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Existence of three nontrivial solutions for an elastic beam equation via variational approach

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ABSTRACT: Based on the variational methods and critical-point theory, we are concerned with the existence results for a class of elastic beam differential equation with two parameters λ and μ.

Key Words: Multiple solutions, elastic beam equation, critical point theory, variational methods.

Contents

1. Introduction

The critical point theory has proved to be a very important and efficient tool in mathematics. It readily became one of fundamental building blocks of nonlinear analysis. Especially, the connection with the existence theory for differential equations has attracted a lot of attention. Now, it is a standard way of proving miscellaneous results such as bifurcation theorems, stability results in the theory of dynamical systems and many others (see, [5,7,9,10,11]).
In this article, employing two kinds of critical point theorems, we focus on the existence of solutions for the following problem

$$\begin{cases} u^{(4)}(x) + 2h(x)u'''(x) + \big(h^2(x) + h'(x)\big)u''(x) = \lambda f\big(x, u(x)\big), & x \in [0,1], \\ u(0) = u'(0) = u''(1) = 0, u'''(1) = \mu g\big(u(1)\big), \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\mu \in \mathbb{R}$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, is an L^1-Carathéodory function $g : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function and $h \in C^1[0,1]$ is non-negative.
The problem (1.1) with $h = 0$ describes the static equilibrium of an elastic beam which is fixed at the left end of $x = 0$ and is attached to a bearing device at the right end to $x = 1$, where the corresponding force of the bearing device is given by function g, the nonlinear term f is a continuous load which is attached to elastic beam.
In recent years, with the development of science and technology, more and more scholars are devoted to the study on the existence and multiplicity of solutions for these fourth-order ordinary differential equations. We refer the reader to [2,4,6,12,13] and references therein. In particular, in [4], Cabada et al. investigated the problem

$$\begin{cases} u^{(4)}(x) = f\big(x, u(x)\big), & x \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.2}$$

where the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. They studied the existence and multiplicity of solutions by using the critical point theorems. In [2], Bonanno et al. investigated the following problem

$$\begin{cases} u^{(4)}(x) = \lambda f\big(x, u(x)\big), & x \in [0,1], \\ u(0) = u'(0) = u''(1) = 0, & u'''(1) = \mu g\big(u(1)\big), \end{cases} \tag{1.3}$$

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where $\lambda > 0$, $\mu > 0$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. By using the variational methods, they established the existence result of solutions for problem (1.3).

In the present paper, motivated by the above works, employing two kinds of three critical points theorems obtained in [1,3], we establish the existence of at least three weak solutions for the problem (1.1).

The remaining part of this paper is organized as follows. Some fundamental facts will be given in Section 2 and the main results of this paper will be presented in Section 3.

2. Preliminaries

We begin describing the variational formulation of problem (1.1), which is based on the function space

$$X = \{u \in H^2(0,1) \ : \ u(0) = u'(0) = 0\},$$

where $H^2(0,1)$ is the Sobolev space of all functions $u : [0,1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2(0,1)$. Put

$$H(x) = \int_0^x h(\xi)d\xi, \qquad \text{for all } x \in [0,1].$$

It is not hard to show that X is a Hilbert space equipped with the inner product

$$\langle u, v\rangle = \int_0^1 e^{H(x)} u''(x)v''(x)dx,$$

and norm

$$\|u\|_X = \left(\int_0^1 e^{H(x)}|u''(x)|^2 dx\right)^{\frac{1}{2}},$$

which is equivalent to the usual norm

$$\|u\| = \left(\int_0^1 |u''(x)|^2 dx\right)^{\frac{1}{2}},$$

and

$$\|u\|_0 = \left(\int_0^1 (|u(x)|^2 + |u'(x)|^2 + |u''(x)|^2)dx\right)^{\frac{1}{2}}.$$

Remark 2.1 *For all $u \in X$*

$$\frac{1}{\sqrt{e^{H_1}}}\|u\|_X \leqslant \|u\| \leqslant \frac{1}{\sqrt{e^{H_0}}}\|u\|_X,$$

where $H_0 = \min_{x\in[0,1]} H(x)$, $H_1 = \max_{x\in[0,1]} H(x)$.

