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L^2- Decay of Solutions of Three Dimensional Incompressible MHD Equations: A Modified Approach

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ABSTRACT: In this paper, we adopt a recent and modified approach to prove L^2-decay properties of solutions of incompressible MHD equations following the methods in H. Sohr [“*The Navier-Stokes Equations: An Elementary Functional Analytic Approach*”, Birkhauser Advanced Texts, Birkhauser Verlag, Basel, 2001.] and Duong and Khai [TNU Journal of Science and Technology, Vol. 225 (02), 45-51 (2020)] Also, we compare our results with previously known results in the literature and find that our results (decay rate) are sharper than previously proven results. Last Theorem is an additional property of solutions of MHD equations, namely that they approach solution of homogeneous system as $t \to \infty$

Key Words:: Incompressible Magnetohydrodynamic Equations, L^2- decay of solutions, Homogeneous Stokes System.

Contents

1. Introduction

Magnetohydrodynamics (MHD) deals with dynamics of conducting fluid moving in an electromagnetic field, especially when the current set in the matter by induction, modifies the field, and thus the field and dynamic equations are coupled to govern the moving conducting fluid. The equations describing the motion of a viscous incompressible conducting fluid moving in a magnetic field are described by coupling Navier-Stokes (N-S) equations with Maxwell's electromagnetic equations together with the expression for the Lorentz force. The domain in which the fluid is moving is either a bounded subset of $\mathbb{R}^3$ or the whole space $\mathbb{R}^3$.

Importance of proving qualitative properties of hydrodynamic and hydromagnetic flows like existence and uniqueness of solutions, their stability, regularity and large time behaviour, have been known to theoretical and applied mathematicians since last six decades or more. In spite of many monumental research papers published from time to time by pioneering workers, the global existence problem for three dimensional Navier-Stokes equations and MHD equations is still open. Nevertheless, many important qualitative properties mentioned above have been proved and improved over during last four decades by using methods from Non-linear Functional Analysis.

In particular, large time behaviour and L^2-Decay properties of solutions of Navier-Stokes equations and MHD equations have been proved by various authors (see for example [1,2,3,4,5,6]). S. D. Mohgaonkar and R. V. Saraykar [7] proved such L^2-Decay properties for MHD equations by using method of Fourier transform.

In this paper, we use modified and simple approach to prove L^2-decay properties for MHD equations in general domains. Our results are an improvement over previously known results.

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Thus, in section 2, we give preliminary notations and describe MHD equations with appropriate initial and boundary conditions. We also describe some basic results on the Laplacian operator and state certain inequalities which will be used in proving our main theorems. Herein, we state our main theorems also. In section 3, we state and prove some Lemmas. In section 4, we give proof of our main theorems on L^2-decay properties. We conclude the paper with some remarks about previously known results in various contexts, in section 5.

2. MHD equations, preliminary results and statement of main theorems

Let Ω be a non-empty connected open subset of $\mathbb{R}^3$, bounded or unbounded, with boundary $\partial\Omega$ and $[0,T)$ being time interval, $0<T\leq\infty$. The motion of the viscous incompressible conducting fluid moving in a domain Ω under the influence of magnetic field is described by the system of equations [7]:

$$\begin{cases} u_t + u.\nabla u - B.\nabla B - \vartheta\Delta u + \nabla P = 0 \\ B_t + u.\nabla B - B.\nabla u - \lambda\Delta B = 0 \\ \nabla.u = 0,\ \nabla.B = 0 \\ u(x,0) = u_0,\ B(x,0) = B_0 \\ \text{and } \ u|_{\partial\Omega} = 0,\ B|_{\partial\Omega} = 0 \end{cases} \tag{2.1}$$

where u is the velocity of conducting fluid and B is the magnetic field acting on the conducting moving fluid, $u_t = \frac{\partial u}{\partial t}$, $B_t = \frac{\partial B}{\partial t}$, ϑ is the kinematic coefficient of viscosity, λ is magnetic diffusivity and P is the external pressure. For mathematical convenience, we take $\vartheta = \lambda = 1$.

Now, we briefly describe the requisite function spaces, definitions of weak and strong solutions to (2.1) and introduce some notations required for describing our results. In this paper, sometimes we use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant C. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

The notation $\langle . \ , \ .\rangle_\Omega$ denotes pairing of functions, vector fields etc. on Ω and $\langle . \ , \ .\rangle_{\Omega,T}$ means the corresponding pairing on $[0,T)\times\Omega$.

For $1\leq q\leq\infty$, we use the well known Lebesgue and Sobolev spaces $L^q(\Omega), W^{k,p}(\Omega)$, with norms $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $\|\cdot\|_{W^{k,p}(\Omega)} = \|\cdot\|_{k,p}$. We also need the Bochner spaces $L^s(0,T;L^p(\Omega))$, $1\leq s,\ p\leq\infty$ with the norm

$$||\cdot||_{L^s(0,T;L^p(\Omega))} = \Big(\int_0^T ||\cdot||_p^s dt\Big)^{1/s} = ||\cdot||_{p,s,T}$$

Furthermore, let C_0^∞ denote the space of C^∞ functions with compact support in Ω, and let

$$C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega) \mid div\ u = 0\}$$

Then $L^2_\sigma(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}$ with respect to L^2- norm , $W_0^{1,2}(\Omega) = \overline{C_0^\infty(\Omega)}$ with respect to $W^{1,2}-$ norm and $W_{0,\sigma}^{1,2}(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}$ with respect to $W^{1,2}-$ norm.

$\mathbb{P}$ denotes Helmholtz projection function, $\mathbb{P}: L^2(\Omega)\to L^2_\sigma(\Omega)$. We define Stokes operator A as follows: $A = -\mathbb{P}\Delta : \mathbb{D}(A) \to L^2_\sigma(\Omega)$ with the domain of definition
$\mathbb{D}(A) = \Big\{u\in W_{0,\sigma}^{1,2}(\Omega), \exists\ f\in L^2_\sigma(\Omega) : \langle\nabla u, \nabla\varphi\rangle_\Omega = \langle f,\varphi\rangle_\Omega, \forall\varphi\in W_{0,\sigma}^{1,2}(\Omega)\Big\}$
is defined as $Au = -\mathbb{P}\Delta u = f$, where $u\in\mathbb{D}(A)$.

We define the fractional powers as in [1] $A^\alpha : \mathbb{D}(A^\alpha)\to L^2_\sigma(\Omega), -1\leq\alpha\leq 1$
and we note that $\mathbb{D}(A)\subset\mathbb{D}(A^\alpha)\subset L^2_\sigma(\Omega)$ for $\alpha\in(0,1]$. For any domain $\Omega\subset\mathbb{R}^3$, it is well known that the operator A is self –adjoint and generates a bounded analytic semi group $e^{-tA}, t\geq 0$ on $L^2_\sigma(\Omega)$.

In the theory of Navier-Stokes equations, the following embedding properties play a basic role:

$$\left\|A^{\frac{-\beta}{2}}\mathbb{P}u\right\|_2 \leq C\,\|u\|_q\,, u\in L^q_\sigma(\Omega) \text{ where } \frac{1}{2}\leq\beta<\frac{3}{2},\ \frac{1}{q} = \frac{1}{2}+\frac{\beta}{3}. \tag{2.2}$$

Furthermore, we shall use the Stokes semigroup estimates

$$\left\|A^\alpha e^{-tA}u\right\|_2 \leq t^{-\alpha}\,\|u\|_2 \text{ with } u\in L^2_\sigma(\Omega),\ 0\leq\alpha\leq 1. \tag{2.3}$$

Next, we define weak and strong solutions of MHD system (2.1): In what follows, we use standard norm for vectors in a Cartesian product of Banach spaces:
If X and Y are two Banach spaces with norms $||.||_X$ & $||.||_Y$ respectively, then for $(x, y) \in X \times Y$, we define $||(x, y)||$ by $||(x, y)||^2 = ||x||_X^2 + ||y||_Y^2$.

