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Analysis of Mixed Fractional Integrodifferential Equations with Deviating Argument via Monotone Iterative Technique

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ABSTRACT: This paper deals about study of Riemann-Liouville fractional integrodifferential equations with deviating and integral arguments under integral boundary conditions by developing a monotone interative technique by introducing upper and lower solutions to Riemann Liouville fractional integrodifferential equations.

$$\begin{cases} D^{\alpha}_{0+}q(s) = h\left(s, q(s), q(\phi(s)), \int_0^s K(s,v)q(v)dv, \int_0^{\mathcal{T}} H(s,v)q(v)dv\right),\ s\in\Omega=[0,\mathcal{T}], \\ \quad q(0) = \mu\int_0^T q(v)dv + b,\ b\in\mathbb{R} \end{cases}$$

Key Words: Fractional integrodifferential equations with deviating and integral arguments, integral boundary value conditions, existence of solutions, upper and lower solutions.

Contents

1. Introduction

The theory of fractional calculus has been available and applicable to various fields of study. The investigation of the theory of fractional differential and integral equations has started quite recently. One can see the monographs of Kilbas et.al. [19], Podlubny [27].

Integral boundary conditions are used in blood flow models, population dynamics, cellular systems, chemical engineering, plasma physics, heat transmission, thermo elasticity, etc. Such conditions arise when the values of the dependent variable at the boundary points depend upon its values in the domain. As the integral boundary conditions are more precise than the local boundary conditions, modelling of such physical phenomena with integral boundary conditions are preferred over local boundary conditions. The detailed work in concern with fractional differential and fractional integrodifferential equations can be found in the articles [1]- [6], [8]- [18], [20]- [33] and references cited therein.

In this paper, we are studying the Riemann-Liouville's fractional differential equations with deviating and integral arguments under the integral boundary conditions :

$$\begin{cases} D^{\sigma}_{0+}q(s) = h\Big(s,\ q(s),\ q(\phi(s)),\ \mathcal{K}(q(s)),\ \mathcal{H}(q(s))\Big), s\in\Omega=[0,\mathcal{T}] \\ \quad q(0) = \mu\int_0^{\mathcal{T}} q(v)dv + b,\quad b\in\mathbb{R} \end{cases} \tag{1.1}$$

where $h\in C(\Omega\times\mathbb{R}^4,\mathbb{R}),\ \phi\in C(\Omega,\Omega),\ \phi(s)\le s,\ s\in\Omega,\ \mu\ge 0,\ 0<\sigma<1$, the continuous functions $\mathcal{K},\ \mathcal{H}:\Omega\times\Omega\to\mathbb{R}$ are given by,

$$\mathcal{K}(q(s)) = \int_0^s \mathcal{K}(s,v)q(v)dv, \tag{1.2}$$

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and

$$\mathcal{H}(q(s)) = \int_0^{\mathcal{T}} \mathcal{H}(s,v)q(v)dv, \tag{1.3}$$

where $\mathcal{K}(s,v),\ \mathcal{H}(s,v) \geq 0$ with $\mathcal{K}_{\mathcal{T}} = \sup\{|\mathcal{K}(s,v)| : 0 \leq s,v \leq \mathcal{T}\}$ and $\mathcal{H}_{\mathcal{T}} = \sup\{|\mathcal{H}(s,v)| : 0 \leq s,v \leq \mathcal{T}\}$.

We show that problem defined by (1.1) has the existence, uniqueness of a solution by using Banach fixed point theorem. Also, we consider the existence, uniqueness results for Riemann-Liouville fractional differential equations with deviating and integral arguments under the integral boundary conditions using monotone method.

2. Preliminaries

To discuss main results, we require some definitions, lemmas and hypotheses that is defined as follows:

Definition 2.1 ([19], [27]) The Riemann-Liouville fractional integral of order σ, with right hand side is pointwise defined on $(0,\infty)$ is given by

$$I^{\sigma}_{0+}q(s) = \frac{1}{\Gamma(\sigma)}\int_0^s (s-v)^{\sigma-1}q(v)dv, \quad \sigma > 0, \tag{2.1}$$

Definition 2.2 ([19], [27]) The Riemann-Liouville fractional derivative of order σ with $m-1 < \sigma < m$

$$D^{\sigma}_{0+}q(s) = \big(\frac{d}{ds}\big)^m \big(I^{m-\sigma}_{0+}q(s)\big) = \frac{1}{\Gamma(m-\sigma)}\frac{d^m}{ds^m}\int_0^s (s-v)^{m-\sigma-1}q(v)\,dv,\ \ s>0. \tag{2.2}$$

Lemma 2.1 *([19]) Let $q \in C^m[0,\mathcal{T}], \sigma \in (m-1,m), m \in \mathbb{N}$. Then for $s \in \Omega$,*

$$I^{\sigma}_{0+}D^{\sigma}_{0+}q(s) = q(s) - \sum_{k=0}^{m-1}\frac{s^k}{k!}q^k(0). \tag{2.3}$$

Consider the space $C_{1-\sigma}(\Omega,\mathbb{R}) = \{q \in C((0,\mathcal{T}],\mathbb{R}) : s^{1-\sigma}q(s) \in C(\Omega,\mathbb{R})\}$.

