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Congruences For Simultaneously s-Regular and t-Regular Partition Function

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ABSTRACT: A partition of a positive integer n is said to be simultaneously s-regular and t-regular partition if none of the parts are divisible by s and t. In this paper, we establish many infinite families of congruences for simultaneously s-regular and t-regular partition function by considering some particular values of s and t.

Key Words: simultaneously s-regular and t-regular partition, congruence, theta-functions, q-series.

Contents

1. Introduction

Throughout the paper, we use the standard notations

$$(B)_\infty := (B;q)_\infty := \prod_{j=1}^{\infty}(1-Bq^{j-1}), \tag{1.1}$$

where B and q are complex numbers with $|q|<1$. For brevity, for any integer $k\geq 1$, we write

$$(B_1;q)_\infty(B_2;q)_\infty\cdots(B_k;q)_\infty = (B_1,B_2,\cdots,B_k;q)_\infty$$

and

$$g_k := (q^k;q^k)_\infty.$$

A partition of a positive integer n is a nonincreasing finite sequence of positive integers $\{a_i\}_{i=1}^{k}$ satisfying $n=\sum_{i=1}^{k}a_i$. The number of partitions of an integer $n\geq 1$ is usually denoted by $p(n)$ and its generating function is given by

$$\sum_{n=0}^{\infty}p(n)q^n = \frac{1}{g_1}.$$

Following congruences for the partition function $p(n)$ are due to Ramanujan [2,9,10]:

$$p(5n+4)\equiv 0 \pmod 5,\quad p(7n+5)\equiv 0 \pmod 7,\quad p(11n+6)\equiv 0 \pmod{11}.$$

For any integer $\ell\geq 2$, a partition of an integer $n\geq 1$ is said to be ℓ-regular if none of its parts are divisible by ℓ. If $b_\ell(n)$ denotes the number of ℓ-regular partitions of n, then

$$\sum_{n=0}^{\infty}b_\ell(n)q^n = \frac{g_\ell}{g_1}.$$

For example, $b_2(5)=3$ with the relevant partitions given by 5, $3+1+1$ and $1+1+1+1+1$. For congruence and other arithmetic properties of $b_\ell(n)$, see [4,8,11] and references therein.

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In [7], Keith defined simultaneously s-regular and t-regular partition of an integer $n \geq 1$ in which none of the parts of the partition is divisible by s and t. Suppose $M_{s,t}(n)$ denotes the number of simultaneously s-regular and t-regular partition of n, then the generating function of $M_{s,t}(n)$ [7] is given by

$$\sum_{n=0}^{\infty} M_{s,t}(n)q^n = \prod_{k=1}^{\infty} \frac{(1-q^{sk})(1-q^{tk})}{(1-q^k)(1-q^{[s,t]k})} = \frac{g_s g_t}{g_1 g_{[s,t]}}, \tag{1.2}$$

where $[s,t]$ denotes the least common multiple of s and t. For example, $M_{2,3}(5) = 2$ with the relevant partitions 5 and $1+1+1+1+1$. Keith [7] studied bijection properties of the partition function $M_{s,t}(n)$.

In this paper, we prove several infinite families of congruences for the partition function $M_{s,t}(n)$ for $(s,t) \in \{(3,9),(4,6),(4,9),(4,12),(6,9),(9,12)\}$ by using theta-function and q-series identities. Congruences for $M_{s,t}(n)$ are proved in Section 3. Section 2 is devoted to list some preliminary results.

2. Preliminaries

In this section, we record some q-series and theta-function identities which will be used in the subsequent section.

Ramanujan's general theta-function $\mathfrak{f}(x,y)$ [1, Page 34, Equation 18.1] is defined by

$$\mathfrak{f}(x,y) = \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} y^{n(n-1)/2}, \ |xy| < 1.$$

The three important special cases of $\mathfrak{f}(x,y)$ are the theta-functions $\phi(q)$, $\psi(q)$, and $f(-q)$ [1, p. 36, Entry 22 (i), (ii), (iii)] defined by

$$\phi(q) := \mathfrak{f}(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{g_2^5}{g_1^2 g_4^2}, \tag{2.1}$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{g_2^2}{g_1}, \tag{2.2}$$

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = g_1. \tag{2.3}$$

Lemma 2.1 *[6, (8.1.1)] We have*

$$g_1 = g_{25}\left(R(q^5) - q - q^2/R(q^5)\right), \tag{2.4}$$

where

$$R(q) = \frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}.$$

Lemma 2.2 *[1, p. 303, Entry 17 (v)] We have*

$$g_1 = g_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.5}$$

where

$$A(q) = \mathfrak{f}(-q^3,-q^4), \ B(q) = \mathfrak{f}(-q^2,-q^5) \quad \textit{and} \quad C(q) = \mathfrak{f}(-q,-q^6).$$

Lemma 2.3 *We have*

$$\frac{1}{g_1^2} = \frac{g_8^5}{g_2^5 g_{16}^2} + 2q\frac{g_4^2 g_{16}^2}{g_2^5 g_8}, \tag{2.6}$$

$$g_1^2 = \frac{g_2 g_8^5}{g_4^2 g_{16}^2} - 2q\frac{g_2 g_{16}^2}{g_8}, \tag{2.7}$$

$$\frac{g_2^2}{g_1} = \frac{g_6 g_9^2}{g_3 g_{18}} + q\frac{g_{18}^2}{g_9}, \tag{2.8}$$

$$\frac{g_2}{g_1^2} = \frac{g_6^4 g_9^6}{g_3^8 g_{18}^3} + 2q\frac{g_6^3 g_9^3}{g_3^7} + 4q^2\frac{g_6^2 g_{18}^3}{g_3^6}, \tag{2.9}$$

$$\frac{g_3^3}{g_1} = \frac{g_4^3 g_6^2}{g_2^2 g_{12}} + q\frac{g_{12}^3}{g_4}, \tag{2.10}$$

$$\frac{g_3}{g_1^3} = \frac{g_4^6 g_6^3}{g_2^9 g_{12}^2} + 3q\frac{g_4^2 g_6 g_{12}^2}{g_2^7}, \tag{2.11}$$

$$\frac{g_4}{g_1} = \frac{g_{12} g_{18}^4}{g_3^3 g_{36}^2} + q\frac{g_6^2 g_9^3 g_{36}}{g_3^4 g_{18}^2} + 2q^2\frac{g_6 g_{18} g_{36}}{g_3^3}, \tag{2.12}$$

$$\frac{g_9}{g_1} = \frac{g_{12}^3 g_{18}}{g_2^2 g_6 g_{36}} + q\frac{g_4^2 g_6 g_{36}}{g_2^3 g_{12}}. \tag{2.13}$$

Identity (2.6) follows from (1.9.4) of [6]. Replacing q by $-q$ in (2.6) and the result $(-q;-q)_\infty = g_2^3/(g_1 g_4)$, we arrive at (2.7). Identities (2.8) and (2.10) follow from (14.3.3) and (22.1.14) of [6], respectively. (2.9) is from [**?**, Equation (2)]. Identities (2.11), (2.12), and (2.13) follow from [13, Equation (3.38)], [3, Lemma 2.6], and [12, Lemma 3.5], respectively.

