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Conformal Einstein Solitons on a Perfect Fluid Spacetime

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ABSTRACT: The objective of this research is to investigate the geometrical composition of a perfect fluid spacetime with a torse-forming vector field ζ in connection with conformal Einstein and conformal η-Einstein solitons. Also, We have studied the conditions for a conformal Einstein soliton to be shrinking or expanding or steady. Further, we investigated the perfect fluid spacetime that satisfies $(\zeta,\cdot)W_p\cdot Z=0$, $(\zeta,\cdot)M_p\cdot Z=0$, $(\zeta,\cdot)Z\cdot W_p=0$ and $(\zeta,\cdot)Z\cdot M_p=0$.

Key Words: η-Einstein, conformal Einstein, conformal η-Einstein, M-projective Curvature tensor and Weyl-projective curvature tensor.

Contents

1. Introduction

In 2016, Catino and Mazzieri [6] investigated the concept of Eintein soliton, which generate self similar solution to Einstein flow

$$\frac{\partial}{\partial t}g+2S=rg,$$

where g is Riemannian metric, S is Ricci tensor and r is the scalar curvature. A Riemannian manifold (M,g) of dimension n is Einstein soliton if a vector field V exists, such that [6]

$$\frac{1}{2}L_V g(U_1,U_2)+S(U_1,U_2)=(\frac{r}{2}-\Lambda)g(U_1,U_2), \tag{1.1}$$

where Λ is a real constant, L_V the Lie derivative operator in the direction of V, S is the Ricci tensor and r is the scalar curvature of metric g. If r is constant, then the Ricci and Einstein solitons will be equivalent.
A Riemannian manifold (M,g) of dimension n is said to admit η-Einstein soliton if [1]

$$L_V g(U_1,U_2)+2S(U_1,U_2)+(2\Lambda-r)g(U_1,U_2)+2\mu\eta(U_1)\eta(U_2)=0, \tag{1.2}$$

where Λ and μ are real constants. For $\mu\neq 0$ the data (g,Λ,μ,ζ) will be an η-Einstein soliton. If r is constant, then the η-Einstein soliton will reduces to an η-Ricci soliton $(g,\Lambda-\frac{r}{2},\mu,\zeta)$ and if $\mu=0$,

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then the η-Einstein soliton reduces to Ricci soliton $(g, \Lambda - \frac{r}{2}, \zeta)$. Therefore, the notions of η-Einstein and η-Ricci solitons are distinct on manifolds of non constant scalar curvature. [8,9,12]
The concept of a conformal Einstein soliton and conformal η-Einstein soliton was initiated by Roy et al. [15,16] and the extended work on this was carried out by Ajay Kumar et al. [3], Narasimhamurthy et al. [13], which are defined as

$$L_V g(U_1, U_2) + 2S(U_1, U_2) + (2\Lambda - r - (p + \tfrac{2}{n}))g(U_1, U_2) = 0, \tag{1.3}$$

$$L_V g(U_1, U_2) + 2S(U_1, U_2) + (2\Lambda - r - (p + \tfrac{2}{n}))g(U_1, U_2) + 2\mu\eta(U_1)\eta(U_2) = 0, \tag{1.4}$$

where p is a scalar non-dynamical field (time dependent scalar field). The conformal Einstein soliton is said to be shrinking or steady or expanding accordingly as Λ is negative or zero or positive.
A perfect fluid is one whose rest frame mass density ρ_m and isotropic pressure p_i can completely characterise it. A perfect fluid has no shear stresses, viscosity or heat conduction and it is distinguished by an energy-momentum tensor T of the form [11]

$$T(U_1, U_2) = p_i g(U_1, U_2) + (\rho_m + p_i)\eta(U_1)\eta(U_2), \tag{1.5}$$

for any $U_1, U_2 \in \chi(M)$, $g(U_1, \zeta) = \eta(U_1)$ is $1 - form$ and $g(\zeta, \zeta) = -1$. If $\rho_m = -p_i$, the energy-momentum tensor is Lorentz-invariant. The medium is a radiation fluid if $\rho_m = 3p_i$.
Einstein's gravitational equation is the field equation which governs perfect fluid motion [11]

$$S(U_1, U_2) + (\lambda - \frac{r}{2})g(U_1, U_2) = kT(U_1, U_2), \tag{1.6}$$

where k is the gravitational constant (which can be written as $8\Pi G$, with G representing the universal gravitational constant) and λ is a cosmological constant.
In view of (1.5) and (1.6), yield

$$S(U_1, U_2) = \left(k\rho_m - \lambda + \frac{r}{2}\right) g(U_1, U_2) + k(\rho_m + p_i)\eta(U_1)\eta(U_2). \tag{1.7}$$

In this paper we study conformal Einstein solitons on a $[P.F.St]$, where the perfect fluid spacetime is denoted by $[P.F.St]$.