Lemma 2.2 *([7]) Let $n \in C_{1-\sigma}(\Omega,\mathbb{R})$ where for some $s_1 \in (0,\mathcal{T}]$, $n(s_1) = 0$ with $n(s) \leq 0$, $0 \leq s \leq s_1$, then $D^{\sigma}n(s_1) \geq 0$.*

Lemma 2.3 *Consider $h \in C(\Omega \times \mathbb{R}^4,\mathbb{R})$. If function $q \in C_{1-\sigma}(\Omega,\mathbb{R})$ is a solution of the problem (1.1) if and only if q is a solution of the integral equation*

$$q(s) = \frac{1}{\Gamma(\sigma)}\int_0^s (s-v)^{\sigma-1}h\Big(s,\ q(s),\ q(\phi(s)),\ \mathcal{K}(q(s)),\ \mathcal{H}(q(s))\Big)dv + \mu\int_0^{\mathcal{T}} q(v)dv + b. \tag{2.4}$$

Proof: *Let q is a solution of (1.1). With Lemma 2.3, we get,*

$$I^{\sigma}_{0+}D^{\sigma}_{0+}q(s) = q(s) - \sum_{k=0}^{m-1}\frac{s^k}{k!}q^k(0)$$

If $q \in C_{1-\sigma}(\Omega,\mathbb{R})$, and $m = 1$ in above equation, we get,

$$I^{\sigma}_{0+}D^{\sigma}_{0+}q(t) = q(t) - q(0) \tag{2.5}$$

As we know from problem (1.1), $D^{\sigma}_{0+}q(s) = h\Big(s,\ q(s),\ q(\phi(s)),\ \mathcal{K}(q(s)), \mathcal{H}(q(s))\Big)$, substituting in (2.5) results,

$$I^{\sigma}_{0+}\Big[h\Big(s,\ q(s),\ q(\phi(s)),\ \mathcal{K}(q(s)), \mathcal{H}(q(s))\Big)\Big] = q(s) - q(0) \tag{2.6}$$

By using equation (2.1) in the above equation, we get (2.4).
Conversely, assume that q satisfies equation (2.4). On taking Riemann-Liouville derivative of order σ of both sides, we get first equation of the problem (1.1). On putting $s = 0$ in equation (2.4), we get the integral boundary condition

$$q(0) = \mu \int_0^{\mathcal{T}} q(v)dv + b.$$

This completes the proof.

Lemma 2.4 *Assume that $\{q_\delta\}$ is a set of all continuous functions defined on Ω, for each $\delta > 0$, which satisfies*

$$\begin{cases} D^{\sigma}_{0+} q_\delta(s) = h\Big(s,\ q_\delta(s),\ q_\delta(\phi(s)),\ \mathcal{K}(q_\delta(s)),\ \mathcal{H}(q_\delta(s))\Big), s \in \Omega = [0, \mathcal{T}] \\ \quad q_\delta(0) = \mu \displaystyle\int_0^{\mathcal{T}} q_\delta(v)dv + b, b \in \mathbb{R}, \end{cases}$$

where $\Big|h\Big(s,\ q_\delta(s),\ q_\delta(\phi(s)),\ \mathcal{K}(q_\delta(s)),\ \mathcal{H}(q_\delta(s))\Big)\Big| \leq M$ for all $s \in \Omega$. Further set of $\{q_\delta\}$ is equicontinuous on Ω.

Proof: Consider $0 \leq s_1 < s_2 \leq \mathcal{T}$, with equation (2.4), we have

$$\begin{aligned} |q_\delta(s_1) - q_\delta(s_2)| &= \bigg| \frac{1}{\Gamma(\sigma)} \int_0^{s_1} (s_1 - v)^{\sigma-1} h\Big(s_1, q_\delta(s_1), q_\delta(\phi(s_1)), \mathcal{K}(q_\delta(s_1)), \mathcal{H}(q_\delta(s_1))\Big)dv \\ &\quad - \frac{1}{\Gamma(\sigma)} \int_0^{s_2} (s_2 - v)^{\sigma-1} h\Big(s_2, q_\delta(s_2), q_\delta(\phi(s_2)), \mathcal{K}(q_\delta(s_2)), \mathcal{H}(q_\delta(s_2))\Big)dv \bigg| \\ &\leq \frac{M}{\Gamma(\sigma)} \bigg| \int_0^{s_1} (s_1 - v)^{\sigma-1}dv - \int_0^{s_2} (s_2 - v)^{\sigma-1}dv \bigg| \\ &= \frac{M}{\Gamma(\sigma)} \bigg| \int_0^{s_1} (s_1 - v)^{\sigma-1}dv - \bigg[\int_0^{s_1} (s_2 - v)^{\sigma-1}dv + \int_{s_1}^{s_2} (s_2 - v)^{\sigma-1}dv \bigg] \bigg| \\ &= \frac{M}{\sigma\Gamma(\sigma)} |2(s_2 - s_1)^\sigma + s_1^\sigma - s_2^\sigma| \\ &\leq \frac{2M}{\Gamma(\sigma+1)} |(s_2 - s_1)^\sigma| < \delta, \end{aligned}$$

As $|s_2 - s_1| < \epsilon$, for $\epsilon = \sqrt[\sigma]{\dfrac{\delta\Gamma(\sigma+1)}{2M}}$, we get $|q_\delta(s_1) - q_\delta(s_2)| < \delta$.