Lemma 2.4 *[4, Theorem 2.2] For any prime $p \geq 5$, we have*

$$g_1 = \sum_{\substack{k=-(p-1)/2 \\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} \mathfrak{f}\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} g_{p^2},$$

where

$$\frac{\pm p - 1}{6} := \begin{cases} \dfrac{(p-1)}{6}, & if\ p \equiv 1 \pmod 6 \\ \dfrac{(-p-1)}{6}, & if\ p \equiv -1 \pmod 6). \end{cases}$$

Furthermore, if

$$\frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2} \text{ and } k \neq \frac{(\pm p - 1)}{6},$$

then

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

Lemma 2.5 *[4, Theorem 2.1] For any odd prime p, we have*

$$\psi(q) = \sum_{i=0}^{(p-3)/2} q^{(i^2+i)/2} \mathfrak{f}\left(q^{(p^2+(2i+1)p)/2}, q^{(p^2-(2i+1)p)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2}).$$

Furthermore, $\dfrac{(i^2+i)}{2} \not\equiv \dfrac{(p^2-1)}{8} \pmod p$, *when* $0 \leq i \leq \dfrac{(p-3)}{2}$.

The next lemma is a consequence of binomial expansion and (1.1).

Lemma 2.6 *For any positive integer k, m and prime p, we have*

$$g_{pm}^{p^{k-1}} \equiv g_m^{p^k} \pmod{p^k}. \tag{2.14}$$

3. Congruences for $M_{s,t}(n)$

The section is devoted to proving congruences for $M_{s,t}(n)$. First we define the Legendre symbol $\left(\frac{a}{p}\right)$. Let p be any odd prime and a be any integer relatively prime to p, then

$$\left(\frac{a}{p}\right)=\begin{cases} 1, & \text{if } a \text{ is a quadratic residue of } p,\\ -1, & \text{if } a \text{ is a quadratic non-residue of } p.\end{cases}$$

Theorem 3.1 *Let $p\geq 5$ be any prime such that $\left(\frac{-3}{p}\right)=-1$. Then for any integers $\alpha\geq 0$, $n\geq 0$, and $1\leq j\leq p-1$, we have*

$$M_{4,9}\left(6\cdot p^{2\alpha+2}n+6\cdot p^{2\alpha+1}j+p^{2\alpha+2}+1\right)\equiv 0 \pmod{6}, \tag{3.1}$$
$$M_{4,9}\left(18\cdot p^{2\alpha+2}n+18\cdot p^{2\alpha+1}j+3\cdot p^{2\alpha+2}+1\right)\equiv 0 \pmod{16}, \tag{3.2}$$
$$M_{4,9}\left(72\cdot p^{2\alpha+2}n+72\cdot p^{2\alpha+1}j+12\cdot p^{2\alpha+2}+1\right)\equiv 0 \pmod{8}. \tag{3.3}$$

Proof: Setting $s=4$ and $t=9$ in (1.2), we obtain

$$\sum_{n=0}^{\infty}M_{4,9}(n)q^n=\frac{g_4g_9}{g_1g_{36}}. \tag{3.4}$$

Employing (2.12) in (3.4), we obtain

$$\sum_{n=0}^{\infty}M_{4,9}(n)q^n=\frac{g_9g_{12}g_{18}^4}{g_3^3g_{36}^3}+q\frac{g_6^2g_9^4}{g_3^4g_{18}^2}+2q^2\frac{g_6g_9g_{18}}{g_3^3}. \tag{3.5}$$

Extracting the terms involving q^{3n+2} from both sides of (3.5), dividing by q^2 and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty}M_{4,9}(3n+2)q^n=2\frac{g_2g_3g_6}{g_1^3}. \tag{3.6}$$

Employing (2.14) in (3.6), we obtain

$$\sum_{n=0}^{\infty}M_{4,9}(3n+2)q^n\equiv 2g_2g_6 \pmod{6}. \tag{3.7}$$

Extracting the terms involving q^{2n} from both sides of (3.7) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}M_{4,9}(6n+2)q^n\equiv 2g_1g_3 \pmod{6}. \tag{3.8}$$

Employing Lemma 2.4 in (3.8), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}M_{4,9}(6n+2)q^n\\ &\equiv 2\Bigg[\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2}(-1)^kq^{(3k^2+k)/2}\mathfrak{f}\left(-q^{(3p^2+(6k+1)p)/2},-q^{(3p^2-(6k+1)p)/2}\right)+(-1)^{(\pm p-1)/6}q^{(p^2-1)/24}g_{p^2}\Bigg]\\ &\quad\times\Bigg[\sum_{\substack{m=-(p-1)/2\\ m\neq(\pm p-1)/6}}^{(p-1)/2}(-1)^mq^{3(3m^2+m)/2}\mathfrak{f}\left(-q^{3(3p^2+(6m+1)p)/2},-q^{3(3p^2-(6m+1)p)/2}\right)\\ &\qquad+(-1)^{(\pm p-1)/6}q^{3(p^2-1)/24}g_{3p^2}\Bigg] \pmod{6}.\end{aligned} \tag{3.9}$$

Consider the congruence

$$\left(\frac{3k^2+k}{2}\right)+3\left(\frac{3m^2+m}{2}\right)\equiv 4\left(\frac{p^2-1}{24}\right) \pmod{p},$$

which is equivalent to

$$(6k+1)^2+3\,(6m+1)^2\equiv 0 \pmod{p}. \tag{3.10}$$

For $\left(\frac{-3}{p}\right)=-1$, the congruence (3.10) has only solution $k=m=(\pm p-1)/6$. Therefore, extracting the terms involving $q^{p^2n+(p^2-1)/6}$ from both sides of (3.9), dividing by $q^{(p^2-1)/6}$ and replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}\left(6\cdot p^2n+(p^2-1)+2\right)q^n\equiv 2g_1g_3 \pmod{6}. \tag{3.11}$$

Iterating (3.11) by employing Lemma 2.4, extracting the terms involving $q^{p^2n+(p^2-1)/6}$, dividing by $q^{(p^2-1)/6}$ and replacing q^{p^2} by q, we deduce that, for integer $\alpha\geq 0$

$$\sum_{n=0}^{\infty} M_{4,9}\left(6\cdot p^{2\alpha}n+(p^{2\alpha}-1)+2\right)q^n\equiv 2g_1g_3 \pmod{6}. \tag{3.12}$$