Lemma 2.1 *([8]) The embedding*

$$X \hookrightarrow C^1([0,1]),$$

is compact and

$$\|u\|_{C^1([0,1])} = \max\{\|u\|_\infty, \|u'\|_\infty\} \leqslant \|u\|,$$

for all $u \in X$.

Lemma 2.2 *([14]) The embedding*

$$X \hookrightarrow L^2([0,1]),$$

is compact and

$$\|u\|_2 \leqslant \frac{1}{2\sqrt{2}}\|u\|, \quad \|u'\|_2 \leqslant \frac{1}{\sqrt{2}}\|u\|,$$

for all $u \in X$, *where* $\|u\|_2 = \left(\int_0^1 |u(x)|^2 dx\right)^{\frac{1}{2}}$.

By Lemma 2.2, it is easy to obtain the following inequalities which is of independent interest.

Lemma 2.3 *For all* $u \in X$,

$$\frac{2\sqrt{2}}{\sqrt{13}}\|u\|_0 \leqslant \|u\| \leqslant \|u\|_0.$$

Here we prove that a weak solution of problem (1.1) is a function $u \in X$ such that

$$\int_0^1 e^{H(x)}u''(x)\nu''(x)dx - \lambda\int_0^1 e^{H(x)}f\big(x,u(x)\big)\nu(x)dx + \mu e^{H(1)}g\big(u(1)\big)\nu(1) = 0,$$

for every $\nu \in X$. To this end, an integration by parts gives

$$\begin{aligned}
&\int_0^1 e^{H(x)}u''(x)\nu''(x)dx\\
&= \int_0^1 e^{H(x)}u''(x)d\nu'(x)\\
&= e^{H(x)}u''(x)\nu'(x)|_{x=0}^{x=1} - \int_0^1 \big(e^{H(x)}u'''(x) + e^{H(x)}u''(x)h(x)\big)d\nu(x)\\
&= e^{H(x)}u''(x)\nu'(x)|_{x=0}^{x=1} - \big(e^{H(x)}u'''(x) + e^{H(x)}u''(x)h(x)\big)\nu(x)|_{x=0}^{x=1}\\
&+ \int_0^1 \big(e^{H(x)}u^{(4)}(x) + 2e^{H(x)}u'''(x)h(x) + e^{H(x)}u''(x)h^2(x) + e^{H(x)}u''(x)h'(x)\big)\nu(x)dx.
\end{aligned}$$

Then, by choosing $\nu \in H^2(0,1) \cap H_0^2(0,1)$, we have

$$\int_0^1 e^{H(x)}[u^{(4)}(x) + 2u'''(x)h(x) + u''(x)h^2(x) + u''(x)h'(x) - \lambda f(x,u(x)]\nu(x)dx = 0,$$

and

$$u^{(4)}(x) + 2u'''(x)h(x) + u''(x)h^2(x) + u''(x)h'(x) = \lambda f\big(x,u(x)\big), \quad \textit{a.e. on } [0,1].$$

Next, we show that u satisfies the boundary condition of problem (1.1). For any $\nu \in X$, we can obtain

$$u''(1)e^{H(1)}[\nu'(1) - h(1)\nu(1)] + [\mu g\big(u(1)\big) - u'''(1)]e^{H(1)}\nu(1) = 0.$$

Since $\nu(1)$ and $\nu'(1)$ are arbitrary, we deduce that $u''(1) = 0$, $u'''(1) = \mu g\big(u(1)\big)$. Therefore, u is a weak solution for the problem (1.1).
Now, for every $u \in X$, we define the functionals Φ and Ψ as follows

$$\begin{cases}\Phi(u) = \frac{1}{2}\|u\|_X^2,\\ \Psi(u) = \int_0^1 e^{H(x)}F\big(x,u(x)\big)dx - \frac{\mu}{\lambda}e^{H(1)}G\big(u(1)\big).\end{cases} \tag{2.1}$$