2. $[P.F.St]$ with torse-forming vector field

Let (M, g) be a relativistic perfect fluid spacetime$[P.F.St]$ which satisfies (1.7). Then we can contract (1.7) and take $g(\zeta, \zeta) = -1$ to obtain

$$r = 4\lambda + k(\rho_m - 3p_i). \tag{2.1}$$

Therefore from equation (1.7), we get

$$S(U_1, U_2) = \left(\lambda + k\left(\tfrac{\rho_m - p_i}{2}\right)\right) g(U_1, U_2) + k(\rho_m + p_i)\eta(U_1)\eta(U_2), \tag{2.2}$$

$$QU_1 = \left(\lambda + k\left(\tfrac{\rho_m - p_i}{2}\right)\right) U_1 + k(\rho_m + p_i)\eta(U_1)\zeta, \tag{2.3}$$

where Q is the Ricci operator defined as $g(QU_1, U_2) = S(U_1, U_2)$.
A radiation fluid is a perfect fluid with $\rho_m = 3p_i$ and so the energy momentum tensor T becomes

$$T(U_1, U_2) = p_i[g(U_1, U_2) + 4\eta(U_1)\eta(U_2)]. \tag{2.4}$$

Now consider a specific scenario in which ζ is a torse-forming vector field of the form [2] and [10]

$$\nabla_{U_1}\zeta = U_1 + \eta(U_1)\zeta. \tag{2.5}$$

If ζ the vector field is torse-forming, then the following results hold [2]

$$\nabla_\zeta \zeta = 0, \tag{2.6}$$

$$(\nabla_{U_1}\eta)(U_2) = g(U_1, U_2) + \eta(U_1)\eta(U_2), \tag{2.7}$$

$$R(U_1, U_2)\zeta = \eta(U_2)U_1 - \eta(U_1)U_2, \tag{2.8}$$

$$\eta(R(U_1, U_2)U_3) = \eta(U_1)g(U_2, U_3) - \eta(U_2)g(U_1, U_3), \tag{2.9}$$

for any vector fields $U_1, U_2, U_3 \in \chi(M)$.

3. Conformal Einstein Soliton on $[P.F.St]$ with Torse-Forming Vector Field

Consider V as a torse-forming vector field ζ in equation (1.3) and taking $n = 4$, we have

$$L_\zeta g(U_1, U_2) + 2S(U_1, U_2) + \left(2\Lambda - r - \left(p + \frac{1}{2}\right)\right) g(U_1, U_2) = 0. \tag{3.1}$$

By virtue of (2.9) and (3.1), we get

$$2[g(U_1, U_2) + \eta(U_1)\eta(U_2)] + 2S(U_1, U_2) + \left(2\Lambda - r - \left(p + \frac{1}{2}\right)\right) g(U_1, U_2) = 0. \tag{3.2}$$

Using (2.2), we get

$$\left[\Lambda - \frac{r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right) + \lambda + \frac{k(\rho_m + p_i)}{2} + 1\right] g(U_1, U_2) + [k(\rho_m + p_i) + 1]\,\eta(U_1)\eta(U_2) = 0. \tag{3.3}$$

Put $U_1 = U_2 = \zeta$ in (3.3) to obtain

$$\Lambda = \frac{r}{2} + \frac{1}{2}\left(p + \frac{1}{2}\right) - \lambda + \frac{k(\rho_m + p_i)}{2}. \tag{3.4}$$

Using (2.1), (3.4) reduces to

$$\Lambda = \frac{1}{2}\left(p + \frac{1}{2}\right) + \lambda + k(\rho_m - p_i). \tag{3.5}$$

As a result, the following theorem can be stated:

Theorem 3.1 *Let $[P.F.St]$ admit the conformal Einstein soliton with a torse-forming vector field ζ. Then, as a result, soliton is shrinking or steady or expanding accordingly as $\frac{1}{2}\left(p + \frac{1}{2}\right) + \lambda + k(\rho_m - p_i) < 0$ or $\frac{1}{2}\left(p + \frac{1}{2}\right) + \lambda + k(\rho_m - p_i) = 0$ or $\frac{1}{2}\left(p + \frac{1}{2}\right) + \lambda + k(\rho_m - p_i) > 0$.*

If $p + \frac{1}{2} = 0$, then equation (3.5) reduces to $\Lambda = k(\rho_m - p_i) + \lambda$.
As a result, the following corollary can be stated:

Corollary 3.1 *Let $[P.F.St]$ admit the conformal Einstein soliton with a torse-forming vector field ζ. Then, as a result, soliton is shrinking or steady or expanding accordingly as $k(\rho_m - p_i) + \lambda < 0$ or $k(\rho_m - p_i) + \lambda = 0$ or $k(\rho_m - p_i) + \lambda > 0$.*

A vector field V is known as conformal-Killing vector field if and only if the following result holds

$$(L_V g)(U_1, U_2) = 2\Phi g(U_1, U_2), \tag{3.6}$$

where Φ is some function of the coordinates (conformal scalar). If Φ is non-constant, then the conformal-Killing vector field V is called proper. If Φ is constant, then V is said to be homothetic vector field and when the constant $\Phi \neq 0$, V is known as proper homothetic vector field. Also if $\Phi = 0$, then V is said to be Killing vector field.
Let us consider that in (1.3), the potential vector field V is conformal-Killing vector field. Then from (1.3) and (3.6), we obtain

$$S(U_1, U_2) = -\left[\Lambda + \Phi - \frac{r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right] g(U_1, U_2), \tag{3.7}$$

which implies that the spacetime is Einstein. In contrast, consider the $[P.F.St]$ with torse-forming vector field ζ is Einstein spacetime $[S(U_1, U_2) = \theta g(U_1, U_2)]$. Then the equation (1.3) is therefore reduced to

$$(L_V g)(U_1, U_2) = 2\Psi g(U_1, U_2), \tag{3.8}$$

where $\Psi = -\left[\Lambda + \theta - \frac{r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right]$.
Hence from (3.8), we conclude that V is a conformal-Killing vector field,
which leads to the following theorem:

Theorem 3.2 *Let [P.F.St] with a torse-forming vector field ζ admits a conformal Einstein soliton (M, Λ, ζ, g) and the potential vector field V is a conformal-Killing vector field iff the spacetime is Einstein.*

Now in consequence of (2.2) and (3.7), we obtain

$$\left[\Lambda + \Phi + \lambda + \frac{k(\rho_m - p_i)}{2} - \frac{r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right] g(U_1, U_2) + k(\rho_m + p_i)\eta(U_1)\eta(U_2) = 0. \tag{3.9}$$

Putting $U_2 = \zeta$ in (3.9) and considering $\eta(\zeta) = -1$, equation (3.9) reduces to

$$\left[\Lambda + \Phi + \lambda - \frac{k(\rho_m + 3p_i)}{2} - \frac{r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right] \eta(U_1) = 0. \tag{3.10}$$

Since $\eta(U_1) \neq 0$, we have

$$\Lambda + \Phi + \lambda - \frac{k(\rho_m + 3p_i)}{2} - \frac{r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right) = 0. \tag{3.11}$$

By substituting the value of r from (2.1) in (3.11), we get

$$\Phi = k\rho_m + \lambda + \frac{1}{2}\left(p + \frac{1}{2}\right) - \Lambda. \tag{3.12}$$

As a result, the following theorem can be stated:

Theorem 3.3 *Let [P.F.St] with torse-forming vector field ζ admits a conformal Einstein soliton (M, Λ, ζ, g) and the potential vector field V is a conformal-Killing vector field. Then V is*

- *homothetic vector field if p is constant.*
- *proper conformal-Killing vector field if p is non constant.*

Using the property of Lie derivative we can write

$$(L_V g)(U_1, U_2) = g(\nabla_{U_1} V, U_2) + g(\nabla_{U_2} V, U_1). \tag{3.13}$$