Again, employing Lemma 2.4 in (3.12) and then extracting the terms involving $q^{pn+(p^2-1)/6}$, dividing by $q^{(p^2-1)/6}$ and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}\left(6\cdot p^{2\alpha+1}n+(p^{2\alpha+2}-1)+2\right)q^n\equiv 2g_pg_{3p} \pmod{6}. \tag{3.13}$$

Extracting the terms involving q^{pn+j}, for $1\leq j\leq p-1$ from both sides of (3.13), we arrive at (3.1). Extracting the terms involving q^{3n+1} from both sides of (3.5), dividing by q and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(3n+1)q^n=\frac{g_2^2g_3^4}{g_1^4g_6^2}. \tag{3.14}$$

Employing (2.9) in (3.14), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(3n+1)q^n=\frac{g_6^6g_9^{12}}{g_3^{12}g_{18}^6}+4q\frac{g_6^5g_9^9}{g_3^{11}g_{18}^3}+12q^2\frac{g_6^4g_9^6}{g_3^{10}}+16q^3\frac{g_6^3g_9^3g_{18}^3}{g_3^9}+16q^4\frac{g_6^2g_{18}^6}{g_3^8}. \tag{3.15}$$

Extracting the terms involving q^{3n+1} from both sides of (3.15), dividing by q and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(9n+4)q^n\equiv 4\frac{g_2^5g_3^9}{g_1^{11}g_6^3} \pmod{16}. \tag{3.16}$$

Using (2.14) with $\{m=1,\ p=2,\ k=2\}$ and $\{m=3,\ p=2,\ k=2\}$ in (3.16), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(9n+4)q^n\equiv 4\frac{g_2g_3g_6}{g_1^3} \pmod{16}. \tag{3.17}$$

Using (2.11) in (3.17), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(9n+4)q^n\equiv 4\frac{g_4^6g_6^4}{g_2^8g_{12}^2}+12q\frac{g_4^2g_6^2g_{12}^2}{g_2^6} \pmod{16}. \tag{3.18}$$

Extracting the terms involving q^{2n} from both sides of (3.18) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(18n+4)q^n \equiv 4\frac{g_2^6 g_3^4}{g_1^8 g_6^2} \pmod{16}. \tag{3.19}$$

Using (2.14) with $\{m=3,\ p=2,\ k=2\}$ and $\{m=1,\ p=2,\ k=2\}$ in (3.19), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(18n+4)q^n \equiv 4g_2^2 \pmod{16}. \tag{3.20}$$

Using (2.2) in (3.20), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(18n+4)q^n \equiv 4g_1\psi(q) \pmod{16}. \tag{3.21}$$

Employing Lemmas 2.4 and 2.5 in (3.21), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} M_{4,9}(18n+4)q^n \equiv 4\Bigg[&\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} \mathfrak{f}\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right)\\ &+(-1)^{(\pm p-1)/6} q^{(p^2-1)/24} g_{p^2}\Bigg]\\ \times\Bigg[&\sum_{m=0}^{(p-3)/2} q^{(m^2+m)/2}\mathfrak{f}\left(q^{(p^2+(2m+1)p)/2}, q^{(p^2-(2m+1)p)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2})\Bigg] \pmod{16}.\end{aligned} \tag{3.22}$$

Consider the congruence

$$\left(\frac{3k^2+k}{2}\right)+\left(\frac{m^2+m}{2}\right)\equiv 4\left(\frac{p^2-1}{24}\right) \pmod{p}. \tag{3.23}$$

The congruence (3.23) is equivalent to

$$(6k+1)^2+3\,(2m+1)^2\equiv 0 \pmod{p}. \tag{3.24}$$

For $\left(\frac{-3}{p}\right)=-1$, the congruence (3.24) has only solution $k=(\pm p-1)/6$ and $m=(p-1)/2$. Therefore, extracting the terms involving $q^{p^2n+(p^2-1)/6}$ from both sides of (3.22), dividing by $q^{(p^2-1)/6}$ and replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}\left(18\cdot p^2 n+3\cdot(p^2-1)+4\right)q^n \equiv 4g_1\psi(q) \pmod{16}. \tag{3.25}$$

Iterating (3.25) by employing Lemmas 2.4 and 2.5, extracting the terms involving $q^{p^2n+(p^2-1)/6}$, dividing by $q^{(p^2-1)/6}$, and replacing q^{p^2} by q, we deduce that, for integer $\alpha\geq 0$

$$\sum_{n=0}^{\infty} M_{4,9}\left(18\cdot p^{2\alpha} n+3\cdot(p^{2\alpha}-1)+4\right)q^n \equiv 4g_1\psi(q) \pmod{16}. \tag{3.26}$$

Employing Lemmas 2.4 and 2.5 in (3.26) and then extracting the terms involving $q^{pn+(p^2-1)/6}$, dividing by $q^{(p^2-1)/6}$ and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}\left(18\cdot p^{2\alpha+1} n+3\cdot(p^{2\alpha+2}-1)+4\right)q^n \equiv 4g_p\psi(q^p) \pmod{16}. \tag{3.27}$$

Extracting the terms involving q^{pn+j}, for $1 \le j \le p-1$ from both sides of (3.27), we arrive at (3.2). Extracting the terms involving q^{2n+1} from (3.18), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(18n+13)q^n \equiv 12\frac{g_2^2 g_3^2 g_6^2}{g_1^6} \pmod{16}. \tag{3.28}$$

Employing (2.14) with $\{p=2,\ k=1\}$ in (3.28), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(18n+13)q^n \equiv 4\frac{g_6^3}{g_2} \pmod{8}. \tag{3.29}$$

Bringing out the terms containing q^{2n} from both sides of (3.29) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(36n+13)q^n \equiv 4\frac{g_3^3}{g_1} \pmod{8}. \tag{3.30}$$

Employing (2.10) in (3.30), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(36n+13)q^n \equiv 4\frac{g_4^3 g_6^2}{g_2^2 g_{12}} + 4q\frac{g_{12}^3}{g_4} \pmod{8}. \tag{3.31}$$

Bringing out the terms involving q^{2n} from both sides of (3.31) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(72n+13)q^n \equiv 4\frac{g_2^3 g_3^2}{g_1^2 g_6} \pmod{8}. \tag{3.32}$$

Employing (2.14) in (3.32), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(72n+13)q^n \equiv 4g_2^2 \pmod{8}. \tag{3.33}$$

Using (2.2) in (3.33), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(72n+13)q^n \equiv 4g_1\psi(q) \pmod{8}. \tag{3.34}$$

The remaining proof of the identity (3.3) is similar to the proof of (3.2) and the desired result can be obtained by appealing to Lemmas 2.4 and 2.5. So, the details are omited. □

