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The Strongly $(M, N)_h$-Convex Functions and Hadamard Inequalities

Lakhlifa Sadek

ABSTRACT: In this study, we introduce the concept of highly $(M, N)_h$-convex functions with respect to a parameter $\alpha > 0$ and explore their properties. These functions exhibit certain distinct characteristics which we investigate thoroughly. Drawing inspiration from the notion of strongly $(M, N)_h$-convex functions, we provide a framework for characterizing inner product spaces (IPS). By leveraging the concept of strongly $(M, N)_h$-convex functions, we present insightful results that shed light on the structure and properties of IPS. Furthermore, we establish that strongly $(M, N)_h$-convex functions satisfy Hadamard inequalities, which offer valuable insights into their behavior and relationships with other mathematical structures. These inequalities provide bounds and constraints that contribute to a deeper understanding of the geometric and analytical properties of strongly $(M, N)_h$-convex functions. Overall, our study not only introduces and investigates the properties of highly $(M, N)_h$-convex functions but also demonstrates their utility in characterizing inner product spaces and establishing fundamental inequalities, thereby contributing to the advancement of mathematical theory and its applications.

Key Words: $(M, N)_h$-convex functions ($(M, N)_h$-CF), h-convex functions (h-CF), convex functions (CF), strongly $(M, N)_h$-convex functions (S-$(M, N)_h$-CF), Hadamard inequalities (HI).

Contents

1. Introduction

The HI's [1] for CF's state that if $\psi : J \subset \mathbb{R} \to \mathbb{R}$ is a CF on J and $k_1, k_2 \in J$ with $k_1 < k_2$, then

$$\psi\left(\frac{k_1 + k_2}{2}\right) \leq \frac{1}{k_2 - k_1}\int_{k_1}^{k_2} \psi(\tau)d\tau \leq \frac{\psi(k_1) + \psi(k_2)}{2}. \tag{1.1}$$

In the last decades, inequality (1.1) has evoked the interest of many mathematicians. Numerous generalizations of this inequality have been obtained see [2,3,4,5,6,7,8,9,10,11].

Let the function $h : (0, 1) \to (0, \infty)$. A function ψ is said to be h-CF [12] on J if

$$\psi(\beta t_1 + (1 - \beta)t_2) \leq h(\beta)\psi(t_1) + h(1 - \beta)\psi(t_2), \forall t_1, t_2 \in J, \forall \beta \in (0, 1). \tag{1.2}$$

In the specific case:

- If $h(k) = k$, the function ψ on J is convex functions.
- If $h(k) = k^s$, the function ψ on J is s-convex functions.
- If $h(k) = \frac{1}{k}$, the function ψ on J is Gudunova-Levin functions.
- If $h(k) = 1$, the function ψ on J is P-functions.

For the properties of them can be found see [13,14,15,16,17].

Let us consider the functions $M, N : [k_1, k_2] \to [k_1, k_2]$ with $[k_1, k_2] \subset \mathbb{R}$. Youness [18] defined the M-CF:

Definition 1.1 *Let $\psi : [k_1, k_2] \to \mathbb{R}$. the function ψ is said to be M-CF on $[k_1, k_2]$ if*

$$\psi(\beta M(t_1) + (1 - \beta)M(t_2)) \leq \beta\psi(M(t_1)) + (1 - \beta)\psi(M(t_2)), \forall t_1 \in [k_1, k_2], t_2 \in [k_1, k_2], \forall \beta \in (0, 1).$$

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If $M(t) = t$ for all $t \in [k_1, k_2]$, then the classical convexity. For the properties of the M-CF's can be found see [19,20].

Saleh [21] defined the (M, N)-CF's:

Definition 1.2 *Let the function $\psi : [k_1, k_2] \to \mathbb{R}$. ψ is said to be (M, N)-CF on $[k_1, k_2]$ if*

$$\psi(\beta M(t_1) + (1-\beta)N(t_2)) \leq \beta\psi(M(t_1)) + (1-\beta)\psi(N(t_2)), \forall t_1 \in [k_1, k_2], t_2 \in [k_1, k_2], \beta \in [0, 1].$$

Definition 1.3 *The ψ is called strongly CF on J with modulus $\alpha > 0$, if*

$$\psi(\beta t_1 + (1-\beta)t_2) \leq \beta\psi(t_1) + (1-\beta)\psi(t_2) - \alpha\beta(1-\beta)(t_1 - t_2)^2,$$

for all $t_1, t_2 \in J$ and $\beta \in (0, 1)$.