In consequence of (2.2) and (3.13), equation (1.3) implies

$$g(\nabla_{U_1} V, U_2) + g(\nabla_{U_2} V, U_1) + \left[2\Lambda - r - (p + \tfrac{1}{2}) + 2\lambda + k(\rho_m - p_i)\right] g(U_1, U_2) + 2k(\rho_m + p_i)\eta(U_1)\eta(U_2) = 0. \tag{3.14}$$

Let Ω is a $1-form$ which is metrically equivalent to V and is given by $\Omega(U_1) = g(U_1, V)$ for an arbitrary vector field U_1. Then the exterior derivative of Ω can be written as

$$2(d\Omega)(U_1, U_2) = g(\nabla_{U_1} V, U_2) - g(\nabla_{U_2} V, U_1). \tag{3.15}$$

As $d\Omega$ is skew-symmetric, so if we define a tensor field E of type $(1, 1)$ by

$$(d\Omega)(U_1, U_2) = g(U_1, EU_2), \tag{3.16}$$

then E is skew self adjoint $[g(U_1, EU_2) = -g(EU_1, U_2)]$.
Therefore equation (3.15) becomes

$$g(\nabla_{U_1} V, U_2) - g(\nabla_{U_2} V, U_1) = -2g(EU_1, U_2). \tag{3.17}$$

On adding (3.17) and (3.14) and factoring out U_2, we obtain

$$\nabla_{U_1} V = -EU_1 - \left[\Lambda + \lambda + \frac{k(\rho_m - p_i)}{2} - \frac{r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right] U_1 - k(\rho_m + p_i)\eta(U_1)\zeta. \tag{3.18}$$

Use the above equation in
$R(U_1, U_2)V = \nabla_{U_1}\nabla_{U_2}V - \nabla_{U_2}\nabla_{U_1}V - \nabla_{[U_1,U_2]}V$ to obtain

$$\begin{aligned} R(U_1, U_2)V = (\nabla_{U_2}E)U_1 - (\nabla_{U_1}E)U_2 \\ + k(\rho_m + p_i)[\eta(U_1)U_2 - \eta(U_2)U_1]. \end{aligned} \tag{3.19}$$

Since $d\Omega$ is closed, we have

$$g(U_1, (\nabla_{U_3}E)U_2) + g(U_2, (\nabla_{U_1}E)U_3) + g(U_3, (\nabla_{U_2}E)U_1) = 0. \tag{3.20}$$

Contracting (3.19) with respect to U_3, we have

$$\begin{aligned} &g(R(U_1, U_2)V, U_3) = g((\nabla_{U_2}E)U_1, U_3) - g((\nabla_{U_1}E)U_2, U_3) \\ &+k(\rho_m + p_i)[\eta(U_1)g(U_2, U_3) - \eta(U_2)g(U_1, U_3)]. \end{aligned} \tag{3.21}$$

As E is skew self adjoint, then $\nabla_{U_1}E$ is also skew self adjoint. Then by virtue of (3.20), (3.21) becomes

$$\begin{aligned} g(R(U_1, U_2)V, U_3) =&k(\rho_m + p_i)[\eta(U_1)g(U_2, U_3) - \eta(U_2)g(U_1, U_3)] \\ &- g(U_1, (\nabla_{U_3}E)U_2). \end{aligned} \tag{3.22}$$

Taking $U_1 = U_3 = e_i$ in (3.22), we have

$$S(U_2, V) = -3k(\rho_m + p_i)\eta(U_2) - (divE)U_2, \tag{3.23}$$

where $divE$ is the divergence of the tensor field E.
In view of (2.2) and (3.23), we obtain

$$(divE)U_2 = -k(\rho_m + p_i)[3 + \eta(V)]\eta(U_2) - \left[\lambda + \frac{k(\rho_m - p_i)}{2}\right]\Omega(U_2). \tag{3.24}$$

Now we find the covariant derivative of the squared $g - norm$ of V using (3.18) as

$$\begin{aligned} \nabla_{U_1}|V|^2 \quad &= -2g(EU_1, V) - [2\Lambda - r - (p + \tfrac{1}{2}) + 2\lambda + k(\rho_m - p_i)] \\ &g(U_1, V) - 2k(\rho_m + p_i)\eta(U_1)\eta(V). \end{aligned} \tag{3.25}$$