Corollary 3.1 *For $n \ge 0$, we have*

$$M_{4,9}(9n+7) \equiv 0 \pmod{12}, \tag{3.35}$$
$$M_{4,9}(36n+31) \equiv 0 \pmod{8}, \tag{3.36}$$
$$M_{4,9}(36n+34) \equiv 0 \pmod{48}, \tag{3.37}$$
$$M_{4,9}(288n+157) \equiv 0 \pmod{8}. \tag{3.38}$$

Proof: Extracting the terms involving q^{3n+2} from both sides of (3.15), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(9n+7)q^n = 12\frac{g_2^4 g_3^6}{g_1^{10}}. \tag{3.39}$$

The desired result (3.35) follows easily from (3.39). Extracting the terms involving q^{2n+1} from both sides of (3.29), we arrive at (3.36). Employing (2.14) with $\{m=1,\ p=2,\ k=3\}$ in (3.39), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(9n+7)q^n \equiv 12\frac{g_3^6}{g_1^2} \pmod{96}. \tag{3.40}$$

Employing (2.10) in (3.40), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(9n+7)q^n \equiv 12\frac{g_4^6 g_6^4}{g_2^4 g_{12}^2} + 24q\frac{g_4^2 g_6^2 g_{12}^2}{g_2^2} + 12q^2\frac{g_{12}^6}{g_4^2} \pmod{96}. \tag{3.41}$$

Extracting the terms involving q^{2n+1} from both sides of (3.41), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(18n+16)q^n \equiv 24\frac{g_2^2 g_3^2 g_6^2}{g_1^2} \pmod{96}. \tag{3.42}$$

Employing (2.14) in (3.42), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(18n+16)q^n \equiv 24 g_2 g_6^3 \pmod{48}. \tag{3.43}$$

Extracting the terms involving q^{2n+1} from both sides of (3.43), we arrive at (3.37). Extracting the terms involving q^{2n} from both sides of (3.33) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(144n+13)q^n \equiv 4g_1^2 \pmod{8}. \tag{3.44}$$

Employing (2.7) in (3.44), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(144n+13)q^n \equiv 4\frac{g_2 g_8^5}{g_4^2 g_{16}^2} - 8q\frac{g_2 g_{16}^2}{g_8} \pmod{8}. \tag{3.45}$$

Extracting the terms involving q^{2n+1} from (3.45), we arrive at (3.38). □

Theorem 3.2 *For any prime $p \geq 5$, integers $\alpha \geq 0$, $n \geq 0$, and $1 \leq j \leq p-1$, we have*

$$M_{4,9}\left(72\cdot p^{2\alpha+2}n + 72\cdot p^{2\alpha+1}j + 3p^{2\alpha+2}+1\right) \equiv 0 \pmod{8}, \tag{3.46}$$

$$M_{4,9}\left(288\cdot p^{2\alpha+2}n + 288\cdot p^{2\alpha+1}j + 12p^{2\alpha+2}+1\right) \equiv 0 \pmod{8}. \tag{3.47}$$

Proof: Extracting the terms involving q^{2n} from both sides of (3.20) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(36n+4)q^n \equiv 4g_1^2 \pmod{16}. \tag{3.48}$$

Employing (2.14) with $\{m=1,\ p=2,\ k=1\}$ into (3.48) and then extracting the terms involving q^{2n} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(72n+4)q^n \equiv 4g_1 \pmod{8}. \tag{3.49}$$

Employing Lemma 2.4 in (3.49), we obtain

$$\sum_{n\geq 0} M_{4,9}(72n+4)q^n \equiv 4\Bigg[\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} \mathfrak{f}\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} g_{p^2}\Bigg] \pmod{8}. \tag{3.50}$$

Bringing out the terms involving $q^{p^2n+(p^2-1)/24}$ from (3.50), dividing by $q^{(p^2-1)/24}$ and replacing q^{p^2} by q, we obtain

$$\sum_{n\geq 0} M_{4,9}\left(72p^2n+3(p^2-1)+4\right)q^n \equiv 4g_1 \pmod 8. \tag{3.51}$$

Iterating (3.51) by employing Lemma 2.4, extracting the terms involving $q^{p^2n+(p^2-1)/24}$, dividing by $q^{(p^2-1)/24}$ and replacing q^{p^2} by q, we deduce that, for integer $\alpha\geq 0$,

$$\sum_{n\geq 0} M_{4,9}\left(72p^{2\alpha}n+3(p^{2\alpha}-1)+4\right)q^n \equiv 4g_1 \pmod 8. \tag{3.52}$$

Employing Lemma 2.4 in (3.52), collecting the terms involving $q^{pn+(p^2-1)/24}$, dividing by $q^{(p^2-1)/24}$ and replacing q^p by q, we obtain

$$\sum_{n\geq 0} M_{4,9}\left(72p^{2\alpha+1}n+3(p^{2\alpha+2}-1)+4\right)q^n \equiv 4g_p \pmod 8. \tag{3.53}$$

Extracting the terms involving q^{pn+j}, for $1\leq j\leq p-1$ and simplifying, we complete the proof of (3.46). Employing (2.14) in (3.33), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(72n+13)q^n \equiv 4g_4 \pmod 8. \tag{3.54}$$

Extracting the terms involving q^{4n} from both sides of (3.54) and replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(288n+13)q^n \equiv 4g_1 \pmod 8. \tag{3.55}$$

The remaining proof of the identity (3.47) is identitcal to the proof of (3.46) and the desired result can be obtained by appealing to Lemma 2.4. So, the details are omited. □

Theorem 3.3 *Let $p\geq 5$ be any prime such that $\left(\frac{-12}{p}\right)=-1$. Then for any integers $\alpha\geq 0$, $n\geq 0$ and $1\leq j\leq p-1$, we have*

$$M_{4,9}\left(72\cdot p^{2\alpha+2}n+72\cdot p^{2\alpha+1}j+39p^{2\alpha+2}+1\right)\equiv 0 \pmod{16}. \tag{3.56}$$

Proof: Employing (2.7) in (3.48), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(36n+4)q^n \equiv 4\frac{g_2g_8^5}{g_4^2g_{16}^2} - 8q\frac{g_2g_{16}^2}{g_8} \pmod{16}. \tag{3.57}$$

Extracting the terms involving q^{2n+1} from both sides of (3.57), dividing by q and replacing q^2 by q and then using (2.2), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(72n+40)q^n \equiv 8\frac{g_1g_8^2}{g_4} \equiv 8g_1\psi(q^4) \pmod{16}. \tag{3.58}$$