Definition 2.1 Weak Solution
A vector field

$$(u, B) \in L^\infty(0, T, L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)) \cap L^2([0, T); W^{1,2}_{0,\sigma}(\Omega) \times W^{1,2}_{0,\sigma}) \tag{2.4}$$

is called a weak solution in the sense of Leary-Hopf of system (2.1) with initial value $u(0, x) = u_0, B(0, x) = B_0$ *if the relations*

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} + \langle u.\nabla u, w\rangle_{\Omega,T} - \langle B.\nabla B, w\rangle_{\Omega,T} = \langle u_0, w\rangle_\Omega \tag{2.5}$$

and

$$-\langle B, w_t\rangle_{\Omega,T} + \langle \nabla B, \nabla w\rangle_{\Omega,T} + \langle u.\nabla B, w\rangle_{\Omega,T} - \langle B.\nabla u, w\rangle_{\Omega,T} = \langle B_0, w\rangle_\Omega \tag{2.6}$$

are satisfied for all test functions $w \in C_0^\infty([0, T]; C^\infty_{0,\sigma}(\Omega))$, *and additionally the energy inequality*

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau + 2\int_0^t \|\nabla B(\tau)\|_2^2 \, d\tau \le \|u_0\|_2^2 + \|B_0\|_2^2 \tag{2.7}$$

is satisfied for all $t \in [0, T)$. *The energy inequality (2.7) can be written in product norms,*

$$\|(u(t), B(t))\|_2^2 + 2\int_0^t \|(\nabla u(\tau), \nabla B(\tau))\|_2^2 \, d\tau \le \|(u(t_0), B(t_0))\|_2^2 \tag{2.8}$$

Definition 2.2 Strong Solution
A weak solution (u, B) *is called a strong solution of MHD equations (2.1) if additionally local Serrin's condition*

$$(u, B) \in L^s_{loc}([0, T); L^q(\Omega) \times L^q(\Omega)) \tag{2.9}$$

is satisfied with $2 < s < \infty, \; 3 < q < \infty$ *where* $\frac{2}{s} + \frac{3}{q} \le 1$.

If the domain Ω is bounded, by routine methods, we can prove the existence of weak solution (u, B) as in definition (2.1) which additionally satisfies the strong energy inequality, given as

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 + 2\int_{t'}^t \|\nabla u(s)\|_2^2 \, ds + 2\int_{t'}^t \|\nabla B(s)\|_2^2 \, ds \le \|u(t')\|_2^2 + \|B(t')\|_2^2 \tag{2.10}$$

for almost all $t' \in [0, T)$ and all $t \in [t', T)$ [1]. The energy inequality (2.10) can be written in product norms,

$$\|(u(t), B(t))\|_2^2 + 2\int_{t'}^t \|(\nabla u(s), \nabla B(s))\|_2^2 \, ds \le \|(u(t'), B(t'))\|_2^2 \tag{2.11}$$

For results in the case of unbounded domain for Navier-Stokes equations, we refer the reader to [8]. These results are easily extendible to MHD case and we omit the details.

In what follows, we shall need the following standard inequalities:
(1) **Holder inequality**: - if $f_1 \in L^p$ and $f_2 \in L^q$ with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\int |f_1.f_2| \le \|f_1\|_p \cdot \|f_2\|_q$
Variant of Holder inequality [8] :-

$$\text{If } p, q, r > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \text{ then } \|f_1.f_2\|_r \le \|f_1\|_p \cdot \|f_2\|_q \tag{2.12}$$

(2) **Gagliardo – Nirenberg Interpolation inequality**:-
We need a special case of this, called generalized Ladyzhensakaya's inequality which is stated as follows:-
Whenever $n = dim.\Omega = 2$ or $3, p > q \geq 1$,
$s > n\left(\frac{1}{2} - \frac{1}{p}\right)$ and $\frac{1}{p} = \frac{\alpha}{q} + (1-\alpha)\left(\frac{1}{2} - \frac{s}{n}\right)$, we have, $\|u\|_p \leq C.\|u\|_q^{\alpha}\|u\|_{H_0^s}^{1-\alpha}$
In what follows, we shall take $p = \frac{3}{\beta}, \alpha = \beta - \frac{1}{2}$, $\frac{1}{2} \leq \beta < \frac{3}{2}$, $q = 2, s = 1$ and $n = 3$. Then above conditions are satisfied and we get

$$\|u\|_{\frac{3}{\beta}} \leq C\|u\|_2^{\beta-\frac{1}{2}}\|\nabla u\|_2^{\frac{3}{2}-\beta} \tag{2.13}$$

since $\|u\|_{H_0^1} \leq C_1\|\nabla u\|_2$

We now state our main Theorems:

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a general domain, $(u_0, B_0) \in L_\sigma^2 \times L_\sigma^2$ and (u, B) is a weak solution of the MHD system (2.1) satisfying the strong energy inequality (2.11). Then*

- *a) If $\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 = O(t^{-\alpha})$ for some $0 \leq \alpha \leq \frac{3}{4}$,*
  then $\|(u(t), B(t))\|_2 = O(t^{-\alpha})$ as $t \to \infty$
- *b) If $\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 = o(t^{-\alpha})$ for some $0 \leq \alpha \leq \frac{3}{4}$,*
  then $\|(u(t), B(t))\|_2 = o(t^{-\alpha})$ as $t \to \infty$

Here O and o denote big order and small order respectively.

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^3$ be a general domain, $(u_0, B_0) \in L_\sigma^2(\Omega) \times L_\sigma^2(\Omega)$ and (u, B) is a weak solution of the MHD system (2.1) satisfying strong energy inequality (2.11).*
If $(u_0, B_0) \in (L^q(\Omega) \times L^q(\Omega)) \cap \left(L_\sigma^2(\Omega) \times L_\sigma^2(\Omega)\right)$, $1 < q < 2$ then

$$\|(u(t), B(t))\|_2 = o\left(t^{\frac{-1}{2}\left(\frac{1}{q} - \frac{1}{2}\right)}\right) \text{ as } t \to \infty$$

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^3$ be a general domain $(u_0, B_0) \in L_\sigma^2(\Omega) \times L_\sigma^2(\Omega)$ and (u, B) is a weak solution of MHD system (2.1) satisfying the strong energy inequality (2.11). If there exist positive constants t_0, C_1, C_2 such that*
$C_1 t^{-\alpha_1} \leq \left\|(e^{-tA}u_0, e^{-ta}B_0)\right\| \leq C_2 t^{-\alpha_2}$ for $t \geq t_0$
where α_1 and α_2 are constants satisfying $0 \leq \alpha_2 < \frac{1}{2}$ and $\alpha_2 \leq \alpha_1 < \alpha_2 + \frac{1}{4}$
Then

$$\lim_{t\to\infty} \frac{\left\|(u(t), B(t)) - (e^{-tA}u_0, e^{-tA}B_0)\right\|_2}{\|(u(t), B(t))\|_2} = 0$$

Proofs of these theorems will be given in Section 4.