3. Existence Results

Theorem 3.1 *Assume that*

1. $h \in C(\Omega \times \mathbb{R}^4, \mathbb{R}),\ \phi(s) \in C(\Omega, \Omega),\ \phi(s) \leq s,\ s \in \Omega$
2. *there exist non-negative constants $\mathcal{M}, \mathcal{N}, \mathcal{L}$ and $\mathcal{Q}$ such that the function satisfies* $|h(s, q_1, q_2, q_3, q_4) - h(s, r_1, r_2, r_3, r_4)| \leq \mathcal{M}|q_1 - r_1| + \mathcal{N}|q_2 - r_2| + \mathcal{L}|q_3 - r_3| + \mathcal{Q}|q_4 - r_4|$ $\forall\ s \in \Omega$ *and* $q_i,\ r_i \in \mathbb{R}, i = 1, 2, 3, 4.$
If

$$\mu < \frac{1}{\mathcal{T}} - \frac{1}{\Gamma(\sigma+1)} \bigg[(\mathcal{M} + \mathcal{N})\mathcal{T}^{\sigma-1} + \frac{(\mathcal{L}\mathcal{K}_{\mathcal{T}} + \mathcal{Q}\mathcal{H}_{\mathcal{T}})}{(\sigma+1)} \mathcal{T}^\sigma \bigg],$$

then the problem (1.1) has unique solution.

Proof: *Define an operator $\mathcal{G}$ given by*

$$(\mathcal{G}q)(s) = \frac{1}{\Gamma(\sigma)} \int_0^s (s - v)^{\sigma-1} h\Big(s,\ q(s),\ q(\phi(s)),\ \mathcal{K}(q(s)), \mathcal{H}(q(s))\Big)dv + \mu \int_0^{\mathcal{T}} q(v)dv + b.$$

Now, we show that operator $\mathcal{G} : C_{1-\sigma}(\Omega,\mathbb{R}) \to C_{1-\sigma}(\Omega,\mathbb{R})$ is contraction map.

For any $q,\ r \in C_{1-\sigma}(\Omega,\mathbb{R})$, consider

$$
\begin{aligned}
&||\mathcal{G}q-\mathcal{G}r||\\
&=\max_{s\in\Omega}\Big|(\mathcal{G}q)(t)-(\mathcal{G}r)(t)\Big|\\
&=\max_{s\in\Omega}\Big|\frac{1}{\Gamma(\sigma)}\int_0^s (s-v)^{\sigma-1}h\Big(s,\ q(s),\ q(\phi(s)),\ \mathcal{K}(q(s)),\mathcal{H}(q(s))\Big)dv+\mu\int_0^{\mathcal{T}} q(v)dv+b\\
&-\Big[\frac{1}{\Gamma(\sigma)}\int_0^s (s-v)^{\sigma-1}h\Big(s,\ r(s),\ r(\phi(s)),\ \mathcal{K}(r(s)),\mathcal{H}(r(s))\Big)dv+\mu\int_0^{\mathcal{T}} r(v)dv+b\Big]\Big|\\
&=\max_{s\in\Omega}\Big|\frac{1}{\Gamma(\sigma)}\int_0^s (s-v)^{\sigma-1}\Big[h\Big(s,\ q(s),\ q(\phi(s)),\ \mathcal{K}(q(s)),\mathcal{H}(q(s))\Big)\\
&-h\Big(s,\ r(s),\ r(\phi(s)),\ \mathcal{K}(r(s)),\mathcal{H}(r(s))\Big)\Big]dv+\mu\int_0^{\mathcal{T}}[q(v)-r(v)]dv\Big|\\
&\le \max_{s\in\Omega}\mu\int_0^{\mathcal{T}}|q(v)-r(v)|dv\\
&+\frac{1}{\Gamma(\sigma)}\int_0^s (s-v)^{\sigma-1}\Big|\Big[h\Big(s,\ q(s),\ q(\phi(s)),\ K(q(s)),H(q(s))\Big)\\
&-h\Big(s,\ r(s),\ r(\phi(s)),\ \mathcal{K}(r(s)),\mathcal{H}(r(s))\Big)\Big]\Big|dv\\
&\le \max_{s\in\Omega}\mu\int_0^{\mathcal{T}}|q(v)-r(v)|dv+\\
&\frac{1}{\Gamma(\sigma)}\int_0^s (s-v)^{\sigma-1}\\
&\times\Big[\mathcal{M}|q(v)-r(v)|+\mathcal{N}|q(\phi(v))-r(\phi(v))|+\mathcal{L}|\mathcal{K}(q(s))-\mathcal{K}(r(s))|+\mathcal{Q}|\mathcal{H}(q(s))-\mathcal{H}(r(s))|\Big]dv\\
&\le \max_{s\in\Omega}\Big[\mu\int_0^{\mathcal{T}}dv+\frac{1}{\Gamma(\sigma)}[\mathcal{M}+\mathcal{N}]\int_0^s(s-v)^{\sigma-1}dv+\frac{\mathcal{L}\mathcal{K}_{\mathcal{T}}}{\Gamma(\sigma)}\int_0^s(s-v)^{\sigma-1}v\ dv\\
&+\frac{\mathcal{Q}\mathcal{H}_{\mathcal{T}}}{\Gamma(\sigma)}\int_0^{\mathcal{T}}(s-v)^{\sigma-1}v\ dv\Big]||q-r||_C\\
&\le ||q-r||_C\Big[\mu\mathcal{T}+\frac{1}{\Gamma(\sigma)}[\mathcal{M}+\mathcal{N}]\frac{\mathcal{T}^{\sigma}}{\sigma}+\frac{\mathcal{L}\mathcal{K}_{\mathcal{T}}}{\Gamma(\sigma)}\frac{\mathcal{T}^{\sigma+1}}{(\sigma+1)\sigma}+\frac{\mathcal{Q}\mathcal{H}_{\mathcal{T}}}{\Gamma(\sigma)}\frac{\mathcal{T}^{\sigma+1}}{(\sigma+1)\sigma}\Big]\\
&=||q-r||_C\Big[\mu\mathcal{T}+[\mathcal{M}+\mathcal{N}]\frac{\mathcal{T}^{\sigma}}{\Gamma(\sigma+1)}+\frac{\mathcal{T}^{\sigma+1}}{\Gamma(\sigma+2)}[\mathcal{L}\mathcal{K}_{\mathcal{T}}+\mathcal{Q}\mathcal{H}_{\mathcal{T}}]\Big]
\end{aligned}
$$