The function ψ is called strongly h-CF [14] with modulus $\alpha > 0$, if

$$\psi(\beta t_1 + (1-\beta)t_2) \leq h(\beta)\psi(t_1) + h(1-\beta)\psi(t_2) - \alpha\beta(1-\beta)(t_1 - t_2)^2.$$

In [22] have been introduced strongly convex functions, they play an important role in mathematical economics and optimization theory. (see [14,15,22,23,24]).

In this study, we define S-$(M, N)_h$-CF's defined in normed spaces and discuss some of their characteristics. We provide a representation of highly $(M, N)_h$-CF's in IPS, and among normed spaces, we give a characterization of IPS that includes the concept of S-$(M, N)_h$-CF. Inequalities of the HI are introduced for S-$(M, N)_h$-CF's as a final section. The HI for S-$(M, N)_h$-CF's are generalized by this result.

2. Main result

The following uses $(E, \|.\|)$ to symbolize a real normed space, F to represent a (M, N)-convex subset of E (Definition 2.1), $M, N : F \to F$, $h : (0, 1) \to (0, \infty)$ are provided functions, and α is a positive constant.

Definition 2.1 *F is called (M, N)-convex set if*

$$\beta M(t_1) + (1-\beta)N(t_2) \in F, \forall t_1, t_2 \in F, \forall \beta \in [0, 1].$$

Definition 2.2 *Let the function $\psi : F \to [0, \infty)$. ψ is S-$(M, N)_h$-CF on F with modulus α if*

$$\psi(\beta M(t_1) + (1-\beta)N(t_2)) \leq h(\beta)\psi(M(t_1)) + h(1-\beta)\psi(N(t_2)) - \alpha\beta(1-\beta)\|M(t_1) - N(t_2)\|^2, \quad (2.1)$$

for all $t_1, t_2 \in F$ and $\beta \in (0, 1)$.

In the specific case:

- If $h(t) = t$, the function ψ on F is strongly (M, N)-convex functions.
- If $h(t) = t^s, (s \in (0, 1))$, the function ψ on F is strongly $(M, N)_s$-convex functions.
- If $h(t) = \frac{1}{t}$, the function ψ on F is strongly (M, N)-Gudunova-Levin functions.
- If $h(t) = 1$, the function ψ on F is (M, N)-P-functions.
- If $\alpha = 0$, the function ψ on F is $(M, N)_h$-convex function.

Remark 2.1 *Assume that $h(\beta) \geq \beta$ for all $\beta \in (0, 1)$. If ψ is S-(M, N)-CF on J, then for $t_1, t_2 \in J$ and $\beta \in (0, 1)$, we have*

$$\begin{aligned}\psi(\beta M(t_1) + (1-\beta)N(t_2)) &\leq \beta\psi(M(t_1)) + (1-\beta)\psi(N(t_2)) - \alpha\beta(1-\beta)\|M(t_1) - N(t_2)\|^2 \\ &\leq h(\beta)\psi(M(t_1)) + h(1-\beta)\psi(N(t_2)) - \alpha\beta(1-\beta)\|M(t_1) - N(t_2)\|^2,\end{aligned}$$

so ψ is S-$(M, N)_h$-CF on J.