3. Lemmas and their proofs

For the proofs of the theorems 2.1 – 2.3 ,we need the following Lemmas:

Lemma 3.1 *Let $\gamma, \theta \in \mathbb{R}$ and $t > 0$, then*

- *a) If $\theta < 1$ then $\int\limits_0^{\frac{t}{2}} (t-\tau)^{-\gamma}\tau^{-\theta}d\tau = K_1\ t^{1-\gamma-\theta}$*

  where $K_1 = \int\limits_0^{\frac{1}{2}} (1-\tau)^{-\gamma}\tau^{-\theta}d\tau < \infty$

- *b) If $\gamma < 1$ then $\int\limits_{\frac{t}{2}}^{t} (t-\tau)^{-\gamma}\tau^{-\theta}d\tau = K_2\ t^{1-\gamma-\theta}$*

  where $K_2 = \int\limits_{\frac{1}{2}}^{1} (1-\tau)^{-\gamma}\tau^{-\theta}d\tau < \infty$

For proof of this lemma, we refer the reader to the book by Sohr [1].

Lemma 3.2 *Let* $(u, B) \in L^2(\Omega) \times L^2(\Omega)$ *and* $(\nabla u, \nabla B) \in L^2(\Omega) \times L^2(\Omega)$.
Then we have the following inequalities:

- *a)* $\left\|e^{-tA}\mathbb{P}(u.\nabla u)\right\|_2 \lesssim C\, t^{\frac{-\beta}{2}} \|u\|_2^{\beta-\frac{1}{2}} \|\nabla u\|_2^{\frac{5}{2}-\beta}$
- *b)* $\left\|e^{-tA}\mathbb{P}(B.\nabla B)\right\|_2 \lesssim C\, t^{\frac{-\beta}{2}} \|B\|_2^{\beta-\frac{1}{2}} \|\nabla B\|_2^{\frac{5}{2}-\beta}$
- *c)* $\left\|e^{-tA}\mathbb{P}(B.\nabla u)\right\|_2 \lesssim C\, t^{\frac{-\beta}{2}} \|B\|_2^{\beta-\frac{1}{2}} \|\nabla B\|_2^{\frac{3}{2}-\beta} \|\nabla u\|_2$
- *d)* $\left\|e^{-tA}\mathbb{P}(u.\nabla B)\right\|_2 \lesssim C\, t^{\frac{-\beta}{2}} \|u\|_2^{\beta-\frac{1}{2}} \|\nabla u\|_2^{\frac{3}{2}-\beta} \|\nabla B\|_2$
  where β *is a positive constant such that* $\frac{1}{2} \leq \beta < \frac{3}{2}$

Proof: a):
We first write $\left\|e^{-tA}\, \mathbb{P}(u, \nabla u)\right\|_2 = \left\|A^{\frac{\beta}{2}} e^{-tA} A^{\frac{-\beta}{2}} \mathbb{P}(u.\nabla u)\right\|_2$
Then, applying inequalities (2.2) and (2.3), we get

$$\left\|e^{-tA}\mathbb{P}(u.\nabla u)\right\|_2 \leq t^{\frac{-\beta}{2}} \left\|A^{\frac{-\beta}{2}}\mathbb{P}(u.\nabla u)\right\|_2 \leq C.t^{\frac{-\beta}{2}} \|(u.\nabla u)\|_q. \tag{3.1}$$

We now apply variant of Holder inequality with $\frac{1}{q} = \frac{1}{2} + \frac{\beta}{3}$, therefore inequality (3.1) takes the form,

$$\left\|e^{-tA}\mathbb{P}(u.\nabla u)\right\|_2 \lesssim C.t^{\frac{-\beta}{2}} \|u\|_{\frac{3}{\beta}} \|\nabla u\|_2. \tag{3.2}$$

Next, we apply Interpolation inequality (2.13) to $\|u\|_{\frac{3}{\beta}}$,
then, inequality (3.2) becomes

$$\left\|e^{-tA}\mathbb{P}(u.\nabla u)\right\|_2 \lesssim C.t^{\frac{-\beta}{2}} \|u\|_2^{\beta-\frac{1}{2}} \|\nabla u\|_2^{\frac{3}{2}-\beta} \|\nabla u\|_2$$
$$\lesssim C.t^{\frac{-\beta}{2}} \|u\|_2^{\beta-\frac{1}{2}} \|\nabla u\|_2^{\frac{5}{2}-\beta}$$

Thus we have the required result:

$$\left\|e^{-tA}\mathbb{P}(u.\nabla u)\right\|_2 \lesssim C.t^{\frac{-\beta}{2}} \|u\|_2^{\beta-\frac{1}{2}} \|\nabla u\|_2^{\frac{5}{2}-\beta}.$$

This proof appears in reference [2]. We have elaborated it here, to prove remaining inequalities of the Lemma (3.2)
b):
Using proof of (a), inequality (b) can be proved similarly for B. Thus, we get
$\left\|e^{-tA}\mathbb{P}(B.\nabla B)\right\|_2 \lesssim C.t^{\frac{-\beta}{2}} \|B\|_2^{\beta-\frac{1}{2}} \|\nabla B\|_2^{\frac{5}{2}-\beta}$
c):
To prove inequality (c), we proceed as follows:-
$\left\|e^{-tA}\mathbb{P}(B.\nabla u)\right\|_2 = \left\|A^{\frac{\beta}{2}} e^{-tA} A^{\frac{-\beta}{2}} \mathbb{P}(B.\nabla u)\right\|_2$
Applying inequality (2.3) first, and then (2.2), above inequality becomes

$$\begin{aligned}\left\|e^{-tA}\mathbb{P}(B.\nabla u)\right\|_2 &\leq t^{\frac{-\beta}{2}} \left\|A^{\frac{-\beta}{2}}\mathbb{P}(B.\nabla u)\right\|_2 \\ &\leq C.t^{\frac{-\beta}{2}} \|B.\nabla u\|_q\end{aligned}$$

Further using variant of Holder inequality (2.12), we have

$$\begin{aligned}\left\|e^{-tA}\mathbb{P}(B.\nabla u)\right\|_2 &\lesssim C.t^{\frac{-\beta}{2}} \|B\|_{\frac{3}{\beta}} \|\nabla u\|_2 \\ &\lesssim C.t^{\frac{-\beta}{2}} \|B\|_2^{\beta-\frac{1}{2}} \|\nabla B\|_2^{\frac{3}{2}-\beta} .||\nabla u||_2\end{aligned}$$

by interpolation inequality (2.13). Thus

$$\left\|e^{-tA}\mathbb{P}(B.\nabla u)\right\|_2 \lesssim C.t^{\frac{-\beta}{2}}\left\|B\right\|_2^{\beta-\frac{1}{2}}\left\|\nabla B\right\|_2^{\frac{3}{2}-\beta}\left\|\nabla u\right\|_2$$

d):
On the lines of proof (c), we obtain the proof of (d),

$$\left\|e^{-tA}\mathbb{P}(u.\nabla B)\right\|_2 \lesssim C.t^{\frac{-\beta}{2}}\left\|u\right\|_2^{\beta-\frac{1}{2}}\left\|\nabla u\right\|_2^{\frac{5}{2}-\beta}\left\|\nabla B\right\|_2$$

Thus, Lemma (3.2) is proved. □

Lemma 3.3 *There exist a positive constant* δ *such that if* $(u_0, B_0) \in \mathbb{D}(A^{\frac{1}{4}}) \times \mathbb{D}(A^{\frac{1}{4}})$ *and* $\left\|(A^{\frac{1}{4}}u_0, A^{\frac{1}{4}}B_0)\right\|_2 \leq \delta$ *then the MHD system (2.1) has a strong solution with the initial value* (u_0, B_0) *satisfying* $\|(\nabla u, \nabla B)t\|_2^2 \leq t^{\frac{-1}{2}}$ *for all* $t \geq 0$.