Here,

$$F(x,t)=\int_0^t f(x,\xi)d\xi, \quad \text{for all } (x,t)\in[0,1]\times\mathbb{R},$$
$$G(t)=\int_0^t g(\xi)d\xi, \quad \text{for all } t\in\mathbb{R}.$$

It is not hard to show that if $u\in X$ is a critical point of the variational functional $I_\lambda=\Phi-\lambda\Psi$, then it is a weak solution for the problem (1.1).
Our main tools, to prove our main results, are ([3], Theorem 3.3) and ([1], Theorem 2.6) that we recall here in a convenient form.
For all $r, r_1, r_2 > \inf_X \Phi$, $r_2 > r_1$, $r_3 > 0$, we define

$$\varphi(r)=\inf_{u\in\Phi^{-1}(-\infty,r)}\frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)-\Psi(u)}{r-\Phi(u)},$$

$$\beta(r_1,r_2)=\inf_{u\in\Phi^{-1}(-\infty,r_1)}\sup_{v\in\Phi^{-1}[r_1,r_2)}\frac{\Psi(v)-\Psi(u)}{\Phi(v)-\Phi(u)},$$

$$\gamma(r_2,r_3)=\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3},$$

and

$$\alpha(r_1,r_2,r_3)=\max\{\varphi(r_1),\varphi(r_2),\gamma(r_2,r_3)\}.$$

Theorem 2.1 *([3], Theorem 3.3). Let X be a reflexive real Banach space. Let $\Phi : X \longrightarrow \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*. Let $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*
(a_1) $\Phi(0)=\Psi(0)=0$.
(a_2) for every u_1, u_2 such that $\Psi(u_1)\geqslant 0$ and $\Psi(u_2)\geqslant 0$ one has

$$\inf_{t\in[0,1]}\Psi\big(tu_1+(1-t)u_2\big)\geqslant 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$ such that
(a_3) $\varphi(r_1)<\beta(r_1,r_2)$;
(a_4) $\varphi(r_2)<\beta(r_1,r_2)$;
(a_5) $\gamma(r_2,r_3)<\beta(r_1,r_2)$.
Then for each $\lambda\in\left]\frac{1}{\beta(r_1,r_2)},\frac{1}{\alpha(r_1,r_2,r_3)}\right[$, the functional $\Phi-\lambda\Psi$ admits at least three distinct critical points u_1, u_2, u_3 such that $u_1\in\Phi^{-1}(-\infty,r_1)$, $u_2\in\Phi^{-1}[r_1,r_2)$ and $u_3\in\Phi^{-1}(-\infty,r_2+r_3)$.

Theorem 2.2 *([1], Theorem 2.6). Let X be a reflexive real Banach space; $\Phi : X\to\mathbb{R}$ be a sequentially weakly lower semi-continuous, coercive and continuously Gateaux differentiable functional whose Gateaux derivative admits a continuous inverse on X^* and $\Psi : X\to\mathbb{R}$ be a continuously Gateaux differentiable functional whose Gateaux derivative is compact, such that*

$$\Phi(0)=\Psi(0)=0.$$

Assume that there exist $r>0$ and $\bar\nu\in X$, with $r<\Phi(\bar\nu)$ such that
(i) $\dfrac{\sup_{\Phi(u)\leqslant r}\Psi(u)}{r}<\dfrac{\Psi(\bar\nu)}{\Phi(\bar\nu)}$,
(ii) for each $\lambda\in\Lambda_r=\left]\frac{\Phi(\bar\nu)}{\Psi(\bar\nu)},\frac{r}{\sup_{\Phi(u)\leqslant r}\Psi(u)}\right[$ the functional $I_\lambda=\Phi-\lambda\Psi$ is coercive.
Then, for each $\lambda\in\Lambda_r$ the functional I_λ has at least three distinct critical points in X.