Hence, $\mathcal{G}$ is a contraction mapping. Therefore by fixed point theorem of Banach, the problem (1.1) has a unique solution.

Corollary 3.1 *Consider $\mathcal{M},\mathcal{N},\mathcal{L}$ and $\mathcal{Q}$ are constants, $\lambda \in C_{1-\sigma}(\Omega,\mathbb{R})$. Then the linear problem*

$$
\begin{cases}
D^{\sigma}_{0+}q(s)+\mathcal{M}q(s)+\mathcal{N}q(\phi(s))+\mathcal{L}\int_0^s\mathcal{K}(s,v)q(v)dv+Q\int_0^{\mathcal{T}}\mathcal{H}(s,v)q(v)dv=\lambda(s),\ 0<\sigma<1,\ s\in\Omega,\\
q(0)=\mu\int_0^{\mathcal{T}}q(v)dv+b, b\in\mathbb{R}
\end{cases}
\tag{3.1}
$$

has a unique solution.

Proof: *The proof holds from Theorem 3.1.*

4. Monotone Iterative Method

In this section using monotone iterative method with the method of upper and lower solutions, we show that existence, uniqueness of solution for the problem (1.1) .

We show that the following lemma as comparison result which will be used in proving the main results.

Lemma 4.1 *Let $\phi \in C(\Omega, \Omega)$ where $\phi(s) \leq s$ on Ω. Let $z \in C_{1-\sigma}(\Omega, \mathbb{R})$ holds the inequalities*

$$\begin{cases} D^{\sigma}_{0+} z(s) \leq -\mathcal{M} z(s) - \mathcal{N} z(\phi(s)) - \mathcal{L} \int_0^s \mathcal{K}(s,v) z(v) dv - \mathcal{Q} \int_0^{\mathcal{T}} \mathcal{H}(s,v) z(v) dv \equiv F z(s), \\ z(0) \leq 0, \end{cases} \tag{4.1}$$

where $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$ and $\mathcal{Q}$ are constants. If

$$-(1+\mathcal{T}^{\sigma})[\mathcal{M}+\mathcal{N}] < \Gamma(\sigma+1) \tag{4.2}$$

then, $z(s) \leq 0, \ \forall \ s \in \Omega$.

Proof: *Let a function $z_\delta(s) = z(s) - \delta(1+s^{\sigma}), \ \delta > 0$. Furthermore*

$$\begin{aligned} D^{\sigma}_{0+} z_\delta(s) &= D^{\sigma}_{0+} z(s) - D^{\sigma}_{0+} \delta(1+s^{\sigma}) \\ &\leq F z(s) - \frac{\delta}{s^{\sigma}\Gamma(\sigma-1)} - \delta\Gamma(\sigma+1) \\ &= F[z_\delta(s) + \delta(1+s^{\sigma})] - \delta\left[\frac{1}{s^{\sigma}\Gamma(\sigma-1)} + \Gamma(\sigma+1)\right] \\ &= F z_\delta(s) + \delta\Big[-\mathcal{M}(1+s^{\sigma}) - \mathcal{N}(1+s^{\sigma}) - \mathcal{L}\int_0^s K(s,v)(1+v^{\sigma}) dv - \mathcal{Q}\int_0^{\mathcal{T}} H(s,v)(1+v^{\sigma}) dv \\ &\quad - \frac{1}{s^{\sigma}\Gamma(\sigma-1)} - \Gamma(\sigma+1)\Big] \\ &< F z_\delta(s), \end{aligned}$$

with $F z_\delta(s) = z(0) - \delta < 0$.