Proof: This Lemma follows from extending Theorem 2 of Kozono and Ogawa [9] to MHD case with value $\mu = 0$ and $\alpha = \frac{1}{2}$ there. We recall that $\left\|A^{\frac{1}{2}}u\right\| \cong \|\nabla u\|$. We omit the details of the proof here. □

Lemma 3.4 *Let* (u, B) *be a weak solution of the MHD system (2.1) with initial value* $(u_0, B_0) \in L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)$. *Then there exist a positive value* t_0, *large enough, such that* $\|(\nabla u(t), \nabla B(t))\|_2^2 \leq t^{-\frac{1}{2}}$ *for all* $t \geq t_0$

Proof: We use spectral representation theorem for a positive self–adjoint operator A. Let the spectral representation of A be given by

$$A = \int_0^\infty \lambda \, dE_\lambda \tag{3.3}$$

with $\mathbb{D}(A) = \left\{u \in L^2_\sigma(\Omega) : \|Au\|_2^2 = \int_0^\infty \lambda^2 \, d\,\|E_\lambda u\|_2^2 < \infty\right\}$
Therefore, we have representations

$$A^{1/2} = \int_0^\infty \lambda^{1/2} \, dE_\lambda \tag{3.4}$$

$$A^{1/4} = \int_0^\infty \lambda^{1/4} \, dE_\lambda \tag{3.5}$$

as $A^{1/2}$, $A^{1/4}$ are also positive self-adjoint operators since A is so, with their domains
$\mathbb{D}(A^{1/2}) = \left\{u \in L^2_\sigma(\Omega) : \left\|A^{\frac{1}{2}}u\right\|_2^2 = \int_0^\infty \lambda \, d\,\|E_\lambda u\|_2^2 < \infty\right\}$ and
$\mathbb{D}(A^{1/4}) = \left\{u \in L^2_\sigma(\Omega) : \left\|A^{\frac{1}{4}}u\right\|_2^2 = \int_0^\infty \lambda^{1/2} \, d\,\|E_\lambda u\|_2^2 < \infty\right\}$

Taking norm of (3.5), we have

$$\left\|A^{\frac{1}{4}}u\right\|_2^2 = \int_0^\infty \lambda^{1/2} d\,\|E_\lambda u\|_2^2$$

and applying Holder inequality to

$$\int_0^\infty \left(\lambda^{1/2} \, d||E_\lambda u||_2\right).(d||E_\lambda u||_2)\,, \text{ further, we obtain}$$

$$\begin{aligned}
\left\| A^{\frac{1}{4}} u \right\|_2^2 &\le \left(\int_0^\infty \lambda d \left\| E_\lambda u \right\|_2^2 \right)^{1/2} \left(\int_0^\infty d \left\| E_\lambda u \right\|_2^2 \right)^{1/2} \\
&\le \left(\left\| A^{\frac{1}{2}} u \right\|_2^2 \right)^{1/2} \left(\|u\|_2^2 \right)^{1/2} \\
\text{Thus, } \left\| A^{\frac{1}{4}} u \right\|_2^2 &\le \left\| A^{\frac{1}{2}} u \right\|_2 \|u\|_2 \qquad (3.6)
\end{aligned}$$

wherein E_λ are projection operators in the spectral representation, so that $||E_\lambda|| \le 1$. Replacing u by B, similarly we get,

$$\left\| A^{\frac{1}{4}} B \right\|_2^2 \le \left\| A^{\frac{1}{2}} B \right\|_2 \|B\|_2 \qquad (3.7)$$

Combining inequalities (3.6) and (3.7), we obtain

$$\begin{aligned}
\left\| A^{\frac{1}{4}} u \right\|_2^2 + \left\| A^{\frac{1}{4}} B \right\|_2^2 &\le \left\| A^{\frac{1}{2}} u \right\|_2 \|u\|_2 + \left\| A^{\frac{1}{2}} B \right\|_2 \|B\|_2 \\
&\le \left\| A^{\frac{1}{2}} u \right\|_2 \|(u,B)\|_2 + \left\| A^{\frac{1}{2}} B \right\|_2 \|(u,B)\|_2 \\
&\le \left(\left\| A^{\frac{1}{2}} u \right\|_2 + \left\| A^{\frac{1}{2}} B \right\|_2 \right) \|(u,B)\|_2 \qquad (3.8)
\end{aligned}$$

Further, using equivalent product norms, inequality (3.8) can be written as

$$||(A^{1/4}u, A^{1/4}B)||_2^2 \le ||(A^{1/2}u, A^{1/2}B)||_2 . ||(u,B)||_2 \qquad (3.9)$$

Now, we consider the weak solution of MHD system satisfying energy inequality, in product norms,

$$||(u(t), B(t))||_2^2 + 2\int_{t_0}^t ||(\nabla u(s), \nabla B(s))||_2^2 \, ds \le ||(u(t_0), B(t_0))||_2^2 \qquad (3.10)$$

for all $t_o \in [0,\infty) \setminus N$ where N is a null set.
Let δ be a positive constant in Lemma (3.3) then using (3.9) and (3.10), it follows that there exists large enough $t_0 \in [0,\infty) \setminus N$ such that $||(u(t_0), B(t_0))||_{D(A^{1/4})} \le \delta$. Thus, combining Lemma (3.3), inequality (3.10) and Serrin's uniqueness of solutions, we get, finally,

$$\|\nabla u(t), \nabla B(t)\|_2^2 \le t^{-\frac{1}{2}} \text{ for all } t \ge t_0 \qquad (3.11)$$

The proof of Lemma (3.4) is thus completed. □

Lemma 3.5 *Let* $(u_0, B_0) \in L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)$ *then*

a) $\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 \to 0$ *as* $t \to \infty$

b) If $(u_0, B_0) \in (L^q(\Omega) \cap L^2_\sigma(\Omega)) \times (L^q(\Omega) \cap L^2_\sigma(\Omega))$*, for some* $1 < q \le 2$ *then*
$\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 = o\left(t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\right)$ *as* $t \to \infty$

Proof: (a) Let $(u_0, B_0) \in L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)$. The function

$$(u,B) : [0,\infty) \to L^2_\sigma(\Omega) \times L^2_\sigma(\Omega) \text{ is defined by } (u,B)(t) = (S(t)u_0, S(t)B_0) \text{ for } t \ge 0 \qquad (3.12)$$

where $S(t)$, a semi-group operator is given by $S(t) = e^{-tA}$. The Lemma 1.5.1 (page 204) from Sohr's book [1], can easily be extended to prove
$\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 \to 0$ as $t \to \infty$. We omit the proof.