3. Main results

Now, we formulate our main results as follows.

Theorem 3.1 *Assume that there exist four positive constants c_1, d, c_2, c_3 with*

$$c_1 < d < \sqrt{\frac{e^{H_1}}{e^{H_0}}}d < c_2 < c_3,$$

such that

$$(A_1) \quad \max\left\{\frac{\int_0^1 F(x,c_1)dx}{c_1^2}, \frac{\int_0^1 F(x,c_2)dx}{c_2^2}, \frac{\int_0^1 F(x,c_3)dx}{c_3^2 - c_2^2}\right\} < \frac{(\frac{3}{2})^3 e^{H_1}\int_{3/4}^1 F(x,d)dx - \int_0^1 F(x,c_1)dx}{8\pi^4 d^2}.$$

Then for every

$$\lambda \in \left]\frac{4\pi^4 d^2(\frac{2}{3})^3 e^{H_1}}{\int_{3/4}^1 F(x,d)dx - \int_0^1 F(x,c_1)dx}, \min\left\{\frac{32\pi^4 e^{H_0}c_1^2}{27\int_0^1 F(x,c_1)dx}, \frac{32\pi^4 e^{H_0}c_2^2}{27\int_0^1 F(x,c_2)dx}, \frac{32\pi^4 e^{H_0}(c_3^2 - c_2^2)}{27\int_0^1 F(x,c_3)dx}\right\}\right[,$$

and for each non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta_{\lambda,g} > 0$ given by

$$\min\left\{\min\left\{\frac{32\pi^4 e^{H_0}c_1^2 - 27\lambda\int_0^1 F(x,c_1)dx}{G^{c_1}}, \frac{32\pi^4 e^{H_0}c_2^2 - 27\lambda\int_0^1 F(x,c_2)dx}{G^{c_2}},\right.\right.$$
$$\left.\frac{32\pi^4 e^{H_0}(c_3^2 - c_2^2) - 27\lambda\int_0^1 F(x,c_3)dx}{G^{c_3}}\right\},$$
$$\left.\frac{4\pi^4 d^2(\frac{2}{3})^3 e^{H_1} - \lambda\left(\int_{3/4}^1 F(x,d)dx - \int_0^1 F(x,c_1)dx\right)}{G^d - G^{c_1}}\right\},$$

such that for each $\mu \in [0, \delta_{\lambda,g}[$, problem (1.1) admits at least three weak solutions $u_i (i = 1, 2, 3)$ such that $\|u_i\|_\infty < c_3,\ i = 1, 2, 3$.

Proof: Our goal is to apply Theorem 2.1 to problem (1.1). To this end, fix λ and μ as in the conclusion. Now, let $X = \{u \in H^2(0,1) :\ u(0) = u'(0) = 0\}$ and take the functionals Φ, Ψ defined in (2.1). At first, we prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose Gateaux derivative at the point $u \in X$ is

$$\langle \Psi'(u), v\rangle = \int_0^1 e^{H(x)} f\big(x, u(x)\big)v(x)dx - \frac{\mu}{\lambda}e^{H(1)}g\big(u(1)\big)v(1),$$

for every $v \in X$, and $\Psi' : X \to X^*$ is compact. Indeed, it is enough to show that Ψ' is strongly continuous on X; see ([15], Proposition 26.2). For this end, fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$. Then $u_n(x)$ converges uniformly to $u(x)$ on $[0,1]$ as $n \to \infty$, since f, g are continuous, we have

$$e^{H(x)} f\big(x, u_n(x)\big) - \frac{\mu}{\lambda}e^{H(1)}g\big(u_n(1)\big) \to e^{H(x)} f\big(x, u(x)\big) - \frac{\mu}{\lambda}e^{H(1)}g\big(u(1)\big),$$

for every $x \in [0,1]$, as $n \to \infty$. Thus $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$ and we proved that Ψ' is a compact operator. Now, for every $u \in X$, we have

$$\frac{1}{2}\|u\|^2 e^{H_0} \leqslant \Phi(u) \leqslant \frac{1}{2}\|u\|^2 e^{H_1}, \tag{3.1}$$