By virtue of (2.2), (1.3) implies

$$\begin{aligned} (L_V g)(U_1, U_2) \quad = \quad &-[2\Lambda - r - (p + \frac{1}{2}) + 2\lambda + k(\rho_m - p_i)]g(U_1, U_2) \\ &-2k(\rho_m + p_i)\eta(U_1)\eta(U_2). \end{aligned} \tag{3.26}$$

In view of (3.25) and (3.26), we have

$$\nabla_{U_1}|V|^2 + 2g(EU_1, V) - (L_V g)(U_1, V) = 0. \tag{3.27}$$

As a result, we state the following theorem:

Theorem 3.4 *Let [P.F.St] with a torse-forming vector field ζ admits a conformal Einstein soliton (M, Λ, ζ, g), then the vector field V and its metric dual $1 - form$ Ω satisfies the relations*
$(divE)U_2 = -k(\rho_m + p_i)[3 + \eta(V)]\eta(U_2) - \left[\lambda + \frac{k(\rho_m - p_i)}{2}\right]\Omega(U_2)$
and
$\nabla_{U_1}|V|^2 + 2g(EU_1, V) - (L_V g)(U_1, V) = 0.$

4. Conformal η-Einstein Soliton in $[P.F.St]$

Let (M, g) be a general relativistic $[P.F.St]$ and (g, μ, Λ, ζ) be a conformal $\eta-$Einstein soliton in M. Then, by virtue of (1.4),(2.2) and (3.13), we have

$$[\Lambda \quad -\frac{r}{2} - \frac{1}{2}(p + \frac{1}{2}) + \lambda + k(\frac{\rho_m - p_i}{2})]g(U_1, U_2) + [k(\rho_m + p_i) + \mu] \eta(U_1)\eta(U_2) + \frac{1}{2}[g(\nabla_{U_1}\zeta, U_2) + g(\nabla_{U_2}\zeta, U_1)] = 0. \tag{4.1}$$

Consider e_i $(1 \leq i \leq 4)$ an orthonormal frame field and $\zeta = \sum_{n=1}^{4} \zeta^i e_i$. From [1] we have $\sum_{i=1}^{4} \epsilon_{ii}(\zeta^i)^2 = -1$ and $\eta(e_i) = \epsilon_{ii}\zeta^i$.
Multiply ϵ_{ii} with (4.1) and summing over i for $U_1 = U_2 = e_i$, equation (4.1) reduces to

$$4\Lambda - \mu = 4\lambda + 2(p + \frac{1}{2}) + k(\rho_m - 3p_i) - (div\zeta), \tag{4.2}$$

where $(div\zeta)$ is the divergence of the vector field ζ.
Taking $U_1 = U_2 = \zeta$ in (4.1) we obtain

$$\Lambda - \mu = \lambda + \frac{1}{2}(p + \frac{1}{2}) + k\rho_m. \tag{4.3}$$

On solving for Λ and μ from (4.2)and (4.3), we get

$$\Lambda = \lambda + \frac{1}{2}(p + \frac{1}{2}) - kp_i - \frac{1}{3}(div\zeta),$$
$$\mu = -k(\rho_m + p_i) - \frac{1}{3}(div\zeta),$$

which leads the following:

Theorem 4.1 *Let $(M, \mu, \Lambda, \zeta, g)$ be a 4-dimensional pseudo-Riemannian manifold and η be the $g-dual$ 1-form of the gradient vector field $\zeta = grad(f)$, with $g(\zeta, \zeta) = -1$, where f is a smooth function. If (g, μ, Λ, ζ) be a conformal $\eta-$Einstein soliton on M, then Laplacian equation satisfied by f becomes $\Delta(f) = -3[\mu + k(\rho_m + p_i)]$.*

Example 4.1 Conformal η-Einstein soliton (g, μ, Λ, ζ) in a radiation fluid is given by,

$$\Lambda = \lambda + \frac{1}{2}(p + \frac{1}{2}) - kp_i - \frac{1}{3}(div\zeta),$$
and
$$\mu = -k(\rho_m + p_i) - \frac{1}{3}(div\zeta).$$

Definition 4.1 *The concept of Z-tensor was investigated by Mantica et al. [5] in 2012, which is defined as*

$$Z(U_1, U_2) = S(U_1, U_2) + \phi_1 g(U_1, U_2), \tag{4.4}$$

where ϕ_1 is an arbitrary scalar function.