Lemma 2.1 *Let $h_1, h_2 : (0,1) \to (0,\infty)$ be given functions such that $h_2(\beta) \le h_1(\beta)$ for all $\beta \in (0,1)$. If ψ is S-$(M,N)_{h_2}$-CF on J, then ψ is S-$(M,N)_{h_1}$-CF on J.*

Proof: Since ψ is S-$(M,N)_{h_2}$-CF on J, thus for $t_1, t_2 \in J$ and $\beta \in (0,1)$, so

$$\begin{aligned}\psi(\beta M(t_1) + (1-\beta)N(t_2)) &\le h_2(\beta)\psi(M(t_1)) + h_2(1-\beta)\psi(N(t_2)) - \alpha\beta(1-\beta)\|M(t_1) - N(t_2)\|^2 \\ &\le h_1(\beta)\psi(M(t_1)) + h_1(1-\beta)\psi(N(t_2)) - \alpha\beta(1-\beta)\|M(t_1) - N(t_2)\|^2.\end{aligned}$$

□

Lemma 2.2 *If $\psi_1, \psi_2 : J \to [0,\infty)$ are S-$(M,N)_h$-CF's on J and $\alpha > 0$, then for all $\beta \in (0,1)$, $\psi_1 + \psi_2$ and $\alpha\psi_1$ are S-$(M,N)_h$-CF on J.*

Proof: The proof is evident from the definition of highly S-$(M,N)_h$-convexity. □

The following lemma which gives some relationships between S-$(M,N)_h$-CF's and $(M,N)_h$-CF's in the case where E is a real IPS (that is, the norm $\|.\|$ is induced by an inner product: $\|t\| :=< t \mid t >$).

Lemma 2.3 *Let $(E, \|\cdot\|)$ be a IPS, F be a (M,N)-convex subset of E and $\alpha \in \mathbb{R}^+$.*

(i) If $h(\beta) \le \beta, \beta \in (0,1)$ and $\psi : F \to (0,\infty)$ is S-$(M,N)_h$-CF with modulus α, then the function $\varphi = \psi - \alpha\|\cdot\|^2$ is $(M,N)_h$-CF.

(ii) If $h(\beta) \le \beta, \beta \in (0,1)$ and $\varphi = \psi - \alpha\|\cdot\|^2$ is $(M,N)_h$-CF, then the function $\psi : F \to (0,\infty)$ is S-(M,N)-CF with modulus α.

(iii) If $h(\beta) \ge \beta, \beta \in (0,1)$ and $\psi : F \to (0,\infty)$ is S-$(M,N)_h$-CF with modulus α, then the function $\varphi = \psi - \alpha\|\cdot\|^2$ is $(M,N)_h$-CF.

Proof: (i) ψ is S-$(M,N)_h$-CF with modulus α. Using properties of the inner product and assumption $h(\beta) \le \beta, \beta \in (0,1)$, we have

$$\begin{aligned}&\varphi(\beta M(t_1) + (1-\beta)N(t_2)) \\ &= \psi(\beta M(t_1) + (1-\beta)N(t_2)) - \alpha\|\beta M(t_1) + (1-\beta)N(t_2)\|^2 \\ &\le h(\beta)\psi(M(t_1)) + h(1-\beta)\psi(N(t_2)) - \alpha\beta(1-\beta)\|M(t_1) - N(t_2)\|^2 - \alpha\|\beta M(t_1) + (1-\beta)N(t_2)\|^2 \\ &\le h(\beta)\psi(M(t_1)) + h(1-\beta)\psi(N(t_2)) - \alpha\left(\beta(1-\beta)\left[\|M(t_1)\|^2 - 2 < M(t_1) \mid N(t_2) > + \|N(t_2)\|^2\right]\right. \\ &+ \left.\left[\beta^2\|M(t_1)\|^2 + 2\beta(1-\beta) < M(t_1) \mid N(t_2) > + (1-\beta)^2\|N(t_2)\|^2\right]\right) \\ &= h(\beta)\psi(M(t_1)) + h(1-\beta)\psi(N(t_2)) - \alpha\beta\|M(t_1)\|^2 - \alpha(1-\beta)\|N(t_2)\|^2 \\ &\le h(\beta)\psi(M(t_1)) + h(1-\beta)\psi(N(t_2)) - \alpha h(\beta)\|M(t_1)\|^2 - \alpha h(1-\beta)\|N(t_2)\|^2 \\ &= h(\beta)\varphi(M(t_1)) + h(1-\beta)\varphi(N(t_2)),\end{aligned}$$

which gives that φ is a $(M,N)_h$-CF on D.