We show that $F z_\delta(s) < 0$. Let us suppose $F z_\delta(s) \nless 0$ on Ω. Therefore, for $s_1 \in (0, \mathcal{T}]$ as $F z_\delta(s_1) = 0$ with $z_\delta(s) < 0, \ s \in (0, s_1)$. Hence from Lemma 2.2, $D^{\sigma}_{0+} F z_\delta(s_1) \geq 0$. Then, as $F z_\delta(s_1) = 0$, Hence, consequently we have

$$0 < F z_\delta(s_1) = -\mathcal{N} z_\delta(\phi(s_1)) - \mathcal{L}\int_0^{s_1} \mathcal{K}(s_1,v) z_\delta(v) dv - \mathcal{Q}\int_0^{\mathcal{T}} \mathcal{H}(s_1,v) z_\delta(v) dv \tag{4.3}$$

If $\mathcal{N}=0, \mathcal{L}=0, \mathcal{Q}=0$, it gives absurd result as we get $0<0$.
If $\mathcal{N}=0, \mathcal{L}=0, \mathcal{Q}<0$, it follows if $\mathcal{H}(s_1,v) < 0$ which is impossible.
If $\mathcal{N}=0, \mathcal{L}<0, \mathcal{Q}=0$, then $\mathcal{K}(s_1,v) < 0$ which is impossible.
If $\mathcal{N}=0, \mathcal{L}<0, \mathcal{Q}<0$, then $\mathcal{H}(s_1,v) + \mathcal{K}(s_1,v) < 0$ which is impossible.
If $\mathcal{N}<0, \mathcal{L}=0, \mathcal{Q}=0$, then we must have $z_\delta(\phi(s_1)) > 0$ which is impossible.
If $\mathcal{N}<0, \mathcal{L}=0, \mathcal{Q}<0$ then, the equation (4.3) has negative right hand side, which is not possible.
If $\mathcal{N}<0, \mathcal{L}<0, \mathcal{Q}=0$, then the equation (4.3) has negative right hand side, which is not possible.
If $\mathcal{N}<0, \mathcal{L}<0, \mathcal{Q}<0$, then the equation (4.3) has negative right hand side, which is not possible.

Therefore, It gives $z_\delta(s) < 0$ on Ω. Further $z(s) - \delta(1+s^{\sigma}) < 0$ on Ω. As, $\delta \to 0$, we have $z(s) \leq 0$ on Ω.

Definition 4.1 A pair of functions $[x_0, y_0]$ in $C_{1-\sigma}(\Omega, \mathbb{R})$ are respectively known as lower and upper solutions of the problem (1.1) if the following inequalities hold:

$$D^{\sigma}_{0+}x_0(s) \le h\left(s, x_0(s), x_0(\phi(s)), \int_0^s \mathcal{K}(s,v)x_0(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_0(v)dv\right),\; x_0(0) \le \int_0^{\mathcal{T}} x_0(v)dv + b \tag{4.4}$$

$$D^{\sigma}_{0+}y_0(s) \ge h\left(s, y_0(s), y_0(\phi(s)), \int_0^s \mathcal{K}(s,v)y_0(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)y_0(v)dv\right),\; y_0(0) \ge \int_0^{\mathcal{T}} y_0(v)dv + b \tag{4.5}$$

Theorem 4.1 *Suppose that*

1. $h \in C(\Omega \times \mathbb{R}^4, \mathbb{R}),\ \phi(s) \in C(\Omega, \Omega),\ \phi(s) \le s,\ s \in \Omega,$
2. x_0, y_0 *in* $C_{1-\sigma}(\Omega, \mathbb{R})$ *are lower and upper solutions of the problem (1.1) as* $x_0(s) \le y_0(s)$ *on* Ω,
3. *there exist non-negative constants* $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$ *and* $\mathcal{Q}$ *such that the function* h *satisfies*

$$h(s, q_1, q_2, q_3, q_4) - h(s, r_1, r_2, r_3, r_4) \ge -\mathcal{M}(q_1 - r_1) - \mathcal{N}(q_2 - r_2) - \mathcal{L}(q_3 - r_3) - \mathcal{Q}(q_4 - r_4),$$

for

$$x_0(s) \le r_1 \le q_1 \le y_0(s),\ x_0(\phi(s)) \le r_1 \le q_1 \le y_0(\phi(s)),$$
$$\int_0^s \mathcal{K}(s,v)x_0(v)dv \le r_3 \le q_3 \le \int_0^s \mathcal{K}(s,v)y_0(v)dv$$

,

$$\int_0^{\mathcal{T}} \mathcal{H}(s,v)x_0(v)dv \le r_4 \le q_4 \le \int_0^{\mathcal{T}} \mathcal{H}(s,v)y_0(v)dv,$$

then there exist monotone sequences $\{x_n(s)\}$ *and* $\{y_n(s)\}$ *with* $\{x_n(s)\} \to x(s)$ *and* $\{y_n(s)\} \to y(s)$ *as* $n \to \infty$ *for all* $s \in \Omega$ *in* $C_{1-\sigma}(\Omega, \mathbb{R})$ *, where* x *and* y *are minimal and maximal solutions of the problem (1.1) respectively and* $x(s) \le q(s) \le y(s)$ *on* Ω.

Proof: *For any* $\zeta \in C_{1-\sigma}(\Omega, \mathbb{R})$ *satisfying* $\zeta \in [x_0, y_0]$, *consider the following linear problem :*

$$\begin{cases} D^{\sigma}_{0+}q(s) = h\left(s, \zeta(s), \zeta(\phi(s)), \int_0^s \mathcal{K}(s,v)\zeta(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)\zeta(v)dv\right) \\ \qquad + \mathcal{M}[\zeta(s) - q(s)] + \mathcal{N}[\zeta(\phi(s)) - q(\phi(s))] \\ \qquad + \mathcal{L}\left[\int_0^s \mathcal{K}(s,v)\zeta(v)dv - \int_0^s K(s,v)q(v)dv\right] + \mathcal{Q}\left[\int_0^{\mathcal{T}} \mathcal{H}(s,v)\zeta(v)dv - \int_0^{\mathcal{T}} \mathcal{H}(s,v)q(v)dv\right] \\ q(0) = \int_0^{\mathcal{T}} q(v)dv + b. \end{cases} \tag{4.6}$$

Then, by the Corollary 3.1, the linear problem (4.6) possesses a solution $q(t)$.