Employing Lemmas 2.4 and 2.5 in (3.58), we obtain

$$\begin{aligned}\sum_{n\geq 0} M_{4,9}(72n+40)q^n \equiv 8\Bigg[&\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2}\mathfrak{f}\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right)\\ &+(-1)^{(\pm p-1)/6}q^{(p^2-1)/24}g_{p^2}\Bigg]\\ \times\Bigg[&\sum_{m=0}^{(p-3)/2} q^{4(m^2+m)/2}\mathfrak{f}\left(q^{4(p^2+(2m+1)p)/2}, q^{4(p^2-(2m+1)p)/2}\right)+q^{4(p^2-1)/8}\psi(q^{4p^2})\Bigg] \pmod{16}.\end{aligned} \tag{3.59}$$

Consider the congruence

$$\left(\frac{3k^2+k}{2}\right)+4\left(\frac{m^2+m}{2}\right)\equiv 13\left(\frac{p^2-1}{24}\right)\pmod{p}. \tag{3.60}$$

The congruence (3.60) is equivalent to

$$(6k+1)^2+12\,(2m+1)^2\equiv 0\pmod{p}. \tag{3.61}$$

For $\left(\frac{-12}{p}\right)=-1$, the congruence (3.61) has only solution $k=(\pm p-1)/6$ and $m=(p-1)/2$. Therefore, extracting the terms involving $q^{p^2n+13(p^2-1)/24}$ from both sides of (3.59), dividing by $q^{13(p^2-1)/24}$ and replacing q^{p^2} by q, we obtain

$$\sum_{n\geq 0} M_{4,9}\left(72\cdot p^2n+39\cdot(p^2-1)+40\right)q^n\equiv 8g_1\psi(q^4)\pmod{16}. \tag{3.62}$$

Iterating (3.62) by employing Lemmas 2.4 and 2.5, bringing out the terms involving $q^{p^2n+13(p^2-1)/24}$, dividing by $q^{13(p^2-1)/24}$, and replacing q^{p^2} by q, we deduce that, for integer $\alpha\geq 0$

$$\sum_{n\geq 0} M_{4,9}\left(72\cdot p^{2\alpha}n+39\cdot(p^{2\alpha}-1)+40\right)q^n\equiv 8g_1\psi(q^4)\pmod{16}. \tag{3.63}$$

Employing Lemmas 2.4 and 2.5 in (3.63) and then extracting the terms involving $q^{pn+13(p^2-1)/24}$, dividing by $q^{13(p^2-1)/24}$, and replacing q^p by q, we obtain

$$\sum_{n\geq 0} M_{4,9}\left(72\cdot p^{2\alpha+1}n+39\cdot(p^{2\alpha+2}-1)+40\right)q^n\equiv 8g_p\psi(q^{4p})\pmod{16}. \tag{3.64}$$

Extracting the terms involving q^{pn+j}, for $1\leq j\leq p-1$ from both sides of (3.64), we arrive at (3.56). □

Theorem 3.4 *For any integers $\alpha\geq 0$ and $n\geq 0$, we have*

$$M_{4,9}\left(36\cdot 5^{2\alpha+2}n+3\cdot 5^{2\alpha+2}+1\right)\equiv 3^{\alpha+1}M_{4,9}\left(36n+4\right)\pmod{16}, \tag{3.65}$$
$$M_{4,9}\left(36\cdot 7^{2\alpha+2}n+3\cdot 7^{2\alpha+2}+1\right)\equiv M_{4,9}\left(36n+4\right)\pmod{16}, \tag{3.66}$$
$$M_{4,9}\left(144\cdot 5^{2\alpha+2}n+12\cdot 5^{2\alpha+2}+1\right)\equiv M_{4,9}\left(144n+13\right)\pmod{8}, \tag{3.67}$$
$$M_{4,9}\left(144\cdot 7^{2\alpha+2}n+12\cdot 7^{2\alpha+2}+1\right)\equiv M_{4,9}\left(144n+13\right)\pmod{8}, \tag{3.68}$$
$$M_{4,9}\left(9\cdot 2^{2\alpha+4}n+3\cdot 2^{2\alpha+4}+1\right)\equiv M_{4,9}\left(36n+13\right)\pmod{8}. \tag{3.69}$$

Proof: Employing (2.4) in (3.48), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(36n+4)q^n\equiv 4g_{25}^2\left[R^2(q^5)-2qR(q^5)-q^2+2q^3/R(q^5)+q^4/R^2(q^5)\right]\pmod{16}. \tag{3.70}$$

Extracting the terms involving q^{5n+2} from both sides of (3.70), dividing by q^2 and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(180n+76)q^n\equiv 12g_5^2\pmod{16}. \tag{3.71}$$

Extracting the terms involving q^{5n} from both sides of (3.71) and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(900n+76)q^n\equiv 12g_1^2\pmod{16}. \tag{3.72}$$

In view of the congruences (3.48) and (3.72), we obtain

$$M_{4,9}(900n+76) \equiv 3 \cdot M_{4,9}(36n+4) \pmod{16}. \tag{3.73}$$

Using (3.73) and by induction on α, we arrive at (3.65). Employing (2.5) in (3.48), we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(36n+4)q^n \equiv 4g_{49}^2\left[\frac{B^2(q^7)}{C^2(q^7)} - 2q\frac{A(q^7)}{C(q^7)} + q^2\frac{A^2(q^7)}{B^2(q^7)} - 2q^2\frac{B(q^7)}{C(q^7)} + 2q^3\frac{A(q^7)}{B(q^7)} + q^4 \right. \\ \left. + 2q^5\frac{B(q^7)}{A(q^7)} - 2q^6\frac{C(q^7)}{B(q^7)} - 2q^7\frac{C(q^7)}{A(q^7)} + q^{10}\frac{C^2(q^7)}{A^2(q^7)}\right] \pmod{16}. \tag{3.74}$$

Extracting the terms involving q^{7n+4} from both sides of (3.74), dividing by q^4 and replacing q^7 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(252n+148)q^n \equiv 4g_7^2 \pmod{16}. \tag{3.75}$$

Extracting the terms involving q^{7n} from both sides of (3.75) and replacing q^7 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(1764n+148)q^n \equiv 4g_1^2 \pmod{16}. \tag{3.76}$$

In view of the congruences (3.48) and (3.76), we obtain

$$M_{4,9}(1764n+148) \equiv M_{4,9}(36n+4) \pmod{16}. \tag{3.77}$$

Using (3.77) and by induction on α, we arrive at (3.66). Employing (2.4) in (3.44) and then extracting the terms involving q^{5n+2} from both sides of the resulting equation, dividing by q^2 and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(720n+301)q^n \equiv 4g_5^2 \pmod{8}. \tag{3.78}$$

Extracting the terms involving q^{5n} from both sides of (3.78) and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(3600n+301)q^n \equiv 4g_1^2 \pmod{8}. \tag{3.79}$$