(ii) Since φ is a $(M,N)_h$-CF, and using the assumption $h(\beta) \le \beta, \beta \in (0,1)$, we get

$$\begin{aligned}\psi(\beta M(t_1) + (1-\beta)N(t_2)) =& \varphi(\beta M(t_1) + (1-\beta)N(t_2)) + \alpha\|\beta M(t_1) + (1-\beta)N(t_2)\|^2 \\ \le& h(\beta)\varphi(M(t_1)) + h(1-\beta)\varphi(N(t_2)) \\ &+ \alpha\left(\beta^2\|M(t_1)\|^2 + 2\beta(1-\beta) < M(t_1) \mid N(t_2) > + (1-\beta)^2\|N(t_2)\|^2\right) \\ \le& \beta\left[\varphi(M(t_1)) + \alpha\|M(t_1)\|^2\right] + (1-\beta)\left[\varphi(N(t_2)) + \alpha\|N(t_2)\|^2\right] \\ &- \alpha\beta(1-\beta)\left[\|M(t_1)\|^2 - 2 < M(t_1) \mid N(t_2) > + \|N(t_2)\|^2\right] \\ =& \beta\psi(M(t_1)) + (1-\beta)\psi(N(t_2)) - \alpha\beta(1-\beta)\|M(t_1) - N(t_2)\|^2,\end{aligned}$$

which shows that ψ is S-(M,N)-CF with modulus α.

(iii) Similar to that, we can demonstrate it. $\square$

The example that follows demonstrates how crucial it is for the above lemma that E be assumed to be an inner product space.

Exapmle 1 *Let the function h such that $h(\beta)=\beta$ for all $\beta\in(0,1)$ and $E=\mathbb{R}^2$. Let the functions $M,N:\mathbb{R}^2\to\mathbb{R}^2$ defined by $M(t)=t$ and $N(t)=t$ for every $t\in\mathbb{R}^2$ and*

$$\|t\|=\max\{|t_1|,|t_2|\},$$

for $t=(t_1,t_2)$. Let $\psi=\|\cdot\|^2$. Then $\varphi=\psi-\|\cdot\|^2$ is $(M,N)_h$-CF. However, ψ is not S-$(M,N)_h$-CF with modulus 1. Indeed, for $t_1=(1,0)$ and $t_2=(0,1)$, so

$$\psi\left(\frac{t_1+t_2}{2}\right)=\frac{1}{2}\geq\frac{3}{4}=\frac{\psi(t_1)+\psi(t_2)}{2}-\frac{1}{4}\|t_1-t_2\|^2,$$

which contradicts (2.1).

The assumption that E is an IPS in the Lemma 2.3 is essential. Moreover, it appears that the fact that for every $(M,N)_h$-CF $\varphi:E\to\mathbb{R}$ the function $\psi=\varphi+\alpha\|.\|^2$ is S-$(M,N)_h$-CF characterizes IPS among normed spaces. Similar characterizations of IPS by strongly CF, strongly-h-CF and strongly M-CF's are presented in [14,15,24], respectively.

Theorem 2.1 *Let $(E,\|.\|)$ be a normed space, $F\subset E$ (M,N)-CF and assume that $h\left(\frac{1}{2}\right)=\frac{1}{2}$. Then the following conditions are equivalent.*

i) *$(E,\|\cdot\|)$ is a IPS.*

ii) *$\forall\alpha>0, h(\beta)\geq\beta,\beta\in(0,1)$, and for every $(M,N)_h$-CF $\varphi:F\to(0,\infty)$ defined on F, $\psi=\varphi+\alpha\|\cdot\|^2$ is S-$(M,N)_h$-CF with modulus α.*

iii) *$\|\cdot\|^2:E\to(0,\infty)$ is S-$(M,N)_h$-CF with modulus 1.*

Proof: From Lemma 2.3 we have The implication i) $\Rightarrow$ ii). To see that ii) $\Rightarrow$ iii) take $\varphi=0$.

Clearly, φ is $(M,N)_h$-CF, whence $\psi=\alpha\|.\|^2$ is S-$(M,N)_h$-CF with modulus α.