(b) Let $(u_0, B_0) \in (L^q(\Omega) \cap L^2_\sigma(\Omega)) \times (L^q(\Omega) \cap L^2_\sigma(\Omega))$, for some $1 < q \le 2$, so, $u_0 \in L^2_\sigma \cap L^q(\Omega)$ then by Lemma (2.5 b) of Duong and Khai [2], we have

$$\left\|e^{-tA}u_0\right\|_2 \le t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\|e^{-\frac{tA}{2}} A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} u_0\right\|_2 \tag{3.13}$$

Similarly for $B_0 \in L^2_\sigma \cap L^q(\Omega)$, we can write

$$\left\|e^{-tA}B_0\right\|_2 \le t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\|e^{-\frac{tA}{2}} A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} B_0\right\|_2 \tag{3.14}$$

Combining inequalities (3.13) and (3.14), we obtain

$$\begin{aligned}
&\left\|e^{-tA}u_0\right\|_2 + \left\|e^{-tA}B_0\right\|_2 \\
&\le t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\|e^{-\frac{tA}{2}} A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} u_0\right\|_2 + t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\|e^{-\frac{tA}{2}} A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} B_0\right\|_2 \\
&\le t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\left\|e^{-\frac{tA}{2}} A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} u_0\right\|_2 + \left\|e^{-\frac{tA}{2}} A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} B_0\right\|_2\right)
\end{aligned} \tag{3.15}$$

Using inequality (2.2),we have $A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}u_0 \in L^2_\sigma(\Omega)$ and also $A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}B_0 \in L^2_\sigma(\Omega)$ where $\frac{1}{2} \le \beta < \frac{3}{2}, \frac{1}{q} = \frac{1}{2} + \frac{\beta}{3}$
Using properties of product norm and part (a) of above Lemma on inequality (3.15), we conclude that $\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 = o\left(t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\right)$ as $t \to \infty$

□

4. Proofs of the Theorems (2.1) − (2.3)

Theorem (2.1)

Let $\Omega \subset \mathbb{R}^3$ be a general domain, $(u_0, B_0) \in L^2_\sigma \times L^2_\sigma$ and (u, B) is a weak solution of the MHD system (2.1) satisfying the strong energy inequality (2.11). Then

a) If $\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 = O(t^{-\alpha})$ for some $0 \le \alpha \le \frac{3}{4}$,
then $\|(u(t), B(t))\|_2 = O(t^{-\alpha})$ as $t \to \infty$

b) If $\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 = o(t^{-\alpha})$ for some $0 \le \alpha \le \frac{3}{4}$,
then $\|(u(t), B(t))\|_2 = o(t^{-\alpha})$ as $t \to \infty$

Proof: a) Let us construct a weak solution of the following integral equations, in expanded form

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathbb{P}(B.\nabla B)ds - \int_0^t e^{-(t-s)A}\mathbb{P}(u.\nabla u)ds \tag{4.1}$$

$$B(t) = e^{-tA}B_0 + \int_0^t e^{-(t-s)A}\mathbb{P}(B.\nabla u)ds - \int_0^t e^{-(t-s)A}\mathbb{P}(u.\nabla B)ds \tag{4.2}$$

Now, using lemma (3.5), we obtain

$$\begin{aligned} \|u(t)\|_2 &\lesssim \left\|e^{-tA}u_0\right\|_2 + \int\limits_0^t (t-s)^{-\beta/2} \|B(s)\|_2^{\beta-\frac{1}{2}} \|\nabla B(s)\|_2^{\frac{5}{2}-\beta} \, ds \\ &+ \int\limits_0^t (t-s)^{-\beta/2} \|u(s)\|_2^{\beta-\frac{1}{2}} \|\nabla u(s)\|_2^{\frac{5}{2}-\beta} \, ds \qquad (4.3) \\ \|B(t)\|_2 &\lesssim \left\|e^{-tA}B_0\right\|_2 + \int\limits_0^t (t-s)^{-\beta/2} \|B(s)\|_2^{\beta-\frac{1}{2}} \|\nabla B(s)\|_2^{\frac{3}{2}-\beta} \|\nabla u(s)\|_2 \, ds \\ &+ \int\limits_0^t (t-s)^{-\beta/2} \|u(s)\|_2^{\beta-\frac{1}{2}} \|\nabla u(s)\|_2^{\frac{3}{2}-\beta} \|\nabla B(s)\|_2 \, ds \qquad (4.4) \end{aligned}$$

We have $\|(u,B)\|_2 \leq \|u\|_2 + \|B\|_2$, and using (4.3) and (4.4), it becomes

$$\begin{aligned} \|(u,B)\|_2 &\lesssim \left\|e^{-tA}u_0\right\|_2 + \left\|e^{-tA}B_0\right\|_2 \\ &+ \int\limits_0^t (t-s)^{-\frac{\beta}{2}} \Big\{ \|B(s)\|_2^{\beta-\frac{1}{2}} \|\nabla B(s)\|_2^{\frac{5}{2}-\beta} \\ &+ \|u(s)\|_2^{\beta-\frac{1}{2}} \|\nabla u(s)\|_2^{\frac{5}{2}-\beta} + \|B(s)\|_2^{\beta-\frac{1}{2}} \|\nabla B(s)\|_2^{\frac{3}{2}-\beta} \|\nabla u(s)\|_2 \\ &+ \|u(s)\|_2^{\beta-\frac{1}{2}} \|\nabla u(s)\|_2^{\frac{3}{2}-\beta} \|\nabla B(s)\|_2 \Big\} ds \qquad (4.5) \end{aligned}$$

Further, using $\|u\|_2 + \|B\|_2 \leq C \|(u,B)\|_2$, above inequality takes the form

$$\|(u,B)\|_2 \lesssim C_1 \left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 + \int\limits_0^t I_d \, ds \qquad (4.6)$$

where integrand of the above integral,

$$\begin{aligned} I_d &= (t-s)^{-\frac{\beta}{2}} \Big\{ \|B(s)\|_2^{\beta-\frac{1}{2}} \|\nabla B(s)\|_2^{\frac{5}{2}-\beta} \\ &+ \|u(s)\|_2^{\beta-\frac{1}{2}} \|\nabla u(s)\|_2^{\frac{5}{2}-\beta} + \|B(s)\|_2^{\beta-\frac{1}{2}} \|\nabla B(s)\|_2^{\frac{3}{2}-\beta} \|\nabla u(s)\|_2 \\ &+ \|u(s)\|_2^{\beta-\frac{1}{2}} \|\nabla u(s)\|_2^{\frac{3}{2}-\beta} \|\nabla B(s)\|_2 \Big\} \qquad (4.7) \end{aligned}$$

Now, for all $\frac{1}{2} \leq \beta < \frac{3}{2}$, we divide the integral of equation (4.6) into two different parts,

$$I = \int_0^{t/2} I_d \, ds + \int_{t/2}^t I_d \, ds \equiv I_1 + I_2 \qquad (4.8)$$

Also, we have properties $\|u\| \leq \|(u,B)\|$ and $\|B\| \leq \|(u,B)\|$, so integrals I_1 and I_2 becomes

$$\begin{aligned} I_1 \quad & \leq \int_0^{\frac{t}{2}} (t-s)^{\frac{-\beta}{2}} \Big\{ \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{5}{2}-\beta} + \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{5}{2}-\beta} \\ & + \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{3}{2}-\beta} \, \|(\nabla u, \nabla B(s))\|_2 \\ & + \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{3}{2}-\beta} \, \|(\nabla u, \nabla B(s))\|_2 \Big\} ds \end{aligned} \tag{4.9}$$

$$\begin{aligned} I_2 \quad & \leq \int_{\frac{t}{2}}^{t} (t-s)^{\frac{-\beta}{2}} \Big\{ \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{5}{2}-\beta} + \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{5}{2}-\beta} \\ & + \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{3}{2}-\beta} \, \|(\nabla u, \nabla B(s))\|_2 \\ & + \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{3}{2}-\beta} \, \|(\nabla u, \nabla B(s))\|_2 \Big\} ds \end{aligned} \tag{4.10}$$

On simplifying, we have

$$I_1 \quad \leq \int_0^{\frac{t}{2}} (t-s)^{\frac{-\beta}{2}} \Big\{ 4 \ \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{5}{2}-\beta} \Big\} ds \ \equiv I_3 \tag{4.11}$$

$$I_2 \quad \leq \int_{\frac{t}{2}}^{t} (t-s)^{\frac{-\beta}{2}} \Big\{ 4 \ \|(u,B)\|_2^{\beta-\frac{1}{2}} \, \|(\nabla u, \nabla B(s))\|_2^{\frac{5}{2}-\beta} \Big\} ds \ \equiv I_4 \tag{4.12}$$

and $I = I_1 + I_2 \leq I_3 + I_4$.