Now, we designate the iterates as given below and construct the sequences $\{x_n\}$, $\{y_n\}$ *by*

$$\begin{cases} D^{\sigma}_{0+}x_{n+1}(s) = h\left(s, x_n(s), x_n(\phi(s)), \int_0^s \mathcal{K}(s,v)x_n(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_n(v)dv\right) \\ \qquad - \mathcal{M}[x_{n+1}(s) - x_n(s)] - \mathcal{N}[x_{n+1}(\phi(s)) - x_n(\phi(s))] \\ \qquad - \mathcal{L}\left[\int_0^s \mathcal{K}(s,v)x_{n+1}(v)dv - \int_0^s \mathcal{K}(s,v)x_n(v)dv\right] \\ \qquad - \mathcal{Q}\left[\int_0^{\mathcal{T}} \mathcal{H}(s,v)x_{n+1}(v)dv - \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_n(v)dv\right] \\ x_{n+1}(0) = \int_0^{\mathcal{T}} x_n(v)dv + b, \end{cases} \tag{4.7}$$

$$\begin{cases} D^{\sigma}_{0+}y_{n+1}(s) = h\left(s, y_n(s), y_n(\phi(s)), \int_0^s \mathcal{K}(s,v)y_n(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)y_n(v)dv\right) \\ \qquad - \mathcal{M}[y_{n+1}(s) - y_n(s)] - \mathcal{N}[y_{n+1}(\phi(s)) - y_n(\phi(s))] \\ \qquad - \mathcal{L}\Big[\int_0^s \mathcal{K}(s,v)y_{n+1}(v)dv - \int_0^s \mathcal{K}(s,v)y_n(v)dv\Big] \\ \qquad - \mathcal{Q}\Big[\int_0^{\mathcal{T}} \mathcal{H}(s,v)y_{n+1}(v)dv - \int_0^{\mathcal{T}} \mathcal{H}(s,v)y_n(v)dv\Big] \\ y_{n+1}(0) = \int_0^{\mathcal{T}} y_n(v)dv + b, \end{cases} \tag{4.8}$$

from the arguments made above, (4.7) and (4.8) exists, we get the existence of solutions x_1 *and* y_1 *respectively, for* $n = 0$ *in the problems (4.7) and (4.8). Now, We prove that* $x_0(s) \leq x_1(s) \leq y_1(s) \leq y_0(s)$*. Define* $z(t) = x_1(s) - x_0(s)$*. As, lower solution is* $x_0(s)$ *of the problem (4.7), we get*

$$\begin{aligned} D^{\sigma}_{0+}z(s) &= D^{\sigma}_{0+}x_1(s) - D^{\sigma}_{0+}x_0(s) \\ &\geq h\left(s, x_0(s), x_0(\phi(s)), \int_0^s \mathcal{K}(s,v)x_0(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_0(v)dv\right) \\ &- \mathcal{M}[x_1(s) - x_0(s)] - \mathcal{N}[x_1(\phi(s)) - x_0(\phi(s))] \\ &- \mathcal{L}\Big[\int_0^s \mathcal{K}(s,v)x_1(v)dv - \int_0^s \mathcal{K}(s,v)x_0(v)dv\Big] \\ &- \mathcal{Q}\Big[\int_0^{\mathcal{T}} \mathcal{H}(s,v)x_1(v)dv - \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_0(v)dv\Big] \\ &- h\left(s, x_0(s), x_0(\phi(s)), \int_0^s \mathcal{K}(s,v)x_0(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_0(v)dv\right) \\ &= -\mathcal{M}z(s) - \mathcal{N}z(\phi(s)) - \mathcal{L}\int_0^s \mathcal{K}(s,v)z(v)dv - \mathcal{Q}\int_0^{\mathcal{T}} \mathcal{H}(s,v)z(v)dv, \end{aligned}$$

with $z(0) = x_1(0) - x_0(0) \geq \int_0^{\mathcal{T}} x_0(v)dv + b - \int_0^{\mathcal{T}} x_0(v)dv - b = 0$*. From Lemma 4.1, As* $z(s) \geq 0$ *it gives* $x_1(s) \geq x_0(s)$ *on* Ω*. In similar manner, one can prove that* $x_1(s) \leq y_1(s)$ *with* $y_1(s) \leq y_0(s)$ *on* Ω*. Hence, we obtain* $x_0(s) \leq x_1(s) \leq y_1(s) \leq y_0(s)$*. Suppose that for* $k > 1$ *result is true i.e.* $x_{k-1}(s) \leq x_k(s) \leq y_k(s) \leq y_{k-1}(s)$ *on* Ω*.*