Consequently, $\|.\|^2$ is S-$(M,N)_h$-CF with modulus 1. Finally, to prove iii) $\Rightarrow$ i) observe that by the S-$(M,N)_h$-CF of $\|.\|^2$ and assumption $h\left(\frac{1}{2}\right)=\frac{1}{2}$, we obtain

$$\left\|\frac{M(t_1)+N(t_2)}{2}\right\|^2\leq\frac{\|M(t_1)\|^2}{2}+\frac{\|N(t_2)\|^2}{2}-\frac{1}{4}\|M(t_1)+N(t_2)\|^2,$$

and hence

$$\|M(t_1)+N(t_2)\|^2\leq 2\|M(t_1)\|^2+2\|N(t_2)\|^2, \tag{2.2}$$

for all $t_1,t_2\in E$. Now, putting $x=M(t_1)+N(t_2)$ and $y=M(t_1)-N(t_2)$ in (2.2), so

$$2\|x\|^2+2\|y\|^2\leq\|x+y\|^2+\|x-y\|^2. \tag{2.3}$$

From the conditions (2.2) and (2.3) mean that the norm $\|\cdot\|^2$ satisfies the parallelogram law, which implies, by the Jordan-Von Neumann theorem, that $(E,\|\cdot\|)$ is an IPS. $\square$

For S-$(M,N)_h$-CF's with modulus α, we now provide new HI as follows:

Theorem 2.2 *Let $M,N:[k_1,k_2]\to[k_1,k_2]$ the continuous functions, let the function $\psi:J\to(0,\infty)$ Lebesgue integrable and S-$(M,N)_h$-CF with modulus $\alpha>0$, then*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)}\psi\left(\frac{M(k_1)+N(k_2)}{2}\right)+\frac{\alpha}{24h\left(\frac{1}{2}\right)}(M(k_1)-N(k_2))^2\\
&\leq \frac{1}{N(k_2)-M(k_1)}\int_{M(k_1)}^{N(k_2)} f(\tau)d\tau \\
&\leq [\psi(M(k_1))+\psi(N(k_2))]\int_0^1 h(\beta)d\beta-\frac{\alpha}{6}(M(k_1)-N(k_2))^2.
\end{aligned} \tag{2.4}$$

Proof: Let ψ S-$(M,N)_h$-CF, so

$$\begin{aligned}
\psi\left(\frac{M(k_1)+N(k_2)}{2}\right)=&\psi\left(\frac{\beta M(k_1)+(1-\beta)N(k_2)}{2}+\frac{(1-\beta)M(k_1)+\beta N(k_2)}{2}\right)\\
\leq& h\left(\frac{1}{2}\right)[f(\beta M(k_1)+(1-\beta)N(k_2))+\psi((1-\beta)M(k_1)+\beta N(k_2))]\\
&-\frac{\alpha}{4}(1-2\beta)^2(M(k_1)-N(k_2))^2.
\end{aligned}$$

When we integrate the aforementioned inequality throughout the range $(0,1)$, so

$$\begin{aligned}
&\psi\left(\frac{M(k_1)+N(k_2)}{2}\right)+\frac{\alpha}{12}(M(k_1)-N(k_2))^2\\
&\leq h\left(\frac{1}{2}\right)\left[\int_0^1\psi(\beta M(k_1)+(1-\beta)N(k_2))d\beta+\int_0^1\psi((1-\beta)M(k_1)+\beta N(k_2))d\beta\right].
\end{aligned}$$

we substitute $t=\beta M(k_1)+(1-\beta)N(k_2)$ and $t=(1-\beta)M(k_1)+\beta N(k_2)$, the first and second integral respectfully, so

$$\psi\left(\frac{M(k_1)+N(k_2)}{2}\right)+\frac{\alpha}{12}(M(k_1)-N(k_2))^2\leq\frac{2h\left(\frac{1}{2}\right)}{N(k_2)-M(k_1)}\int_{M(k_1)}^{N(k_2)}\psi(t)dt.$$

