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# On some new generalized structures of Fibonacci numbers

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ABSTRACT: Various structures of Fibonacci numbers, Bronze numbers have been studied. The main structure of this paper is to introduce a new generalised sequence of Gaussian Bronze Fibonacci (GBF) numbers. Some basic properties will be studied concerning it. Also, we will establish the generalized Binet formula.

Key Words: Bronze Ratio; Shift Operator; Cassini identity.

## Contents



## 1. Introduction

The Fibonacci sequence is an integer sequence and plays a vital role for many fascinating identities. In nature it shows its presence, even if certain fruits are looked at, the number of little bumps around each ring are counted or the sand on the beach and how waves hit it is watched out, the Fibonacci sequence is seen there. It was studied by many authors in the well-known systematic manner and attractive investigations have been witnessed as can be seen in [1], [4], [5], [10], [19], [28]. Further, several recurrence sequences of natural numbers have been object of study for many researchers. Illustrations of these are the Fibonacci, Lucas, Pell, Pell-Lucas, Modified Pell, Jacobhstal, Jacobsthal-Lucas sequences among others as can be seen in [3], [6]- [14], [16], [20]- [22], [24], [25] and many others.

The authors in [15] have defined the Gaussian Fibonacci numbers $\mathfrak{G}\mathfrak{F}_n$ by the following recurrence relation:

$$\mathfrak{G}\mathfrak{F}_0 = i, \mathfrak{G}\mathfrak{F}_1 = 1,\ \mathfrak{G}\mathfrak{F}_n = \mathfrak{G}\mathfrak{F}_{n-1} + \mathfrak{G}\mathfrak{F}_{n-2},\ n > 1.$$

It is obvious that

$$\mathfrak{G}\mathfrak{F}_n = \mathfrak{F}_n + i\mathfrak{F}_{n-1},$$

where, $\mathfrak{F}_n$ is the usual Fibonacci number. In the same study the Gaussian Lucas numbers are defined as

$$\mathfrak{G}\mathfrak{L}_0 = 2 - i, \mathfrak{G}\mathfrak{L}_1 = 1 + 2i,\ \mathfrak{G}\mathfrak{L}_n = \mathfrak{G}\mathfrak{L}_{n-1} + \mathfrak{G}\mathfrak{L}_{n-2},\ n > 1,$$

where, $\mathfrak{L}_n$ is the $n$th Lucas number. Then some relationships are extended for these numbers.

It is important to note that if we consider the sequence

$$\{1, 1, 4, 13, 43, 142, 469, 1549, 5116, \cdots\},$$

then the following pattern exists:

$$\begin{aligned} 4 &= 3(1) + 1 \\ 13 &= 3(4) + 1 \\ 43 &= 3(13) + 4 \\ 142 &= 3(43) + 13 \\ &\vdots \cdots . \end{aligned}$$

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So, the terms of this sequence form a recurrence relation given by

$$u_{n+2} = 3u_{n+1} + u_n, \quad with\ u_1 = 1\ and\ u_2 = 1. \tag{1.1}$$

Using the property of shift operator, (1.1) can be written as

$$\left(E^2 - 3E - 1\right) u_n = 0.$$

Its auxiliary equation is

$$r^2 - 3r - 1 = 0$$

and has roots $r = \frac{3 \pm \sqrt{13}}{2}$, which are real and unequal. Among these two roots, the number

$$\frac{3 + \sqrt{13}}{2},$$

is called as 'Bronze Ratio'.

Gaussian Fibonacci numbers were found interesting and studied by many authors. The author in [16] has extended the results in [17] to complex Fibonacci numbers.

For an integer $n$ and a nonnegative integer $m$, Gaussian Fibonacci number $\mathfrak{G}\mathfrak{F}_{n+mi}$ is defined and some identities are studied in [18] and were further studied in [23].

Quite recently, Bronze Fibonacci numbers are studied and by observing the following relation:

$$\mathfrak{B}_{n+2} = 3\mathfrak{B}_{n+1} + \mathfrak{B}_n, \ n \geq 0$$

where $\mathfrak{B}_0 = 0$ and $\mathfrak{B}_1 = 1$.

## 2. Main results

In this section, we introduce generalized Gaussian Bronze Fibonacci numbers and compute its generating function.