Claim: $x_k(s) \leq x_{k+1}(s) \leq y_{k+1}(s) \leq y_k(s)$ *on* Ω*.*

Define $z(s) = x_{k+1}(s) - x_k(s)$*. Then*

$$\begin{aligned} D^{\sigma}_{0+}z(s) &= D^{\sigma}_{0+}x_{k+1}(s) - D^{\sigma}_{0+}x_k(s) \\ &\geq h\left(s, x_k(s), x_k(\phi(s)), \int_0^s \mathcal{K}(s,v)x_k(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_k(v)dv\right) \\ &- \mathcal{M}[x_{k+1}(s) - x_k(s)] - \mathcal{N}[x_{k+1}(\phi(s)) - x_k(\phi(s))] \\ &- \mathcal{L}\Big[\int_0^s \mathcal{K}(s,v)x_{k+1}(v)dv - \int_0^s \mathcal{K}(s,v)x_k(v)dv\Big] \\ &- \mathcal{Q}\Big[\int_0^{\mathcal{T}} \mathcal{H}(s,v)x_{k+1}(v)dv - \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_k(v)dv\Big] \end{aligned}$$

$$
\begin{aligned}
&- h\left(s, x_{k-1}(s), x_{k-1}(\phi(s)), \int_0^s \mathcal{K}(s,v)x_{k-1}(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_{k-1}(v)dv\right)\\
&+ \mathcal{M}[x_k(s) - x_{k-1}(s)] + \mathcal{N}[x_k(\phi(s)) - x_{k-1}(\phi(s))]\\
&+ \mathcal{L}\left[\int_0^s \mathcal{K}(s,v)x_k(v)dv - \int_0^s \mathcal{K}(s,v)x_{k-1}(v)dv\right]\\
&+ \mathcal{Q}\left[\int_0^{\mathcal{T}} \mathcal{H}(s,v)x_k(v)dv - \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_{k-1}(v)dv\right]\\
&\geq -\mathcal{M}z(s) - \mathcal{N}z(\phi(s)) - \mathcal{L}\int_0^s \mathcal{K}(s,v)z(v)dv - \mathcal{Q}\int_0^{\mathcal{T}} \mathcal{H}(s,v)z(v)dv
\end{aligned}
$$

$$
\begin{aligned}
z(0) = x_{k+1}(0) - x_k(0) &= \int_0^{\mathcal{T}} x_k(v)dv + b - \int_0^{\mathcal{T}} x_{k-1}(v)dv - b\\
&\geq \int_0^{\mathcal{T}} [x_k(s) - x_k(v)]dv = 0.
\end{aligned}
$$

Hence, we obtain $z(t) \geq 0$ from the Lemma 4.1, implying that $x_{k+1}(s) \geq x_k(s)$ for all k on Ω. Similarly, we can show that $x_{k+1}(s) \leq y_{k+1}(s)$ with $y_{k+1}(s) \leq y_k(s)$ for all s on Ω. Therefore, by mathematical induction, we get

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq y_k \leq \cdots \leq y_2 \leq y_1 \leq y_0$$

on Ω.

Therefore, $\{x_n\}$ and $\{y_n\}$ are the sequences monotonic and uniformly bounded. Hence, $\{D^{\sigma}_{0+}x_n\}$ and $\{D^{\sigma}_{0+}y_n\}$ are uniformly bounded on Ω in context of (4.7) and (4.8). By using the Lemma 2.4 the sequences $\{x_n\}$ and $\{y_n\}$ are equicontinuous. Hence, by theorem of Arzela-Ascoli sequences $\{x_n\}$ and $\{y_n\}$ converge uniformly to x and y on Ω respectively.

Let us prove that x and y are the minimal and maximal solutions of the problem (1.1). Let q be any solution of (1.1) different from x and y. For k as positive integer, $x_k(s) \leq q(s) \leq y_k(s)$ on Ω. Consider $z(s) = q(s) - x_{k+1}(s)$. Therefore ,

$$
\begin{aligned}
D^{\sigma}_{0+}z(s) &= D^{\sigma}_{0+}q(s) - D^{\sigma}_{0+}x_{k+1}(s)\\
&\geq h\left(s, q(s), q(\phi(s)), \int_0^s \mathcal{K}(s,v)q(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)q(v)dv\right)\\
&- h\left(s, x_0(s), x_0(\phi(s)), \int_0^s \mathcal{K}(s,v)x_0(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_0(v)dv\right)\\
&+ \mathcal{M}[x_{k+1}(s) - x_k(s)] + \mathcal{N}[x_{k+1}(\phi(s)) - x_k(\phi(s))]\\
&+ \mathcal{L}\left[\int_0^s \mathcal{K}(s,v)x_{k+1}(v)dv - \int_0^s \mathcal{K}(s,v)x_k(v)dv\right]\\
&+ \mathcal{Q}\left[\int_0^{\mathcal{T}} \mathcal{H}(s,v)x_{k+1}(v)dv - \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_k(v)dv\right]\\
&\geq -\mathcal{M}[q(s) - x_{k+1}(s)] - \mathcal{N}[q(\phi(s)) - x_{k+1}(\phi(s))]\\
&- \mathcal{L}\left[\int_0^s \mathcal{K}(s,v)q(v)dv - \int_0^s \mathcal{K}(s,v)x_{k+1}(v)dv\right]\\
&- \mathcal{Q}\left[\int_0^{\mathcal{T}} \mathcal{H}(s,v)q(v)dv - \int_0^{\mathcal{T}} \mathcal{H}(s,v)x_{k+1}(v)dv\right]\\
&\geq -\mathcal{M}z(s) - \mathcal{N}z(\phi(s)) - \mathcal{L}\left[\int_0^s \mathcal{K}(s,v)z(v)dv\right] - \mathcal{Q}\left[\int_0^{\mathcal{T}} \mathcal{H}(s,v)z(v)dv\right]
\end{aligned}
$$