Clearly, Φ is convex and coercive. Furthermore, Φ is continuously differentiable whose Gateaux derivative at the point $u \in X$ is

$$\langle \Phi'(u), v\rangle = \int_0^1 e^{H(x)} u''(x)v''(x)dx \quad \forall\ v \in X.$$

Also, $\Phi' : X \to X^*$ admits a continuous inverse on X^*. Indeed, for every $u \in E \setminus \{0\}$,

$$\lim_{\|u\|\to\infty} \frac{\langle \Phi'(u), u\rangle}{\|u\|} = \lim_{\|u\|\to\infty} \frac{\int_0^1 e^{H(x)} u''(x)u''(x)dx}{\|u\|}$$
$$\geqslant \lim_{\|u\|\to\infty} \frac{\|u\|^2}{2\|u\|} = \lim_{\|u\|\to\infty} \frac{1}{2}\|u\| = \infty.$$

So, $\Phi'(u)$ is coercive. Now, for any $u, v \in X$, one has

$$\langle \Phi'(u) - \Phi(v), u - v\rangle = \int_0^1 e^{H(x)}\big(u''(x) - v''(x)\big)\big(u''(x) - v''(x)\big)dx$$
$$\geqslant \frac{1}{2}\|u - v\|^2.$$

Thus, $\Phi'(u)$ is uniformly monotone. By ([15], Theorem 26.A), Φ'^{-1} exists and is continuous on X^*. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested of Theorem 2.1, are satisfied. Let

$$\bar{v}(x) = \begin{cases} 0, & x \in [0, 3/8], \\ d\cos^2(\frac{4\pi}{3}x), & x \in]3/8, 3/4[, \\ d, & x \in [3/4, 1]. \end{cases} \tag{3.2}$$

Simple calculations show that $\bar{v} \in X$ and $\|\bar{v}\|^2 = 8\pi^4 d^2(\frac{2}{3})^3$. It follows from (3.1) that

$$\frac{1}{2}\|\bar{v}\|^2 e^{H_0} \leqslant \Phi(\bar{v}) \leqslant \frac{1}{2}\|\bar{v}\|^2 e^{H_1},$$

hence

$$4\pi^4 d^2(\frac{2}{3})^3 e^{H_0} \leqslant \Phi(\bar{v}) \leqslant 4\pi^4 d^2(\frac{2}{3})^3 e^{H_1}. \tag{3.3}$$

Putting $r_1 = \frac{32}{27}\pi^4 e^{H_0} c_1^2$, $r_2 = \frac{32}{27}\pi^4 e^{H_0} c_2^2$, $r_3 = \frac{32}{27}\pi^4 e^{H_0}(c_3^2 - c_2^2)$, the condition

$$c_1 < d < \sqrt{\frac{e^{H_1}}{e^{H_0}}}d < c_2 < c_3,$$

in conjunction with (3.3) yields $r_1 < \Phi(\bar{v}) < r_2$, and also we observe that $r_3 > 0$. Since f and g are non-negative, we obtain

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)} \int_0^1 \Big[F\big(x, u(x)\big)dx - \frac{\mu}{\lambda}G\big(u(1)\big)\Big] \leqslant \int_0^1 \max F(x,t)dx - \frac{\mu}{\lambda}G^{c_1}$$
$$= \int_0^1 F(x, c_1)dx - \frac{\mu}{\lambda}G^{c_1}.$$