Following the authors as in [2], [7], [8], [11], [16], [26]-[29], we define for $s \geq 0$ the generalized Gaussian Bronze Fibonacci numbers by the following recurrence relation:

$$\mathfrak{G}\mathfrak{B}_{n+2} = 3^s \mathfrak{G}\mathfrak{B}_{n+1} + \mathfrak{G}\mathfrak{B}_n, \ for\ n \geq 0,$$

where, $\mathfrak{G}\mathfrak{B}_0 = i$ and $\mathfrak{G}\mathfrak{B}_1 = 1$.

**Theorem 2.1** *The generating function of the GBF numbers is*

$$\mathfrak{h}(z) = \sum_{n=0}^{\infty} \mathfrak{G}\mathfrak{B}_n z^n = \frac{z + i(1 - 3^s z)}{1 - 3^s z - z^2}.$$

**Proof:**

To establish the result, we suppose the generating function of GBF numbers be $\mathfrak{h}(z)$, then

$$\begin{aligned}
& \sum_{n=0}^{\infty} \mathfrak{G}\mathfrak{B}_{n+2} z^n = \sum_{n=0}^{\infty} 3^s \mathfrak{G}\mathfrak{B}_{n+1} z^n + \sum_{n=0}^{\infty} \mathfrak{G}\mathfrak{B}_n z^n \\
\Rightarrow \quad & \sum_{n=2}^{\infty} \mathfrak{G}\mathfrak{B}_n z^{n-2} = \sum_{n=1}^{\infty} 3^s \mathfrak{G}\mathfrak{B}_n z^{n-1} + \sum_{n=0}^{\infty} \mathfrak{G}\mathfrak{B}_n z^n \\
\Rightarrow \quad & \frac{1}{z^2} \sum_{n=2}^{\infty} \mathfrak{G}\mathfrak{B}_n z^n = \frac{1}{z} \sum_{n=1}^{\infty} 3^s \mathfrak{G}\mathfrak{B}_n z^n + \sum_{n=0}^{\infty} \mathfrak{G}\mathfrak{B}_n z^n \\
\Rightarrow \quad & \frac{1}{z^2} \left[\mathfrak{h}(z) - i - z\right] = \frac{1}{z} \left[3^s \mathfrak{h}(z) - 3^s i\right] + \mathfrak{h}(z) \\
\Rightarrow \quad & \mathfrak{h}(z) = \frac{z + i(1 - 3^s z)}{1 - 3^s z - z^2},
\end{aligned}$$

as required. □

## 3. Generalized Binet formula

This part of the article deals with the derivation of GBF numbers.

In 1843, it was Binet who derived the interesting formula using Fibonacci numbers:

$$F_n = \frac{a^n - b^n}{a - b},$$

where the values of $a$ and $b$ are $\frac{1 \pm \sqrt{5}}{2}$. Moreover, the attractive Binet formula for generalized Fibonacci numbers are studied in [14].

**Theorem 3.1** *Binet formula for GBF numbers is given by*

$$\mathfrak{GB}_n = \frac{a^n - b^n}{a - b} + i\frac{ab^n - ba^n}{a - b},$$

*where $a, b$ are the roots of the characteristic equation.*

**Proof:** To establish the result, the GBF numbers using notion of difference equations is given by

$$\mathfrak{GB}_n = \xi_1 a^n + \xi_2 b^n,$$

where $\xi_1$ and $\xi_2$ are coefficients and can be computed by taking $n = 0, 1$ as

$$\xi_1 = \frac{1 - ib}{a - b} \quad \text{and} \quad \xi_2 = \frac{-1 + iq}{a - b}.$$

Consequently, utilizing the values of $a$ and $b$, we see that

$$\mathfrak{GB}_n = \frac{a^n - b^n}{a - b} + i\frac{ab^n - ba^n}{a - b},$$

as required. □

**Theorem 3.2** *The sum of GBF numbers is*

$$\sum_{j=1}^{n} \mathfrak{GB}_j = \frac{1}{3^s}\left(\mathfrak{GB}_{n+1} + \mathfrak{GB}_n - (i+1)\right).$$