5. $[P.F.St]$ satisfying $(\zeta, \cdot)W_p \cdot Z = 0$

Let (M^4, g) be a differentiable manifold with the g metric and the Riemannian connection ∇. The Weyl projective curvature tensor is defined by [4,14,17,18]

$$W_p(U_1, U_2)U_3 = R(U_1, U_2)U_3 - \frac{1}{(3)}(U_1 \Lambda_S U_2)U_3, \tag{5.1}$$

where S is the Ricci tensor and R is the curvature tensor of the manifold.
The $[P.F.St]$ satisfies the condition $(\zeta, \cdot)W_p \cdot Z = 0$ is equivalent to

$$Z(W_p(\zeta, U_1)U_2, \zeta) + Z(U_2, W_p(\zeta, U_1)\zeta) = 0. \tag{5.2}$$

By virtue of (4.4), (5.2) implies

$$\begin{aligned}&S(W_p(\zeta,U_1)U_2,\zeta)+S(U_2,W_p(\zeta,U_1)\zeta)\\&+\phi_1[\eta(W_p(\zeta,U_1)U_2)+g(U_2,W_p(\zeta,U_1)\zeta)]=0.\end{aligned} \tag{5.3}$$

Using (2.2) and (5.1) in (5.3), we obtain

$$[g(U_1,U_2)+\eta(U_1)\eta(U_2)]\,[p_i(k(3+2\phi_1))-2\lambda\phi_1-3k\rho_m]=0. \tag{5.4}$$

Since $g(U_1,U_2)+\eta(U_1)\eta(U_2)\neq 0$, we have

$$p_i=\frac{2\lambda\phi_1-3k\rho_m}{k(3+2\phi_1)}. \tag{5.5}$$

As a result, we state the following theorem:

Theorem 5.1 *Let (M,g) be a general relativistic $[P.F.St]$ satisfying $(\zeta,\cdot)W_p\cdot Z=0$ with torse-forming vector field ζ. Then $p_i=\frac{2\lambda\phi_i-3k\rho_m}{k(3+2\phi_i)}$.*

If $\phi_1=0$, then $(\zeta,\cdot)W_p\cdot Z=(\zeta,\cdot)W_p\cdot S$ and $p_i=-\rho_m$, which leads to the following corollary:

Corollary 5.1 *Let (M,g) be a general relativistic $[P.F.St]$ satisfying $(\zeta,\cdot)W_p\cdot S=0$ with torse-forming vector field ζ. Then $p_i=-\rho_m$.*

6. $[P.F.St]$ satisfying $(\zeta,\cdot)M_p\cdot Z=0$

Let (M^4,g) be a differentiable manifold with the g metric and the Riemannian connection ∇. The M-projective curvature tensor is defined as [7]

$$\begin{aligned}M_p(U_1,U_2)U_3 &= R(U_1,U_2)U_3-\frac{1}{6}(U_1\Lambda_S U_2)U_3+g(U_2,U_3)QU_1\\&\quad -g(U_1,U_3)QU_2],\end{aligned} \tag{6.1}$$

The $[P.F.St]$ satisfies the condition $(\zeta,\cdot)M_p\cdot Z=0$ is equivalent to

$$Z(M_p(\zeta,U_1)U_2,\zeta)+Z(U_2,M_p(\zeta,U_1)\zeta)=0. \tag{6.2}$$

By virtue of (4.4), (6.2) implies

$$\begin{aligned}&S(M_p(\zeta,U_1)U_2,\zeta)+S(U_2,M_p(\zeta,U_1)\zeta)+\phi_1[\eta(M_p(\zeta,U_1)U_2)\\&+g(U_2,M_p(\zeta,U_1)\zeta)]=0.\end{aligned} \tag{6.3}$$

Using (2.2) and (6.1) in (6.3), we get

$$[g(U_1,U_2)+\eta(U_1)\eta(U_2)]\,[kp_i-\lambda+3]=0. \tag{6.4}$$

Since $g(U_1,U_2)+\eta(U_1)\eta(U_2)\neq 0$, we obtain

$$p_i=\frac{\lambda-3}{k}. \tag{6.5}$$

Thus, we can state the following theorem:

Theorem 6.1 *Let (M,g) be a general relativistic $[P.F.St]$ satisfying $(\zeta,\cdot)M_p\cdot Z=0$ with torse-forming vector field ζ. Then $p_i=\frac{\lambda-3}{k}$.*

7. $[P.F.St]$ satisfying $(\zeta,\cdot)Z\cdot W_p=0$

The $[P.F.St]$ satisfies the condition $(\zeta,\cdot)Z\cdot W_p=0$. The condition is equivalent to

$$\begin{aligned}
&Z(U_1,W_p(U_2,U_3)U_4)\zeta-Z(\zeta,W_p(U_2,U_3)U_4)U_1\\
&+Z(U_1,U_2)W_p(\zeta,U_3)U_4-Z(\zeta,U_2)W_p(U_1,U_3)U_4\\
&+Z(U_1,U_3)W_p(U_2,\zeta)U_4-Z(\zeta,U_3)W_p(U_2,U_1)U_4\\
&+Z(U_1,U_4)W_p(U_2,U_3)\zeta-Z(\zeta,U_4)W_p(U_2,U_3)U_1=0.
\end{aligned}\tag{7.1}$$

Contracting the above equation with respect to ζ, we obtain

$$\begin{aligned}
&-Z(U_1,W_p(U_2,U_3)U_4)-Z(\zeta,W_p(U_2,U_3)U_4)\eta(U_1)\\
&+Z(U_1,U_2)\eta(W_p(\zeta,U_3)U_4)-Z(\zeta,U_2)\eta(W_p(U_1,U_3)U_4)\\
&+Z(U_1,U_3)\eta(W_p(U_2,\zeta)U_4)-Z(\zeta,U_3)\eta(W_p(U_2,U_1)U_4)\\
&+Z(U_1,U_4)\eta(W_p(U_2,U_3)\zeta)-Z(\zeta,U_4)\eta(W_p(U_2,U_3)U_1)=0.
\end{aligned}\tag{7.2}$$

In view of (4.4),(2.8),(2.9) and (5.1), (7.2) becomes

$$\begin{aligned}
&-(A+\phi_1)[(g(U_1,U_2)g(U_3,U_4)-g(U_1,U_3)g(U_2,U_4))(1-\frac{A}{3})\\
&+\frac{B}{3}(\eta(U_3)\eta(U_4)g(U_1,U_2)-\eta(U_2)\eta(U_4)g(U_1,U_3))]-(A+\phi_1)\\
&\eta(U_1)\eta(W_p(U_2,U_3)U_4)-(A-B+\phi_1)[\eta(U_2)\eta(W_p(U_1,U_3)U_4)\\
&+\eta(U_3)\eta(W_p(U_2,U_1)U_4)+\eta(U_4)\eta(W_p(U_2,U_3)U_1)]+(\frac{A}{3}-1)\\
&[(A+\phi_1)g(U_1,U_2)+B\eta(U_1)\eta(U_2)][g(U_3,U_4)+\eta(U_3)\eta(U_4)]\\
&(1-\frac{A}{3})[(A+\phi_1)g(U_1,U_3)+B\eta(U_1)\eta(U_3)]\\
&[g(U_2,U_4)+\eta(U_2)\eta(U_4)]=0,
\end{aligned}\tag{7.3}$$

where $A=\lambda+\frac{k(\rho_m-p_i)}{2}$ and $B=k(\rho_m+p_i)$.
Putting $U_3=U_4=\zeta$ in the above equation to obtain

$$[g(U_1,U_2)+\eta(U_1)\eta(U_2)](A+\phi_1)\left[2(1-\frac{A}{3})+\frac{B}{3}\right]+B(\frac{A}{3}-1)=0.\tag{7.4}$$

Since $g(U_1,U_2)+\eta(U_1)\eta(U_2)\neq 0$, we get

$$p_i=\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

where $a=-\frac{3k^2}{2}$, $b=k^2\rho_m+2k(2\lambda+\phi_1-3)$ and $c=\frac{k^2\rho_m^2}{2}-2\lambda^2+\lambda+2\phi_1(3-\lambda)$.
As a result, we can state the following theorem:

Theorem 7.1 *Let (M,g) be a general relativistic $[P.F.St]$ satisfying $(\zeta,\cdot)Z\cdot W_p=0$ with torse-forming vector field ζ. Then $p_i=\frac{-b+\sqrt{b^2-4ac}}{2a}$ or $p_i=\frac{-b-\sqrt{b^2-4ac}}{2a}$.*

If $\phi_1=0$, then $(\zeta,\cdot)Z\cdot W_p=(\zeta,\cdot)S\cdot W_p$ and

$$p_i=\frac{-b_1\pm\sqrt{b_1^2-4ac_1}}{2a},$$

where $b_1=k^2\rho_m+2k(2\lambda-3)$ and $c_1=\frac{k^2\rho_m^2}{2}-2\lambda^2+\lambda$. Thus, the following corollary can be stated:

Corollary 7.1 *Let (M,g) be a general relativistic $[P.F.St]$ satisfying $(\zeta,\cdot)S\cdot W_p=0$ with torse-forming vector field ζ. Then $p_i=\frac{-b_1+\sqrt{b_1^2-4ac_1}}{2a}$ or $p_i=\frac{-b_1-\sqrt{b_1^2-4ac_1}}{2a}$.*

8. $[P.F.St]$ satisfying $(\zeta,\cdot)Z\cdot M_p=0$

The $[P.F.St]$ satisfies the condition $(\zeta,\cdot)Z\cdot M_p=0$ is equivalent to

$$\begin{aligned}&Z(U_1,M_p(U_2,U_3)U_4)\zeta-Z(\zeta,M_p(U_2,U_3)U_4)U_1\\&+Z(U_1,U_2)M_p(\zeta,U_3)U_4-Z(\zeta,U_2)M_p(U_1,U_3)U_4\\&+Z(U_1,U_3)M_p(U_2,\zeta)U_4-Z(\zeta,U_3)M_p(U_2,U_1)U_4\\&+Z(U_1,U_4)M_p(U_2,U_3)\zeta-Z(\zeta,U_4)M_p(U_2,U_3)U_1=0.\end{aligned}\tag{8.1}$$

Contracting the above equation with respect to ζ, we have

$$\begin{aligned}&-Z(U_1,M_p(U_2,U_3)U_4)-Z(\zeta,M_p(U_2,U_3)U_4)\eta(U_1)\\&+Z(U_1,U_2)\eta(M_p(\zeta,U_3)U_4)-Z(\zeta,U_2)\eta(M_p(U_1,U_3)U_4)\\&+Z(U_1,U_3)\eta(M_p(U_2,\zeta)U_4)-Z(\zeta,U_3)\eta(M_p(U_2,U_1)U_4)\\&+Z(U_1,U_4)\eta(M_p(U_2,U_3)\zeta)-Z(\zeta,U_4)\eta(M_p(U_2,U_3)U_1)=0.\end{aligned}\tag{8.2}$$

Using (4.4),(2.8),(2.9) and (6.1) in (8.2) and then taking $U_3=U_4=\zeta$, we have

$$[g(U_1,U_2)+\eta(U_1)\eta(U_2)]\left[\left(1-\frac{2A-B}{6}\right)(2A-B+2\phi_1)\right]=0.\tag{8.3}$$

Since $g(U_1,U_2)+\eta(U_1)\eta(U_2)\neq 0$, we obtain
$p_i=\frac{\lambda-3}{k}$ or $p_i=\frac{\lambda+\phi_1}{k}$.
Thus, the following theorem can be stated:

Theorem 8.1 *Let (M,g) be a general relativistic $[P.F.St]$ satisfying $(\zeta,\cdot)Z\cdot M_p=0$ with torse-forming vector field ζ. Then $p_i=\frac{\lambda-3}{k}$ or $p_i=\frac{\lambda+\phi_1}{k}$.*

If $\phi_1=0$, then $(\zeta,\cdot)Z\cdot M_p=(\zeta,\cdot)S\cdot M_p$ and $p_i=\frac{\lambda-3}{k}$ or $p_i=\frac{\lambda}{k}$.
As a result, we can state the following corollary:

Corollary 8.1 *Let (M,g) be a general relativistic $[P.F.St]$ satisfying $(\zeta,\cdot)S\cdot M_p=0$ with torse-forming vector field ζ. Then $p_i=\frac{\lambda-3}{k}$ or $p_i=\frac{\lambda}{k}$.*

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