In a similar way, we get

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)} \int_0^1 \Big[F\big(x, u(x)\big)dx - \frac{\mu}{\lambda}G\big(u(1)\big)\Big] \leqslant \int_0^1 F(x, c_2)dx - \frac{\mu}{\lambda}G^{c_2},$$

and

$$\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \int_0^1 \Big[F\big(x, u(x)\big)dx - \frac{\mu}{\lambda}G\big(u(1)\big)\Big] \leqslant \int_0^1 F(x, c_3)dx - \frac{\mu}{\lambda}G^{c_3}.$$

Therefore, since $0 \in \Phi^{-1}(-\infty, r_1)$, one has

$$\varphi(r_1) \leqslant \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u)}{r_1} \leqslant \frac{\int_0^1 e^{H(x)}F(x,c_1)dx - \frac{\mu}{\lambda}e^{H(1)}G^{c_1}}{\frac{32}{27}\pi^4 e^{H_0}c_1^2},$$

$$\varphi(r_2) \leqslant \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u)}{r_2} \leqslant \frac{\int_0^1 e^{H(x)}F(x,c_2)dx - \frac{\mu}{\lambda}e^{H(1)}G^{c_2}}{\frac{32}{27}\pi^4 e^{H_0}c_2^2},$$

$$\gamma(r_2,r_3) \leqslant \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3} \leqslant \frac{\int_0^1 e^{H(x)}F(x,c_3)dx - \frac{\mu}{\lambda}e^{H(1)}G^{c_3}}{\frac{32}{27}\pi^4 e^{H_0}(c_3^2-c_2^2)}.$$

On the other hand,

$$\int_0^1 F\big(x,\bar{v}(x)\big) - \frac{\mu}{\lambda}G(d) \geqslant \int_{3/4}^1 F(x,d) - \frac{\mu}{\lambda}G(d).$$

For each $u \in \Phi^{-1}(-\infty, r_1)$, one has

$$\beta(r_1,r_2) \geqslant \frac{\int_{3/4}^1 F(x,d)dx - \frac{\mu}{\lambda}G(d) - \int_0^1 F(x,c_1)dx + \frac{\mu}{\lambda}G^{c_1}}{4\pi^4 d^2(\frac{2}{3})^3 e^{H_1}}.$$

From (A_1), we obtain

$$\alpha(r_1,r_2,r_3) < \beta(r_1,r_2).$$

Finally, we verify that Ψ satisfies the assumption (a_2) of Theorem 2.1. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions of problem (1.1). Since f and g are non-negative, for fixed $\lambda > 0$ and $\mu \geqslant 0$, we have $(\lambda f + \mu g)\big(x, su_1 + (1-s)u_2\big) \geqslant 0$ for all $s \in [0,1]$. Consequently, $\Psi\big(su_1 + (1-s)u_2\big) \geqslant 0$, for every $s \in [0,1]$. Hence, Theorem 2.1 implies that for every

$$\lambda \in \Bigg]\frac{4\pi^4 d^2(\frac{2}{3})^3 e^{H_1}}{\int_{3/4}^1 F(x,d)dx - \int_0^1 F(x,c_1)dx}, \min\left\{\frac{32\pi^4 e^{H_0}c_1^2}{27\int_0^1 F(x,c_1)dx}, \frac{32\pi^4 e^{H_0}c_2^2}{27\int_0^1 F(x,c_2)dx}, \frac{32\pi^4 e^{H_0}(c_3^2-c_2^2)}{27\int_0^1 F(x,c_3)dx}\right\}\Bigg[,$$

and $\mu \in [0, \delta_{\lambda,g})$, the functional $I_\lambda = \Phi - \lambda\Psi$ has three critical points u_i, $i = 1,2,3$, in X such that $\Phi(u_i) < r_1 + r_2$, that is, $\|u_i\|_\infty < c_3$, $i = 1,2,3$. Then, taking into account the fact that the weak solutions of problem (1.1) are exactly critical points of the functional I_λ, we have the desired conclusion. □