**Proof:** To establish the result, we make use of definition of Bronze Fibonacci numbers to this number sequence as follows:

$$\begin{aligned}
\sum_{j=1}^{n} \mathfrak{GB}_j &= \sum_{j=1}^{n} \left(\mathfrak{B}_j + i\mathfrak{B}_{j-1}\right) \\
&= \sum_{j=1}^{n} \mathfrak{B}_j + i\sum_{j=1}^{n} \mathfrak{B}_{j-1} \\
&= \sum_{j=1}^{n} \mathfrak{B}_j + i\sum_{j=1}^{n} \mathfrak{B}_j - i\mathfrak{B}_n \\
&= \frac{1}{3^s}\left(\mathfrak{B}_{n+1} + \mathfrak{B}_n - 1\right) + \frac{i}{3^s}\left(\mathfrak{B}_{n+1} + \mathfrak{B}_n - 1\right) - i\mathfrak{B}_n \\
&= \frac{1}{3^s}\left(\mathfrak{B}_{n+1} + i\mathfrak{B}_n + i\mathfrak{B}_{n+1} + \mathfrak{B}_n - (i+1)\right) - i\mathfrak{B}_n \\
&= \frac{1}{3^s}\left(\mathfrak{B}_{n+1} + i\mathfrak{B}_n + i\left(3^s\mathfrak{B}_n + \mathfrak{B}_{n-1}\right) + \mathfrak{B}_n - (i+1)\right) - i\mathfrak{B}_n \\
&= \frac{1}{3^s}\left(\mathfrak{B}_{n+1} + i\mathfrak{B}_n + \mathfrak{B}_n + i\mathfrak{B}_{n-1} - (i+1)\right) \\
&= \frac{1}{3^s}\left(\mathfrak{GB}_{n+1} + \mathfrak{GB}_n - (i+1)\right).
\end{aligned}$$

as required. □

From above, we have following results:

**Corollary 3.1** *The sum of even GBF numbers is*

$$\sum_{j=1}^{n} \mathfrak{GB}_{2j} = \frac{1}{3^s} \left( \mathfrak{GB}_{2n+1} - 1 \right).$$

**Corollary 3.2** *The sum of odd GBF numbers is*

$$\sum_{j=0}^{n} \mathfrak{GB}_{2j+1} = \frac{1}{3^s} \left( \mathfrak{GB}_{2n+2} - i \right).$$

We now prove the following identity, called Cassini identity.

**Theorem 3.3** *For $j \geq 1$, the following holds:*

$$\mathfrak{GB}_{j+1}\mathfrak{GB}_{j-1} - \mathfrak{GB}_j^2 = (-1)^j (2 - 3^s i).$$

**Proof:** We induct on $j$. The result is trivial for $j = 1$. So, for $j + 1$, we have

$$\begin{aligned}
\mathfrak{GB}_{j+2}\mathfrak{GB}_j - \mathfrak{GB}_{j+1}^2 &= (3^s \mathfrak{GB}_{j+1}\mathfrak{GB}_j) \left( \frac{1}{3^s}\mathfrak{GB}_{j+1} - \frac{1}{3^s}\mathfrak{GB}_{j-1} \right) - \mathfrak{GB}_{j+1}^2 \\
&= \frac{1}{3^s}\mathfrak{GB}_{j+1}\mathfrak{GB}_j - \frac{1}{3^s}\mathfrak{GB}_j\mathfrak{GB}_{j-1} - \left[ (-1)^j (2 - 3^s i) + \mathfrak{GB}_j^2 \right] \\
&= \frac{1}{3^s}\mathfrak{GB}_{j+1}\mathfrak{GB}_j - \mathfrak{GB}_j \left( \frac{1}{3^s}\mathfrak{GB}_{j-1} + \mathfrak{GB}_j \right) + (-1)^{j+1}(2 - 3^s i) \\
&= (-1)^{j+1}(2 - 3^s i),
\end{aligned}$$

as desired. □

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## 4. Conclusion

It has been seen that the Fibonacci and Bronze numbers have been well structured by various authors and their various generalization have been computed. In this study, we have introduced a new generalised sequence of Gaussian Bronze Fibonacci numbers. Also, in it various generalized and basic properties have been computed. Moreover, the Binet formula have been established.

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