Now, we consider three cases: $i) 0 \leq \alpha \leq \frac{1}{4}$, $ii) \frac{1}{4} \leq \alpha < \frac{1}{2}$, $iii) \frac{1}{2} \leq \alpha < \frac{3}{4}$

Case $i)$ **:-** $0 \leq \alpha \leq \frac{1}{4}$

The integral I_3 using weak energy condition and Holder's inequality can be written as

$$\begin{aligned} I_3 \quad & \leq 4 \, \|(u_0, B_0)\|_2^{\beta-\frac{1}{2}} \, t^{\frac{-\beta}{2}} \int_0^{\frac{t}{2}} \|(\nabla u, \nabla B)\|_2^{\frac{5}{2}-\beta} \, ds \\ & \leq 4 \, \|(u_0, B_0)\|_2^{\beta-\frac{1}{2}} \, t^{\frac{-\beta}{2}} \Big(\int_0^{\frac{t}{2}} ds \Big)^{\frac{2\beta-1}{4}} \ \Big(\int_0^{\frac{t}{2}} \|(\nabla u, \nabla B)\|_2^2 \, ds \Big)^{\frac{5-2\beta}{4}} \\ & \leq 4 \, \|(u_0, B_0)\|_2^{\beta-\frac{1}{2}} \, t^{\frac{-\beta}{2}} \, ([s]_0^{\frac{t}{2}})^{\frac{2\beta-1}{4}} \ \|(u_0, B_0)\|_2^{\frac{5-2\beta}{4}} \\ & \leq 4 \, \|(u_0, B_0)\|_2^{\beta-\frac{1}{2}} \, t^{\frac{-\beta}{2}} \, (\frac{t}{2})^{\frac{2\beta-1}{4}} \ \|(u_0, B_0)\|_2^{\frac{5-2\beta}{4}} \\ & \leq 4 \, \|(u_0, B_0)\|_2^{\beta-\frac{1}{2}} \, t^{\frac{-1}{4}} \ 2^{\frac{1-2\beta}{4}} \ \|(u_0, B_0)\|_2^{\frac{5-2\beta}{4}} \\ \text{Thus, we get } \ I_3 \quad & \leq O\left(t^{\frac{-1}{4}}\right) \end{aligned} \tag{4.13}$$

and by applying Lemma (3.1b) and strong energy condition (2.11), we have

$$\begin{aligned} I_4 \quad & \leq 4 \, \|(u_0, B_0)\|_2^{\beta-\frac{1}{2}} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{\beta}{2}} (s^{-1/2})^{\frac{5}{2}-\beta} ds \\ & \leq 4 \, \|(u_0, B_0)\|_2^{\beta-\frac{1}{2}} \, t^{1-\frac{\beta}{2}-\frac{1}{2}(\frac{5}{2}-\beta)} \text{ by Lemma (3.1(b))} \\ & \leq 4 \, \|(u_0, B_0)\|_2^{\beta-\frac{1}{2}} \, t^{1-\frac{\beta}{2}-\frac{5}{4}+\frac{\beta}{2}}. \\ & \leq 4 \, \|(u_0, B_0)\|_2^{\beta-\frac{1}{2}} \, t^{-\frac{1}{4}}. \\ \text{Thus, } I_4 \quad & \leq O\left(t^{\frac{-1}{4}}\right) \ \text{ for } t \geq 2 \, t_0 \end{aligned} \tag{4.14}$$

and $I \leq I_3 + I_4 \leq O\left(t^{\frac{-1}{4}}\right)$. From equation (4.6) and given hypothesis, we can write $\|(u(t),B(t))\|_2 \lesssim \left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 + I \leq O(t^{-\alpha}) + O\left(t^{\frac{-1}{4}}\right)$ by using Lemmas (3.5 - 3.4). Therefore, $\|(u(t),B(t))\|_2 = O(t^{-\alpha})$ as $t \to \infty$

Case ii):- $\frac{1}{4} \leq \alpha < \frac{1}{2}$

Applying result of **Case (i)** for $\alpha = \frac{1}{4}$ in I_3

$$\begin{aligned}
I_3 &\leq 4t^{\frac{-\beta}{2}} \int_0^{\frac{t}{2}} \|(u,B)\|_2^{\beta-\frac{1}{2}} \|(\nabla u, \nabla B)\|_2^{\frac{5}{2}-\beta} \, ds \\
&\leq 4t^{\frac{-\beta}{2}} \int_0^{\frac{t}{2}} (s^{\frac{-1}{4}})^{\beta-\frac{1}{2}} \|(\nabla u, \nabla B)\|_2^{\frac{5}{2}-\beta} \, ds \\
&\leq 4t^{\frac{-\beta}{2}} \Big(\int_0^{\frac{t}{2}} s^{\frac{-1}{2}} \, ds\Big)^{\frac{2\beta-1}{4}} \Big(\int_0^{\frac{t}{2}} \|(\nabla u, \nabla B)\|_2^2 \, ds\Big)^{\frac{5-2\beta}{4}} \text{ by Holder inequality} \\
&\leq 4t^{\frac{-\beta}{2}} (t^{\frac{1}{2}})^{\frac{2\beta-1}{4}} ||(u_0, B_0)||_2^{\frac{5-2\beta}{4}} \\
\text{Thus, } I_3 &\leq O(t^{\frac{-\beta}{4}-\frac{1}{8}}) \qquad (4.15)
\end{aligned}$$

Using Lemmas (3.4) and (3.1b), we have

$$\begin{aligned}
I_4 &\leq 4 \int_{\frac{t}{2}}^{t} (t-s)^{\frac{-\beta}{2}} \left(s^{\frac{-1}{4}}\right)^{\beta-\frac{1}{2}} (s^{-\frac{1}{2}})^{\frac{5}{2}-\beta} ds \\
&\leq 4 \int_{\frac{t}{2}}^{t} (t-s)^{\frac{-\beta}{2}} s^{\frac{1}{8}-\frac{\beta}{4}} s^{\frac{\beta}{2}-\frac{5}{4}} ds \\
&\leq 4 \int_{\frac{t}{2}}^{t} (t-s)^{\frac{-\beta}{2}} s^{\frac{-9}{8}+\frac{\beta}{4}} \\
&\leq 4\, O(t^{1-\frac{\beta}{2}-\frac{9}{8}+\frac{\beta}{4}}) \\
I_4 &\leq O\left(t^{-\frac{\beta}{4}-\frac{1}{8}}\right) \text{ for } t \geq 2t_0 \qquad (4.16)
\end{aligned}$$

So we have

$$\begin{aligned}
\|(u(t),B(t))\|_2 &\lesssim \left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 + I \\
&\leq O(t^{-\alpha}) + O\left(t^{\frac{-\beta}{4}-\frac{1}{8}}\right) \text{ for } t \geq 2t_0 \qquad (4.17)
\end{aligned}$$