Theorem 3.2 *Assume that:*

(C_1) $F(x,t) \geqslant 0$, *for all* $(x,t) \in \big([0,1/3] \cup [2/3,1]\big) \times \mathbb{R}$;

(C_2) $e^{2(H_0-H_1)} \int_{1/3}^{2/3} F(x,d)dx > \frac{351}{350} \int_0^1 \max\limits_{|t|\leqslant\sqrt{350}d} F(x,t)dx$, *for some* $d > 0$;

(C_3) *there exist* $\varepsilon, h > 0$ *such that* $\varepsilon < \frac{8}{350} \int_0^1 \max\limits_{|t|\leqslant\sqrt{350}d} F(x,t)dx$ *and*

$$F(x,t) \leqslant \varepsilon t^2 + h, \text{ for all } (x,t) \in [0,1] \times \mathbb{R};$$

(C_4) there exist $a, b, c > 0$ such that

$$g(x) \leqslant a + b|x|^c, \text{ for all } x \in \mathbb{R}.$$

Then, for every $\lambda \in \frac{1}{2}\left]\frac{351e^{(H_1-H_0)}}{\int_{1/3}^{2/3} F(x,d)dx}, \frac{350e^{(H_0-H_1)}}{\int_0^1 \max_{|t|\leqslant\sqrt{350}d} F(x,t)dx}\right[$ *and for each* $\mu > 0$, *problem* (1.1) *has at least three distinct, weak solutions in* X.

Proof: We will apply Theorem 2.2. Let $X = \{u \in H^2(0,1) : \ u(0) = u'(0) = 0\}$ and take the functionals Φ, Ψ defined in (2.1). It is well-known that the regularity assumption on Φ and Ψ, as required in Theorem 2.2, are verified. We now look on the existence of at least three distinct critical points of the functional $I_\lambda : \Phi - \lambda\Psi$ in X.
First of all, $\Phi(0) = \Psi(0) = 0$. Put

$$w(x) = \begin{cases} -54d(x^3 - \frac{1}{2}x^2), & x \in [0, 1/3], \\ d, & x \in]1/3, 2/3], \\ \frac{9}{2}d(x - x^2), & x \in]2/3, 1]. \end{cases}$$

Clearly, $w \in X$. It is easy to verify that

$$\begin{aligned}\Phi(w) &= \frac{1}{2}\int_0^1 e^{H(x)}|w''(x)|^2 dx \\ &\geqslant \frac{1}{2}e^{H_0}\int_0^1 |w''(x)|^2 dx \\ &= \frac{351}{2}d^2 e^{H_0}.\end{aligned}$$

Now, put $r = 175d^2e^{H_0}$. This ensures $0 < r < \Phi(w)$. Moreover, taking Remark 2.1 and Lemma 2.1 into account, we see that

$$\begin{aligned}\Phi^{-1}(]-\infty, r]) &= \{u \in X \ : \ \Phi(u) \leqslant r\} \\ &= \{u \in X \ : \ \frac{1}{2}\|u\|_X^2 \leqslant r\} \\ &\subseteq \{u \in X \ : \ |u(x)| \leqslant \sqrt{350}d \text{ for each } x \in [0,1]\}.\end{aligned}$$

So, from (C_4), we get

$$\begin{aligned}\sup_{\Phi(u)\leqslant r}\Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r])}\left[\int_0^1 e^{H(x)}F\big(x,u(x)\big)dx - \frac{\mu}{\lambda}e^{H(1)}G\big(u(1)\big)\right] \\ &\leqslant \int_0^1 e^{H(x)}\max_{|t|\leqslant\sqrt{350}d}F(x,t)dx - \frac{\mu}{\lambda}e^{H(1)}\left[\sqrt{350}ad + \frac{b}{c+1}(\sqrt{350}d)^{c+1}\right] \\ &< \int_0^1 e^{H(x)}\max_{|t|\leqslant\sqrt{350}d}F(x,t)dx \\ &< e^{H_1}\int_0^1 \max_{|t|\leqslant\sqrt{350}d}F(x,t)dx.\end{aligned}$$