In view of the congruences (3.44) and (3.79), we obtain

$$M_{4,9}(3600n+301) \equiv M_{4,9}(144n+13) \pmod{8}. \tag{3.80}$$

Using (3.80) and by induction on α, we arrive at (3.67). Employing (2.5) in (3.44) and then extracting the terms involving q^{7n+4} from both sides of the resulting equation, dividing by q^4 and replacing q^7 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(1008n+589)q^n \equiv 4g_7^2 \pmod{8}. \tag{3.81}$$

Extracting the terms involving q^{7n} from both sides of (3.81) and replacing q^7 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(7056n+589)q^n \equiv 4g_1^2 \pmod{8}. \tag{3.82}$$

Combining (3.44) and (3.82), we obtain

$$M_{4,9}(7056n+589) \equiv M_{4,9}(144n+13) \pmod{8}. \tag{3.83}$$

Using (3.83) and by induction on α, we arrive at (3.68). Extracting the terms involving q^{2n+1} from both sides of (3.31), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(72n+49)q^n \equiv 4\frac{g_6^3}{g_2} \pmod 8. \tag{3.84}$$

Extracting the terms involving q^{2n} from both sides of (3.84) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,9}(144n+49)q^n \equiv 4\frac{g_3^3}{g_1} \pmod 8. \tag{3.85}$$

Combining (3.30) and (3.85), we obtain

$$M_{4,9}(144n+49) \equiv M_{4,9}(36n+13) \pmod 8. \tag{3.86}$$

Using (3.86) and by induction on α, we arrive at (3.69). □

Theorem 3.5 *Let $p \geq 5$ be any prime such that $\left(\frac{-12}{p}\right) = -1$. Then for any integers $\alpha \geq 0$, $n \geq 0$, and $1 \leq j \leq p-1$, we have*

$$M_{3,9}\left(2\cdot p^{2\alpha+2}n + 2\cdot p^{2\alpha+1}j + \frac{13p^{2\alpha+2}-1}{12}\right) \equiv 0 \pmod 3. \tag{3.87}$$

Proof: Setting $s=3$ and $t=9$ in (1.2), we obtain

$$\sum_{n=0}^{\infty} M_{3,9}(n)q^n = \frac{g_3}{g_1}. \tag{3.88}$$

Employing (2.14) with $\{p=3,\ k=1\}$ in (3.88), we obtain

$$\sum_{n=0}^{\infty} M_{3,9}(n)q^n \equiv g_1^2 \pmod 3. \tag{3.89}$$

Employing (2.7) in (3.89), we obtain

$$\sum_{n=0}^{\infty} M_{3,9}(n)q^n \equiv \frac{g_2 g_8^5}{g_4^2 g_{16}^2} - 2q\frac{g_2 g_{16}^2}{g_8} \pmod 3. \tag{3.90}$$

Extracting the terms involving q^{2n+1} from both sides of (3.90), dividing by q and replacing q^2 by q and then using (2.2), we obtain

$$\sum_{n=0}^{\infty} M_{3,9}(2n+1)q^n \equiv g_1\psi(q^4) \pmod 3. \tag{3.91}$$

Employing Lemmas 2.4 and 2.5 in (3.91), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} M_{3,9}(2n+1)q^n \equiv \Bigg[&\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} \mathfrak{f}\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right)\\ &+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} g_{p^2}\Bigg]\\ \times \Bigg[&\sum_{m=0}^{(p-3)/2} q^{4(m^2+m)/2} \mathfrak{f}\left(q^{4(p^2+(2m+1)p)/2}, q^{4(p^2-(2m+1)p)/2}\right) + q^{4(p^2-1)/8}\psi(q^{4p^2})\Bigg] \pmod 3. \end{aligned}\tag{3.92}$$

Consider the congruence

$$\left(\frac{3k^2+k}{2}\right)+4\left(\frac{m^2+m}{2}\right)\equiv 13\left(\frac{p^2-1}{24}\right). \tag{3.93}$$

The congruence (3.93) is equivalent to

$$(6k+1)^2+12\,(2m+1)^2\equiv 0 \pmod{p}. \tag{3.94}$$

For $\left(\frac{-12}{p}\right)=-1$, the congruence (3.94) has only solution $k=(\pm p-1)/6$ and $m=(p-1)/2$. Therefore, extracting the terms involving $q^{p^2n+13(p^2-1)/24}$ from both sides of (3.92), dividing by $q^{13(p^2-1)/24}$ and replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty} M_{3,9}\left(2p^2n+\frac{13(p^2-1)}{12}+1\right)q^n\equiv g_1\psi(q^4) \pmod{3}. \tag{3.95}$$

Iterating (3.95) by employing Lemmas 2.4 and 2.5, extracting the terms involving $q^{p^2n+13(p^2-1)/24}$, dividing by $q^{13(p^2-1)/24}$, and replacing q^{p^2} by q, we deduce that, for integer $\alpha\geq 0$

$$\sum_{n=0}^{\infty} M_{3,9}\left(2p^{2\alpha}n+\frac{13(p^{2\alpha}-1)}{12}+1\right)q^n\equiv g_1\psi(q^4) \pmod{3}. \tag{3.96}$$

Employing Lemmas 2.4 and 2.5 in (3.96) and then extracting the terms involving $q^{pn+13(p^2-1)/24}$, dividing by $q^{13(p^2-1)/24}$ and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} M_{3,9}\left(2p^{2\alpha+1}n+\frac{13(p^{2\alpha+2}-1)}{12}+1\right)q^n\equiv g_p\psi(q^{4p}) \pmod{3}. \tag{3.97}$$

Extracting the terms involving q^{pn+j}, for $1\leq j\leq p-1$ from both sides of (3.97), we arrive at (3.87). □

Theorem 3.6 *Let $p\geq 3$ be any prime such that $\left(\frac{-6}{p}\right)=-1$. Then for any integers $\alpha\geq 0$, $n\geq 0$, and $1\leq j\leq p-1$, we have*

$$M_{4,6}\left(9\cdot p^{2\alpha+2}n+9\cdot p^{2\alpha+1}j+\frac{63p^{2\alpha+2}+1}{8}\right)\equiv 0 \pmod{6}. \tag{3.98}$$

Proof: Setting $s=4$ and $t=6$ in (1.2), we obtain

$$\sum_{n=0}^{\infty} M_{4,6}(n)q^n=\frac{g_4g_6}{g_1g_{12}}. \tag{3.99}$$

Employing (2.12) in (3.99), we obtain

$$\sum_{n=0}^{\infty} M_{4,6}(n)q^n=\frac{g_6g_{18}^4}{g_3^3g_{36}^2}+q\frac{g_6^3g_9^3g_{36}}{g_3^4g_{12}g_{18}^2}+2q^2\frac{g_6^2g_{18}g_{36}}{g_3^3g_{12}}. \tag{3.100}$$