Using properties of real numbers, (Archimedean property), it is not difficult to show that there exist a number β such that $\frac{\beta}{4} + \frac{1}{8} \geq \alpha$ and $\frac{1}{2} \leq \beta < \frac{3}{2}$. So, we select one of such β, then from (4.17) it follows that

$$\|(u(t),B(t))\|_2 = O(t^{-\alpha}) \text{ as } t \to \infty$$

Case iii):- $\frac{1}{2} \leq \alpha < \frac{3}{4}$

Applying **Case (ii)** of part (a), we have

$$\|(u(t),B(t))\|_2 \leq t^{-\gamma} \text{ for } t \geq 0 \qquad (4.18)$$

where γ is a constant such that $0 \le \gamma < \frac{1}{2}$.
Applying inequality (4.18) on I_3, we obtain

$$\begin{aligned}
I_3 \quad & \le 4t^{\frac{-\beta}{2}} \int_0^{\frac{t}{2}} (s^{-\gamma})^{\beta-\frac{1}{2}} \left\|(\nabla u, \nabla B)\right\|_2^{\frac{5}{2}-\beta} ds \\
& \le 4t^{\frac{-\beta}{2}} \left(\int_0^{\frac{t}{2}} s^{-2\gamma} ds \right)^{\frac{2\beta-1}{4}} \left(\int_0^{\frac{t}{2}} \left\|(\nabla u, \nabla B)\right\|_2^2 ds \right)^{\frac{5-2\beta}{4}} \quad \text{by Holder inequality} \\
& \le 4t^{\frac{-\beta}{2}} \left(\left[\frac{s}{-2\gamma+1}^{-2\gamma+1} \right]_0^{\frac{t}{2}} \right)^{\frac{2\beta-1}{4}} \left(\int_0^{\frac{t}{2}} \left\|(\nabla u, \nabla B)\right\|_2^2 ds \right)^{\frac{5-2\beta}{4}} \\
& \le 4t^{\frac{-\beta}{2}} (t^{-2\gamma+1})^{\frac{2\beta-1}{4}} \\
& \le 4t^{[\frac{\gamma}{2}-\gamma\beta-\frac{1}{4}]} \\
I_3 \quad & \le O\left(t^{\frac{\gamma}{2}-\gamma\beta-\frac{1}{4}} \right) \qquad (4.19)
\end{aligned}$$

On applying Lemmas (3.5b) and (3.4) on I_4, we have

$$\begin{aligned}
I_4 \quad & \le \int_{\frac{t}{2}}^{t} (t-s)^{\frac{-\beta}{2}} (s^{-\gamma})^{\beta-\frac{1}{2}} (s^{-\frac{1}{2}})^{\frac{5}{2}-\beta} ds \\
& \le t^{1-\frac{\beta}{2}-\gamma\left(\beta-\frac{1}{2}\right)-\frac{1}{2}\left(\frac{5}{2}-\beta\right)} \int_{\frac{t}{2}}^{t} (1-s)^{\frac{-\beta}{2}} s^{-\gamma\left(\beta-\frac{1}{2}\right)} ds \\
& \le t^{\frac{\gamma}{2}-\gamma\beta-\frac{1}{4}} \int_{\frac{t}{2}}^{t} (1-s)^{\frac{-\beta}{2}} s^{-\gamma\left(\beta-\frac{1}{2}\right)} ds \\
I_4 \quad & \le O\left(t^{\frac{\gamma}{2}-\gamma\beta-\frac{1}{4}} \right) \text{ for } t \ge 2t_0 \qquad (4.20)
\end{aligned}$$

Further, equation (4.6) takes the form

$$\begin{aligned}
\left\|(u(t), B(t))\right\|_2 \quad & \le \left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 + I \\
& \le O(t^{-\alpha}) + O\left(t^{\frac{\gamma}{2}-\gamma\beta-\frac{1}{4}} \right) \text{ for } t \ge 2t_0 \qquad (4.21)
\end{aligned}$$

Now, it is easy to show that there exist, γ and β such that
$\frac{\gamma}{2} - \gamma\beta - \frac{1}{4} \le -\alpha, \ \frac{1}{2} \le \beta < \frac{3}{2}, \quad 0 \le \gamma < \frac{1}{2}$.
Choosing one of such γ and β we conclude that

$$\left\|(u(t), B(t))\right\|_2 = O(t^{-\alpha}) \text{ as } t \to \infty.$$

b) Proof of (b) part follows similarly as above □

Corollary (2.1)

Let $\Omega \subset \mathbb{R}^3$ be a general domain. Given (u_0, B_0) and (u, B) as in theorem (2.1), if $\left\|(u(t), B(t))\right\|_2 = o(t^{-\gamma})$ for some $\gamma \in [0, \frac{1}{2})$ then

$$\left\|(u(t), B(t)) - (e^{-tA}u_0, e^{-tA}B_0)\right\|_2 = o(t^{-(\gamma+\theta)}) \text{ for all } \ \theta \in [0, \frac{1}{4}).$$

Proof: The proof is directly derived from proof of **Case (iii)** in Theorem (2.1). □

Theorem (2.2)

Let $\Omega \subset \mathbb{R}^3$ be a general domain, $(u_0, B_0) \in L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)$ and (u, B) is a weak solution of the MHD system (2.1) satisfying strong energy inequality (2.11).
If $(u_0, B_0) \in (L^q(\Omega) \times L^q(\Omega)) \cap \left(L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)\right)$, $1 < q < 2$ then

$$\|(u(t), B(t))\|_2 = o\left(t^{\frac{-1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\right) \text{ as } t \to \infty$$

Proof: By Theorem (2.1(b)), if $(u_0, B_0) \in L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)$ and (u, B) is a weak solution of MHD system (2.1) satisfying strong energy inequality (2.11), then
if $\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 = o(t^{-\alpha})$ for some $0 \le \alpha \le \frac{3}{4}$,
then $\|(u(t), B(t))\|_2 = o(t^{-\alpha})$ as $t \to \infty$
Thus, under the hypothesis of Theorem (2.2), by Lemma (3.5), we have,
$\left\|(e^{-tA}u_0, e^{-tA}B_0)\right\|_2 = o(t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{2})})$ as $t \to \infty$,
since $1 < q < 2$, $1 > (1/q - 1/2) > 0$ and hence $\frac{1}{2}(\frac{1}{q} - \frac{1}{2}) > 0$ and q can be assumed to be $< 3/4$. Hence, by Theorem (2.1(b)), we get
$\|(u(t), B(t))\|_2 = o\left(t^{\frac{-1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\right)$ as $t \to \infty$ as desired. □

Theorem (2.3)

Let $\Omega \subset \mathbb{R}^3$ be a general domain, $(u_0, B_0) \in L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)$ and (u, B) is a weak solution of MHD equations satisfying the energy inequality (2.11). If there exist positive constants t_0, C_1, C_2 such that $C_1 t^{-\alpha_1} \le \left\|e^{-tA}(u_0, B_0)\right\|_2 \le C_2 t^{-\alpha_2}$ for $t \ge t_0$ where α_1 and α_2 are constants satisfying $0 \le \alpha_2 < \frac{1}{2}$ and $\alpha_2 \le \alpha_1 < \alpha_2 + \frac{1}{4}$ then

$$\lim_{t \to \infty} \frac{\left\|(u(t), B(t)) - (e^{-tA}u_0, e^{-tA}B_0)\right\|_2}{\|(u(t), B(t))\|_2} = 0$$