To demonstrate the second inequality, we begin by assuming that ψ possesses S-$(M,N)_h$-CF, which means that for each $\beta\in(0,1)$, one has

$$\psi(\beta M(k_1)+(1-\beta)N(k_2))\leq h(\beta)\psi(M(k_1))+h(1-\beta)\psi(N(k_2))-\alpha\beta(1-\beta)(M(k_1)-N(k_2))^2.$$

When we integrate the aforementioned inequality throughout the range $(0,1)$, we get

$$\int_0^1\psi(\beta M(k_1)+(1-\beta)N(k_2))d\beta\leq[\psi(M(k_1))+\psi(N(k_2))]\int_0^1 h(\beta)d\beta-\alpha(M(k_1)-N(k_2))^2\int_0^1\beta(1-\beta)d\beta.$$

The prior replacement in this inequality's first side results in

$$\frac{1}{(M(k_1)-N(k_2))}\int_{N(k_2)}^{M(k_1)}\psi(t)dt\leq[\psi(M(k_1))+\psi(N(k_2))]\int_0^1 h(\beta)d\beta-\frac{\alpha}{6}(M(k_1)-N(k_2))^2,$$

the second inequality of which (2.4). □

Remark 2.2 *If $h(\beta)=\beta,\beta\in(0,1)$ and $M=N$, then the inequalities (2.4) coincide with the HI for strongly M-CF's proved by Sarikaya in [15].*

Corollary 2.1 *Let $M, N : [k_1, k_2] \to [k_1, k_2]$ the continuous functions, let the function $\psi : J \to (0, \infty)$ Lebesgue integrable and S-$(M, N)_h$-CF with modulus $\alpha > 0$, with $h(\beta) = \beta^s (s \in (0, 1)), \beta \in (0, 1)$, we have*

$$2^{s-1}\psi\left(\frac{M(k_1)+N(k_2)}{2}\right) + \frac{\alpha 2^s}{24}(M(k_1)-N(k_2))^2 \leq \frac{1}{N(k_2)-M(k_1)}\int_{M(k_1)}^{N(k_2)} \psi(t)\mathrm{d}t$$
$$\leq \frac{\psi(M(k_1))+\psi(N(k_2))}{s+1} - \frac{\alpha}{6}(M(k_1)-N(k_2))^2.$$

These inequalities are associated HI's for S-$(M, N)_s$-CF's.

Corollary 2.2 *Let $M, N : [k_1, k_2] \to [k_1, k_2]$ the continuous functions, let the function $\psi : J \to (0, \infty)$ Lebesgue integrable and S-$(M, N)_h$-CF with modulus $\alpha > 0$, with $h(\beta) = \frac{1}{\beta}, \beta \in (0, 1)$, we have*

$$\frac{1}{4}\psi\left(\frac{M(k_1)+N(k_2)}{2}\right) + \frac{\alpha}{48}(M(k_1)-N(k_2))^2 \leq \frac{1}{N(k_2)-M(k_1)}\int_{M(k_1)}^{N(k_2)} \psi(t)\mathrm{d}t.$$

This inequality is associated HI's for S-(M, N)-Godunova-Levin functions.

Corollary 2.3 *Let $M, N : [k_1, k_2] \to [k_1, k_2]$ the continuous functions, let the function $\psi : J \to (0, \infty)$ Lebesgue integrable and S-$(M, N)_h$-CF with modulus $\alpha > 0$, with $h(\beta) = 1, \beta \in (0, 1)$, we have*

$$\frac{1}{2}\psi\left(\frac{M(k_1)+N(k_2)}{2}\right) + \frac{\alpha}{24}(M(k_1)-N(k_2))^2 \leq \frac{1}{N(k_2)-M(k_1)}\int_{M(k_1)}^{N(k_2)} \psi(t)\mathrm{d}t$$
$$\leq \psi(M(k_1)) + \psi(N(k_2)) - \frac{\alpha}{6}(M(k_1)-N(k_2))^2.$$

These inequalities are associated HI's for S-(M, N)-P-convex functions.