and

$$z(0) = q(0) - x_{k+1}(0) = \int_0^{\mathcal{T}} [q(v) - x_k(v)]dv \geq 0$$

From Lemma 4.1, we get $z(s) \geq 0$, inferring $q(s) \geq x_{k+1}(s)$ $\forall$ k on Ω. Also, we can show that $q(s) \leq y_{k+1}(s)$ $\forall$ k on Ω. Since, $x_0(s) \leq q(s) \leq y_0(s)$ on Ω. With mathematical induction it comes $x_k(s) \leq q(s)$ and $q(s) \leq y_k(s)$ $\forall$ k. Therefore, $x_k(s) \leq q(s) \leq y_k(s)$ on Ω, $k \to \infty$, we get $x(s) \leq q(s) \leq y(s)$ on Ω. Hence, the functions $x(s), y(s)$ are the minimal and maximal solutions of the problem (1.1).

Theorem 4.2 *Suppose that*

1. *If conditions of the Theorem 4.1 satisfies, and*
2. *there exists non-negative constants $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$ and $\mathcal{Q}$ so that the function h holds ,*

$$h(s, q_1, q_2, q_3, q_4) - h(s, r_1, r_2, r_3, r_4) \leq \mathcal{M}(q_1 - r_1) + \mathcal{N}(q_2 - r_2) + \mathcal{L}(q_3 - r_3) + \mathcal{Q}(q_4 - r_4), \quad (4.9)$$

for

$$x_0(s) \leq r_1 \leq q_1 \leq y_0(s), x_0(\phi(s)) \leq r_2 \leq q_2 \leq y_0(\phi(s)),$$
$$\int_0^s \mathcal{K}(s,v)x_0(v)dv \leq r_3 \leq q_3 \leq \int_0^s \mathcal{K}(s,v)y_0(v)dv,$$
$$\int_0^{\mathcal{T}} \mathcal{H}(s,v)x_0(v)dv \leq r_4 \leq q_4 \leq \int_0^{\mathcal{T}} \mathcal{H}(s,v)y_0(v)dv,$$

then the problem (1.1) possesses a solution.
Proof: *As $x(s) \leq y(s)$ on Ω. It is enough to show that $x(s) \geq y(s)$ on Ω. Let $z(s) = y(s) - x(s)$. Let*

$$\begin{aligned}
D^{\sigma}_{0+}z(s) &= D^{\sigma}_{0+}y(s) - D^{\sigma}_{0+}x(s) \\
&= h\left(s, y(s), y(\phi(s)), \int_0^s \mathcal{K}(s,v)y(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)y(v)dv\right) \\
&\quad - h\left(s, x(s), x(\phi(s)), \int_0^s \mathcal{K}(s,v)x(v)dv, \int_0^{\mathcal{T}} \mathcal{H}(s,v)x(v)dv\right) \\
&\leq -\mathcal{M}[x(s) - y(s)] - \mathcal{N}[x(\phi(s)) - y(\phi(s))] \\
&\quad - \mathcal{L}\Big[\int_0^s \mathcal{K}(s,v)x(v)dv - \int_0^s \mathcal{K}(s,v)y(v)dv\Big] - \mathcal{Q}\Big[\int_0^{\mathcal{T}} \mathcal{H}(s,v)x(v)dv - \int_0^{\mathcal{T}} \mathcal{H}(s,v)y(v)dv\Big] \\
&= -\mathcal{M}'z(s) - \mathcal{N}'z(\phi(s)) - L'\Big[\int_0^s \mathcal{K}(s,v)z(v)dv\Big] - \mathcal{Q}'\Big[\int_0^{\mathcal{T}} \mathcal{H}(s,v)z(v)dv\Big],
\end{aligned}$$

where $\mathcal{M}' = -\mathcal{M}$, $\mathcal{N}' = -\mathcal{N}$, $\mathcal{L}' = -\mathcal{L}$, $\mathcal{Q}' = -\mathcal{Q}$.

Let

$$z(0) = \int_0^{\mathcal{T}} y(v)dv + b - \int_0^{\mathcal{T}} x(v)dv - b = \int_0^{\mathcal{T}} [y(s) - x(s)]dv = \int_0^{\mathcal{T}} z(v)dv \leq 0.$$

Therefore, by the Lemma 4.1, we must have $y(s) - x(s) \leq 0$, on Ω, i.e.$y(s) \leq x(s)$ on Ω. Therefore, we have $x(s) = y(s)$ on Ω. Then, the problem (1.1) possesses a solution.

Example: Consider the following problem

$$\begin{cases} D^{\sigma}q(s) &= e^{s\cos^2 q(s)} + \int_0^s q(v)dv + \dfrac{1}{2}\int_0^{ln2} q(v)dv, \quad s \in (0, ln2] \\ q(0) &= \mu \int_0^{\mathcal{T}} q(v)dv \end{cases} \quad (4.10)$$

Solution: Here, $D^{\sigma}q(t)$ is Riemann Liouville fractional derivative of order $0 < \sigma < 1$ with $0 \leq D^{\sigma}q(s) \leq e^{2s}$. Take $y_0(s) = e^{2s}$ and $x_0(s) = 0$ *on* Ω as upper and lower solutions of problem (4.10) respectively which guaranties existence of a solution by Theorem 4.2.

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