Consequently

$$\frac{\sup_{\Phi(u)<r}\Psi(u)}{r} < \frac{2}{350d^2}e^{(H_1-H_0)}\int_0^1 \max_{|t|\leqslant\sqrt{350}d}F(x,t)dx. \tag{3.4}$$

On the other hard, from (C_1) and definition of the function G and w, we have

$$\begin{aligned}\Psi(w) &= \int_0^1 e^{H(x)}F\big(x,w(x)\big)dx - \frac{\mu}{\lambda}e^{H(1)}G\big(w(1)\big)\\ &\geqslant \int_0^1 e^{H(x)}F\big(x,w(x)\big)dx\\ &\geqslant e^{H_0}\int_{1/3}^{2/3}F(x,d)dx,\end{aligned}$$

and

$$\begin{aligned}\Phi(w) &= \frac{1}{2}\int_0^1 e^{H(x)}|w''(x)|^2dx\\ &\leqslant \frac{1}{2}e^{H_1}\int_0^1 |w''(x)|^2dx\\ &= \frac{351}{2}d^2e^{H_1}.\end{aligned}$$

Consequently

$$\frac{\Psi(w)}{\Phi(w)} \geqslant \frac{2}{351d^2}e^{(H_0-H_1)}\int_{1/3}^{2/3}F(x,d)dx. \tag{3.5}$$

Therefore, using (3.4), (3.5) and (C_2), we see that the condition (i) of Theorem 2.2 is fulfilled. Now, fixed $\lambda \in \frac{1}{2}\Bigg]\frac{351e^{(H_1-H_0)}}{\int_{1/3}^{2/3}F(x,d)dx}, \frac{350e^{(H_0-H_1)}}{\int_0^1 \max_{|t|\leqslant\sqrt{350}d}F(x,t)dx}\Bigg[$, and let $\mu > 0$. By using (C_3), bearing in mind Remark 2.1 and Lemma 2.2, one has

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|_X^2 - \lambda\int_0^1 e^{H(x)}F\big(x,u(x)\big)dx + \mu e^{H(1)}G\big(u(1)\big)\\ &\geqslant \frac{1}{2}\|u\|_X^2 - \lambda\int_0^1 e^{H(x)}F\big(x,u(x)\big)dx\\ &\geqslant \frac{1}{2}\|u\|_X^2 - \lambda e^{H_1}\int_0^1 F\big(x,u(x)\big)dx\\ &\geqslant \frac{1}{2}\|u\|_X^2 - \lambda\varepsilon e^{H_1}\|u\|^2_{L^2([0,1])} - \lambda h e^{H_1}\\ &\geqslant \frac{1}{2}\|u\|_X^2 - \frac{\lambda\varepsilon e^{H_1}}{8e^{H_0}}\|u\|_X^2 - \lambda h e^{H_1}\\ &= \left(\frac{1}{2} - \frac{\lambda\varepsilon e^{H_1}}{8e^{H_0}}\right)\|u\|_X^2 - \lambda h e^{H_1}.\end{aligned}$$

Since $\varepsilon < \frac{8}{350}\int_0^1 \max_{|t|\leqslant\sqrt{350}d}F(x,t)dx$, this follows $\lim_{\|u\|_X\to\infty} I_\lambda(u) = +\infty$, which means the functional I_λ is coercive, and the condition (ii) of Theorem 2.2 is satisfied. From (3.4) and (3.5) one also has $\lambda \in \Bigg]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leqslant r}\Psi(u)}\Bigg[$. Finally, since the weak solutions of the problem (1.1) are exactly the critical points of the functional I_λ, Theorem 2.2 (with $\bar{\nu} = w$) concludes the result. □

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