Theorem 2.3 *Let $M, N : [k_1, k_2] \to [k_1, k_2]$ the continuous functions, let the function $\psi : J \to (0, \infty)$ Lebesgue integrable and S-$(M, N)_h$-CF with modulus α, then*

$$\begin{aligned}&\frac{1}{N(k_2)-M(k_1)}\int_{M(k_1)}^{N(k_2)} \psi(t)\psi(k_1+k_2-t)\mathrm{d}t \\ &\leq \left[\psi^2(M(k_1)) + \psi^2(N(k_2))\right]\int_0^1 h(\beta)h(1-\beta)d\beta + 2\psi(M(k_1))\psi(N(k_2))\int_0^1 h^2(\beta)\mathrm{d}\beta \\ &- 2\alpha(M(k_1)-N(k_2))^2[\psi(M(k_1))+\psi(N(k_2))]\int_0^1 \beta(1-\beta)h(\beta)\mathrm{d}\beta + \frac{\alpha^2}{30}(M(k_1)-N(k_2))^4.\end{aligned} \tag{2.5}$$

Proof: ψ S-$(M, N)_h$-CF, so that for all $\beta \in (0, 1)$

$$\psi(\beta M(k_1) + (1-\beta)N(k_2)) \leq h(\beta)\psi(M(k_1)) + h(1-\beta)\psi(N(k_2)) - \alpha\beta(1-\beta)(M(k_1)-N(k_2))^2, \tag{2.6}$$

and

$$\psi((1-\beta)M(k_1) + \beta N(k_2)) \leq h(1-\beta)\psi(M(k_2)) + h(\beta)\psi(N(k_2)) - \alpha\beta(1-\beta)(M(k_1)-N(k_2))^2. \tag{2.7}$$

From Eqs. (2.6) and (2.7), we have

$$\begin{aligned}&\psi(\beta M(k_1) + (1-\beta)N(k_2))\psi((1-\beta)M(k_1) + \beta N(k_2)) \\ &\leq h(\beta)h(1-\beta)\left[\psi^2(M(k_1)) + \psi^2(N(k_2))\right] + \left(h^2(\beta) + h^2(1-\beta)\right)\psi(M(k_1))\psi(N(k_2)) \\ &\quad - \alpha\beta(1-\beta)(M(k_1)-N(k_2))^2[\psi(M(k_1))+\psi(N(k_2))][h(\beta)+h(1-\beta)] \\ &\quad + \alpha^2\beta^2(1-\beta)^2(M(k_1)-N(k_2))^4.\end{aligned} \tag{2.8}$$

Integrating the inequality (2.8) with respect to β over $(0,1)$, we have

$$\begin{aligned}&\int_0^1 \psi(\beta M(k_1)+(1-\beta)N(k_2))\psi((1-\beta)M(k_1)+\beta N(k_2))\mathrm{d}\beta\\&\le\left[\psi^2(M(k_1))+\psi^2(N(k_2))\right]\int_0^1 h(\beta)h(1-\beta)\mathrm{d}\beta+2\psi(M(k_1))\psi(N(k_2))\int_0^1 h^2(\beta)\mathrm{d}\beta\\&-2\alpha(M(k_1)-N(k_2))^2[\psi(M(k_1))+\psi(N(k_2))]\int_0^1\beta(1-\beta)h(\beta)\mathrm{d}\beta\\&+\frac{\alpha^2}{30}(M(k_1)-N(k_2))^4.\end{aligned}$$

We change the variable $t:=\beta M(k_1)+(1-\beta)N(k_2), \beta\in(0,1)$, we get the required inequality in (2.5). □

Theorem 2.4 *Let $M,N:[k_1,k_2]\to[k_1,k_2]$ the continuous functions, let the functions $\psi_1,\psi_2:J\to(0,\infty)$ Lebesgue integrable and S-$(M,N)_h$-CF with modulus $\alpha>0$, then*

$$\begin{aligned}&\frac{1}{N(k_2)-M(k_1)}\int_{M(k_1)}^{N(k_2)}\psi_1(x)\mathrm{d}x\le H_1(k_1,k_2)\int_0^1 h^2(\beta)\mathrm{d}\beta+H_2(k_1,k_2)\int_0^1 h(\beta)h(1-\beta)\mathrm{d}\beta\\&-\alpha(M(k_1)-N(k_2))^2H_3(k_1,k_2)\int_0^1\beta(1-\beta)h(\beta)\mathrm{d}\beta+\frac{\alpha^2}{30}(M(k_1)-N(k_2))^4,\end{aligned}\tag{2.9}$$