Extracting the terms involving q^{3n+2} from both sides of (3.100), dividing by q^2 and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,6}(3n+2)q^n=2\frac{g_2^2g_6g_{12}}{g_1^3g_4}. \tag{3.101}$$

Employing (2.14) with $\{p=3,\ k=1\}$ in (3.101), we obtain

$$\sum_{n=0}^{\infty} M_{4,6}(3n+2)q^n \equiv 2\frac{g_2^5g_4^2}{g_3} \equiv 2\frac{g_2^6g_4^2}{g_2g_3} \pmod 6. \tag{3.102}$$

Employing (2.14) in (3.102), we obtain

$$\sum_{n=0}^{\infty} M_{4,6}(3n+2)q^n \equiv 2\frac{g_4^2g_6^2}{g_2g_3} \pmod 6. \tag{3.103}$$

Employing (2.8) in (3.103), we obtain

$$\sum_{n=0}^{\infty} M_{4,6}(3n+2)q^n \equiv 2\frac{g_6g_{12}g_{18}^2}{g_3g_{36}} + 2q^2\frac{g_6^2g_{36}^2}{g_3g_{18}} \pmod 6. \tag{3.104}$$

Extracting the terms involving q^{3n+2} from both sides of (3.104), dividing by q^2 and replacing q^3 by q and then using (2.2), we obtain

$$\sum_{n=0}^{\infty} M_{4,6}(9n+8)q^n \equiv 2\frac{g_2^2g_{12}^2}{g_1g_6} \equiv 2\psi(q)\psi(q^6) \pmod 6. \tag{3.105}$$

Employing Lemma 2.5 in (3.105), we obtain

$$\sum_{n=0}^{\infty} M_{4,6}(9n+8)q^n \equiv 2\left[\sum_{k=0}^{(p-3)/2} q^{(k^2+k)/2}\mathfrak{f}\left(q^{(p^2+(2k+1)p)/2}, q^{(p^2-(2k+1)p)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2})\right]$$
$$\times\left[\sum_{m=0}^{(p-3)/2} q^{6(m^2+m)/2}\mathfrak{f}\left(q^{6(p^2+(2m+1)p)/2}, q^{6(p^2-(2m+1)p)/2}\right) + q^{6(p^2-1)/8}\psi(q^{6p^2})\right] \pmod 6. \tag{3.106}$$

Consider the congruence

$$\left(\frac{k^2+k}{2}\right) + 6\left(\frac{m^2+m}{2}\right) \equiv 7\left(\frac{p^2-1}{8}\right) \pmod p. \tag{3.107}$$

The congruence (3.107) is equivalent to

$$(2k+1)^2 + 6\,(2m+1)^2 \equiv 0 \pmod p. \tag{3.108}$$

For $\left(\frac{-6}{p}\right) = -1$, the congruence (3.108) has only solution $k=m=(\pm p-1)/2$. Therefore, extracting the terms involving $q^{p^2n+7(p^2-1)/8}$ from both sides of (3.106), dividing by $q^{7(p^2-1)/8}$ and replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,6}\left(9\cdot p^2n + \frac{63(p^2-1)}{8} + 8\right)q^n \equiv 2\psi(q)\psi(q^6) \pmod 6. \tag{3.109}$$

Iterating (3.109) by employing Lemma 2.5, extracting the terms involving $q^{p^2n+7(p^2-1)/8}$, dividing by $q^{7(p^2-1)/8}$, and replacing q^{p^2} by q, we deduce that, for integer $\alpha \geq 0$

$$\sum_{n=0}^{\infty} M_{4,6}\left(9\cdot p^{2\alpha}n + \frac{63(p^{2\alpha}-1)}{8} + 8\right)q^n \equiv 2\psi(q)\psi(q^6) \pmod 6. \tag{3.110}$$

Employing Lemma 2.5 in (3.110) and then extracting the terms involving $q^{pn+7(p^2-1)/8}$, dividing by $q^{7(p^2-1)/8}$ and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,6}\left(9\cdot p^{2\alpha+1}n + \frac{63\cdot(p^{2\alpha+2}-1)}{8} + 8\right)q^n \equiv 2\psi(q^p)\psi(q^{6p}) \pmod 6. \tag{3.111}$$

Extracting the terms involving q^{pn+j}, for $1 \leq j \leq p-1$ from both sides of (3.111), we arrive at (3.98). □

Corollary 3.2 *For $n \geq 0$, we have*

$$M_{4,6}(9n+5) \equiv 0 \pmod{6}. \tag{3.112}$$

Proof: Extracting the terms involving q^{3n+1} from both sides of (3.104), we arrive at (3.112). □

Theorem 3.7 *For any odd prime p, integers $\alpha \geq 0$, $n \geq 0$, and $1 \leq j \leq p-1$, we have*

$$M_{4,12}\left(p^{2\alpha+2}n + p^{2\alpha+1}j + \frac{p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod{2}. \tag{3.113}$$

Proof: Setting $s=4$ and $t=12$ in (1.2), we obtain

$$\sum_{n=0}^{\infty} M_{4,12}(n)q^n = \frac{g_4}{g_1}. \tag{3.114}$$

Employing (2.14) in (3.114) and then using (2.2), we obtain

$$\sum_{n=0}^{\infty} M_{4,12}(n)q^n \equiv \psi(q) \pmod{2}. \tag{3.115}$$

Employing Lemma 2.5 in (3.115), we obtain

$$\sum_{n=0}^{\infty} M_{4,12}(n)q^n \equiv \sum_{i=0}^{(p-3)/2} q^{(i^2+i)/2} \mathfrak{f}\left(q^{(p^2+(2i+1)p)/2}, q^{(p^2-(2i+1)p)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2}) \pmod{2}. \tag{3.116}$$

Extracting the terms involving $q^{p^2n+(p^2-1)/8}$, dividing by $q^{(p^2-1)/8}$, replacing q^{p^2} by q and simplifying, we obtain

$$\sum_{n=0}^{\infty} M_{4,12}\left(p^2n + \frac{p^2-1}{8}\right) q^n \equiv \psi(q) \pmod{2}. \tag{3.117}$$

Iterating (3.117) by employing Lemma 2.5, extracting the terms involving $q^{p^2n+(p^2-1)/8}$, dividing by $q^{(p^2-1)/8}$, replacing q^{p^2} by q and simplifying, we deduce that, for integer $\alpha \geq 0$

$$\sum_{n=0}^{\infty} M_{4,12}\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{8}\right) q^n \equiv \psi(q) \pmod{2}. \tag{3.118}$$