Proof: Applying **Corollary 2.1** for $\gamma = \alpha_2$, $\theta = \frac{\alpha_1 - \alpha_2}{2} + \frac{1}{8}$
There exist a positive constant M_1 such that

$$\begin{aligned} \left\|(u(t), B(t)) - (e^{-tA}u_0, e^{-tA}B_0)\right\|_2 &\le M_1\, t^{-\left(\alpha_2 + \frac{\alpha_1 - \alpha_2}{2} + \frac{1}{8}\right)} \\ &\le M_1\, t^{-\left(\frac{\alpha_1 + \alpha_2}{2} + \frac{1}{8}\right)} \text{ for } t \ge 2t_0 \end{aligned} \tag{4.22}$$

It follows from above inequality that

$$\begin{aligned} \|(u, B)\|_2 &= ||(e^{-tA}u_0, e^{-tA}B_o) - (e^{-tA}u_0, e^{-tA}B_o) + (u(t), B(t))||_2 \\ &\ge ||(e^{-tA}u_0, e^{-tA}B_o||_2 - ||(u(t), B(t)) - (e^{-tA}u_0, e^{-tA}B_o)||_2 \\ &\ge C_1 t^{-\alpha_1} - M_1 t^{-\left(\frac{\alpha_1+\alpha_2}{2}+\frac{1}{8}\right)} \text{ using hypothesis and above inequality (4.22)} \\ &\ge \left(C_1 - M_1 t^{-\left(\frac{\alpha_2-\alpha_1}{2}+\frac{1}{8}\right)}\right) t^{-\alpha_1} \\ &\ge \left(C_1 - \frac{M_1}{t^{\left(\frac{\alpha_2-\alpha_1}{2}+\frac{1}{8}\right)}}\right) t^{-\alpha_1} \\ &\ge \left(C_1 - \frac{M_1}{\left[\left(\frac{2M_1}{C_1}\right)^{\frac{1}{\zeta}}\right]^{\zeta}}\right) t^{-\alpha_1},\ \zeta = \left(\frac{\alpha_2 - \alpha_1}{2} + \frac{1}{8}\right) \text{ with } t \ge t_1 \end{aligned}$$

where $t_1 = max\left\{t_0, \left(\frac{2M_1}{C_1}\right)^{\frac{1}{\zeta}}\right\}$.

It further takes the form,

$$\begin{aligned}\|(u,B)\|_2 \quad &\geq \left(C_1 - \frac{M_1}{\frac{2M_1}{C_1}}\right) t^{-\alpha_1} \\ &\geq \left(C_1 - \frac{C_1}{2}\right) t^{-\alpha_1} \\ &\geq \frac{C_1}{2} t^{-\alpha_1} \quad \text{for } t \geq t_1 \end{aligned} \tag{4.23}$$

From the above two estimates (4.22) and (4.23), we obtain that

$$\begin{aligned}\frac{\left\|(u(t),B(t)) - (e^{-tA}u_0, e^{-tA}B_0)\right\|_2}{\|(u(t),B(t))\|_2} \quad &\leq \frac{M_1 t^{-(\frac{\alpha_1+\alpha_2}{2}+\frac{1}{8})}}{\frac{C_1}{2} t^{-\alpha_1}} \\ &\leq \frac{2M_1}{C_1} t^{-\frac{\alpha_1}{2}-\frac{\alpha_2}{2}-\frac{1}{8}} t^{\alpha_1} \\ &\leq \frac{2M_1}{C_1} t^{-(\frac{\alpha_2-\alpha_1}{2}+\frac{1}{8})} \to 0 \text{ as } t \to \infty \\ \text{since} \qquad & 0 \leq \zeta = \left(\frac{\alpha_2-\alpha_1}{2} + \frac{1}{8}\right) \leq \frac{1}{8} \end{aligned}$$

$$\begin{aligned} &\text{Hence, } \frac{\left\|(u(t),B(t)) - (e^{-tA}u_0, e^{-tA}B_0)\right\|_2}{\|(u(t),B(t))\|_2} \to 0 \text{ as } t \to \infty \\ &\text{i.e. } \lim_{t\to\infty} \frac{\left\|(u(t),B(t)) - (e^{-tA}u_0, e^{-tA}B_0)\right\|_2}{\|(u(t),B(t))\|_2} = 0 \end{aligned} \tag{4.24}$$

Proof of the theorem (2.3) is completed. □

5. Concluding Remarks

In this paper, we have used a modified approach to derive L^2- decay properties of solutions of three dimensional incompressible MHD equations. We could find very few papers in the literature discussing decay properties for MHD equations. Earlier work (Please see references [7] and [10] - [13]) uses Fourier transforms and Fourier splitting method used by Shonbeck [14]. Some of the above authors have proved the results for two dimensional equations only. Our approach does not use Fourier transform method and is thus straightforward. The decay properties of the solutions that we have proved are also sharper in the sense of $O(t^{-\alpha})$ or $o(t^{-\alpha})$ for certain range of α as compared to earlier results.

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References

1. H. Sohr, *The Navier-Stokes Equations: An Elementary Functional Analytic Approach*,Birkhauser Advanced Texts, Birkhauser Verlag, Basel, (2001).

2. V. Duong and D. Khai, L^2 *decay of weak solutions for the Navier – Stokes equations in general domains*,TNU Journal of Science and Technology, Vol. 225 (02), 45-51 (2020)

3. W. Borchers, T. Miyakawa, *Algebraic L^2 - decay for Navier-Stokes flows in exterior domains,* Acta Math, vol. 165, pp. 189-227, (1990).

4. W. Borchers, T. Miyakawa, L^2 *-Decay for Navier-Stokes flows in unbounded domains with application to Exterior Stationary Flows,* Arch. Rational Mech. Anal., vol. 118, pp. 273-295, (1992).

5. R. Kajikiya, T. Miyakawa, *On L^2- decay of weak solutions of theNavier-Stokes equations in Rn* , Math. Z., vol. 192, pp. 135- 148, (1986).

6. Cai Xiaojing, Lei Lihua, *L^2- decay of the incompressible Navier-Stokes equations with Damping* , Acta Mathematica Scientia 30(4): 1235-1248, (2010).

7. Satish D. Mohgaonkar and R. V. Saraykar, *L^2 -Decay for Solutions of the Magnetohydrodynamic Equations* , Jour. Math. Phy. Sci. Vol. 23, No. 1, February (1989).

8. R. Farwig, H. Kozono, and H. Sohr, *An Lq -approach to Stokes and Navier–Stokes equations in general domains,* Acta Math, vol. 195, pp. 21-53, (2005).

9. H. Kozono, T. Ogawa, *Global strong solution and its decay properties for the Navier-Stokes equations in three dimensional domains with non-compact boundaries,* Math Zeitschrift, Vol. 216, pp. 1-30, 1(994).

10. Boling Guo and Linghai Zhang, *Decay of solutions to MHD equations in two space dimensions*, Proc. Roy. Soc. Lond. A. Vol. 449, 79-91 (1995)

11. Bo-Qing Dong, Yan Jia, Jingna Li and Jiahong Wu, *Global regularity and Time decay for the 2D magneto-hydrodynamic equations with fractional dissipation and partial magnetic diffusion*, Jour. Math. Fluid Mech. Vol. 20, 1541-1565 (2018)

12. X. Zhao and M. Zhu, *Global well-posedness and asymptotic behaviour of solutions for the three-dimensional MHD equations with Hall and ion-slip effects* , Z. Angew. Math. Phys. 69:22 (2018)

13. Z. Liu, *Algebraic L^2-decay of weak solutions to the MHD equations, Nonlinear Analysis: Real World Applications*, Vol. 50, 267-289 (Dec 2019)

14. M. E. Schonbek, *L^2- decay for weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Vol. 88, 209-222 (1985)

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