with

$$\begin{aligned}H_1(k_1,k_2)&=\psi_1(M(k_1))\psi_2(M(k_1))+\psi_1(N(k_2))\psi_2(N(k_2)),\\H_2(k_1,k_2)&=\psi_1(M(k_1))\psi_2(N(k_2))+\psi_1(N(k_2))\psi_2(M(k_1)),\\H_3(k_1,k_2)&=\psi_1(M(k_1))+\psi_1(N(k_2))+\psi_2(M(k_1))+\psi_2(N(k_2)).\end{aligned}$$

Proof: $\psi_1,\psi_2:J\to(0,\infty)$ S-$(M,N)_h$-CF, we get

$$\psi_1(\beta M(k_1)+(1-\beta)N(k_2))\le h(\beta)\psi_1(M(k_1))+h(1-\beta)\psi_1(N(k_2))-\alpha\beta(1-\beta)(M(k_1)-N(k_2))^2,\tag{2.10}$$

$$\psi_2(\beta M(k_1)+(1-\beta)N(k_2))\le h(\beta)\psi_2(M(k_1))+h(1-\beta)\psi_2(N(k_2))-\alpha\beta(1-\beta)(M(k_1)-N(k_2))^2.\tag{2.11}$$

From Eqs. (2.10) and (2.11), we have

$$\begin{aligned}&\psi_1(\beta M(k_1)+(1-\beta)N(k_2))\psi_2(\beta M(k_1)+(1-\beta)N(k_2))\\&\le h^2(\beta)\psi_1(M(k_1))\psi_2(M(k_2))+h^2(1-\beta)\psi_1(N(k_2))\psi_1(N(k_2))\\&\quad+h(\beta)h(1-\beta)[\psi_1(M(k_1))\psi_2(N(k_2))+\psi_1(N(k_2))\psi_2(M(k_1))]\\&\quad-\alpha\beta(1-\beta)h(\beta)(M(k_1)-N(k_2))^2[\psi_1(M(k_1))+\psi_2(M(k_1))]\\&\quad-\alpha\beta(1-\beta)h(1-\beta)(M(k_1)-N(k_2))^2[\psi_1(N(k_2))+\psi_2(N(k_2))]\\&\quad+\alpha^2\beta^2(1-\beta)^2(M(k_1)-N(k_2))^4.\end{aligned}$$

Integrating the above inequality over the interval $(0,1)$, we get

$$\begin{aligned}&\int_0^1\psi_1(\beta M(k_1)+(1-\beta)N(k_2))\psi_2(\beta M(k_1)+(1-\beta)N(k_2))\mathrm{d}\beta\\&\le[\psi_1(M(k_1))\psi_2(M(k_1))+\psi_1(N(k_2))\psi_1(N(k_2))]\int_0^1 h^2(\beta)\mathrm{d}\beta\\&\quad+[\psi_1(M(k_1))\psi_2(N(k_2))+\psi_1(N(k_2))\psi_2(M(k_1))]\int_0^1 h(\beta)h(1-\beta)\mathrm{d}\beta\\&\quad-\alpha(M(k_1)-N(k_2))^2[\psi_1(M(k_1))+\psi_2(M(k_1))+\psi_1(N(k_2))+\psi_2(N(k_2))]\int_0^1\beta(1-\beta)h(\beta)\mathrm{d}\beta\\&\quad+\alpha^2(M(k_1)-N(k_2))^4\int_0^1\beta^2(1-\beta)^2d\beta.\end{aligned}$$

In the first integral, we substitute $t=\beta M(k_1)+(1-\beta)N(k_2)$, we obtain (2.9). □

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Lakhlifa Sadek,
[a] *Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India*
[b] *Department of Mathematics, Faculty of Sciences and Technology, Al-Hoceima, Abdelmalek Essaadi University, Tetouan, Morocco.*
E-mail address: lakhlifasadek@gmail.com; l.sadek@uae.ac.ma.