\right\}\right)\right\}. \end{aligned}$$

Now from Lemma 4.1 we obtain

$$\begin{aligned} x_1 + \int_{t_0}^{t} g(s,x(s))ds &\in x_1 + (t - t_0)\overline{con}\left(\{g(s,x(s)) : s \in [t_0,t]\}\right), \\ x_0 + \int_{t_0}^{t} f(s,x(s))ds &\in x_1 + (t - t_0)\overline{con}\left(\{f(s,x(s)) : s \in [t_0,t]\}\right), \end{aligned}$$

and thus,

$$
\begin{aligned}
&\leq \max\left\{\sup_{t\in J}\left\{\chi\Big(\ \{x_1+(t-t_0)\overline{con}\left(\{g(s,x(s)):s\in[t_0,t]\}\right)\}\Big\}\right\},\\
&\quad \sup_{t\in J}\left\{\chi\Big(\ \Big\{x_0+(t-t_0)\overline{con}\left(\{g(s,x(s)):s\in[t_0,t]\}\right)\Big)\Big\}\right\}\\
&\leq \sup_{0\leq\lambda\leq h}\left\{\chi\Big(\ \Big\{x_0+\lambda\overline{con}\left(\{g(J\times K_1)\}\right)\Big)\Big\}\right\},\\
&\quad \sup_{0\leq\lambda\leq h}\left\{\chi\Big(\ \Big\{x_0+\lambda\overline{con}\left(\{f(J\times K_2)\}\right)\Big)\Big\}\right\}\\
&=\max\left\{h\chi(g(J\times K_1)),h\alpha(f(J\times K_2))\right\}\\
&=h\chi\left\{(g(J\times K_1)),(f(J\times K_2))\right\}\\
&\leq\frac{1}{2b}\chi\left\{(g(J\times K_1)),(f(J\times K_2))\right\}\\
&=\varsigma(\chi(K_1\cup K_2))\chi(K_1\cup K_2).
\end{aligned}
$$

which imples that T is a ς cyclic condensing operator. Finally, we show that T is cyclic relatively nonexpansive. For any $(x,y)\in C(J,V_1)\times C(J,V_1)$, we have

$$
\begin{aligned}
||Tx(t)-Ty(t)|| &= \left|\left|\left(x_1+\int_{t_0}^{t}g(s,x(s))ds\right)-\left(x_0+\int_{t_0}^{t}f(s,y(s))ds\right)\right|\right|\\
&\leq||x_1-x_0||+\int_{t_0}^{t}||g(s,x(s))-f(s,y(s))||ds\\
&\leq||x_1-x_0||+\frac{1}{h}\int_{t_0}^{t}(||x(s)-y(s)||-||x_1-x_0||)ds\\
&\leq||x_1-x_0||+(||x(s)-y(s)||_\infty-||x_1-x_0||)=||x-y||_\infty.
\end{aligned}
$$

and thereby, $||Tx-Ty||_\infty\leq||x-y||_\infty$. Now the result follows from Theorem 2.1. □

5. Conclusion

In this paper, we establish a best proximity point theorem for cyclic operators in Banach spaces by using a measure of noncompactness. We also obtain a result for noncyclic operators in strictly convex Banach spaces and an addition of Darbo's fixed point theorem also concluded. We investigate the existence of a optimal solution for a system of ordinary differential equations as an application of our findings.

Conflict of Interest The authors declare that they have not any conflict of interest.

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