Employing Lemma 2.5 in (3.118) and then extracting the terms involving $q^{pn+(p^2-1)/8}$, dividing by $q^{(p^2-1)/8}$, replacing q^p by q and simplifying, we obtain

$$\sum_{n=0}^{\infty} M_{4,12}\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{8}\right) q^n \equiv \psi(q^p) \pmod{2}. \tag{3.119}$$

Extracting the terms involving q^{pn+j}, for $1 \leq j \leq p-1$ from (3.119) and simplifying, we complete the proof. □

Theorem 3.8 *For any integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$M_{4,12}\left(9^\alpha n + \frac{9^\alpha - 1}{8}\right) \equiv M_{4,12}(n) \pmod{4}. \tag{3.120}$$

Proof: Employing (2.12) in (3.114), we obtain

$$\sum_{n=0}^{\infty} M_{4,12}(n)q^n = \frac{g_{12}g_{18}^4}{g_3^3 g_{36}^2} + q\frac{g_6^2 g_9^3 g_{36}}{g_3^4 g_{18}^2} + 2q^2\frac{g_6 g_{18} g_{36}}{g_3^3}. \tag{3.121}$$

Extracting the terms involving q^{3n+1} from both sides of (3.121), dividing by q and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,12}(3n+1)q^n = \frac{g_2^2 g_3^3 g_{12}}{g_1^4 g_6^2} = \left(\frac{g_2}{g_1^2}\right)^2 \frac{g_3^3 g_{12}}{g_6^2}. \tag{3.122}$$

Employing (2.9) in (3.122), we obtain

$$\sum_{n=0}^{\infty} M_{4,12}(3n+1)q^n = \frac{g_6^6 g_9^{12} g_{12}}{g_3^{13} g_{18}^6} + 4q\frac{g_6^5 g_9^9 g_{12}}{g_3^{12} g_{18}^3} + 12q^2\frac{g_6^4 g_9^6 g_{12}}{g_3^{11}} + 16q^3\frac{g_6^3 g_9^3 g_{12} g_{18}^3}{g_3^{10}} + 16q^4\frac{g_6^2 g_{12} g_{18}^6}{g_3^9}. \tag{3.123}$$

Extracting the terms involving q^{3n} from both sides of (3.123) and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} M_{4,12}(9n+1)q^n \equiv \frac{g_2^6 g_3^{12} g_4}{g_1^{13} g_6^6} \pmod{16}. \tag{3.124}$$

Employing (2.14) with $\{p=2,\ k=2\}$, we obtain

$$\sum_{n=0}^{\infty} M_{4,12}(9n+1)q^n \equiv \frac{g_4}{g_1} \pmod{4}. \tag{3.125}$$

In view of the congruences (3.114) and (3.125), we obtain

$$M_{4,12}(9n+1) \equiv M_{4,12}(n) \pmod{4}. \tag{3.126}$$

Using the above relation and by induction on α, we arrive at (3.120). □

Corollary 3.3 *For $n \geq 0$, we have*

$$M_{4,12}(9n+3k+1) \equiv 0 \pmod{4}, \quad k=1,2. \tag{3.127}$$

Proof: Extracting the terms involving q^{3n+k}, for $k=1,2$ from both sides of (3.123), we arrive at (3.127). □

Theorem 3.9 *Let $p \geq 5$ be any prime such that $\left(\frac{-9}{p}\right) = -1$. Then for any integers $\alpha \geq 0$, $n \geq 0$, and $1 \leq j \leq p-1$, we have*

$$M_{6,9}\left(2\cdot p^{2\alpha+2}n + 2\cdot p^{2\alpha+1}j + \frac{5p^{2\alpha+2}+1}{6}\right) \equiv 0 \pmod{2}. \tag{3.128}$$

Proof: Setting $s=6$ and $t=9$ in (1.2), we obtain

$$\sum_{n=0}^{\infty} M_{6,9}(n)q^n = \frac{g_6 g_9}{g_1 g_{18}}. \tag{3.129}$$

Employing (2.13) in (3.129), we obtain

$$\sum_{n=0}^{\infty} M_{6,9}(n)q^n = \frac{g_{12}^3}{g_2^2 g_{36}} + q\frac{g_4^2 g_6^2 g_{36}}{g_2^3 g_{12} g_{18}}. \tag{3.130}$$

Extracting the terms involving q^{2n+1} from both sides of (3.130), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{6,9}(2n+1)q^n = \frac{g_2^2 g_3^2 g_{18}}{g_1^3 g_6 g_9}. \tag{3.131}$$

Employing (2.14) in (3.131), we obtain

$$\sum_{n=0}^{\infty} M_{6,9}(2n+1)q^n \equiv g_1 g_9 \pmod 2. \tag{3.132}$$

The remaining proof of the identity (3.128) is similar to the proof of (3.1) and the desired result can be obtained by appealing to Lemma 2.4. So, the details are omited. □

Theorem 3.10 *Let $p \geq 5$ be any prime such that $\left(\frac{-3}{p}\right) = -1$. Then for any integers $\alpha \geq 0$, $n \geq 0$, and $1 \leq j \leq p-1$, we have*

$$M_{9,12}\left(2 \cdot p^{2\alpha+2}n + 2 \cdot p^{2\alpha+1}j + \frac{p^{2\alpha+2}+2}{3}\right) \equiv 0 \pmod 3. \tag{3.133}$$

Proof: Setting $s = 9$ and $t = 12$ in (1.2), we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(n)q^n = \frac{g_9 g_{12}}{g_1 g_{36}}. \tag{3.134}$$

Employing (2.13) in (3.134), we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(n)q^n = \frac{g_{12}^4 g_{18}}{g_2^2 g_6 g_{36}^2} + q\frac{g_4^2 g_6}{g_2^3}. \tag{3.135}$$

Extracting the terms involving q^{2n+1} from both sides of (3.135), dividing by q, and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(2n+1)q^n = \frac{g_2^2 g_3}{g_1^3}. \tag{3.136}$$

Employing (2.14) in (3.136), we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(2n+1)q^n \equiv g_2^2 \pmod 3. \tag{3.137}$$

Employing (2.2) in (3.137), we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(2n+1)q^n \equiv g_1 \psi(q) \pmod 3. \tag{3.138}$$

The remaining proof of the identity (3.133) is similar to the proof of (3.2) and the desired result can be obtained by appealing to Lemmas 2.4 and 2.5. So, we omit the details. □

Theorem 3.11 *Let $p \geq 5$ be any prime such that $\left(\frac{-12}{p}\right) = -1$. Then for any integers $\alpha \geq 0$, $n \geq 0$, and $1 \leq j \leq p-1$, we have*

$$M_{9,12}\left(8 \cdot p^{2\alpha+2}n + 8 \cdot p^{2\alpha+1}j + \frac{13p^{2\alpha+2}+2}{3}\right) \equiv 0 \pmod 3. \tag{3.139}$$

Proof: Extracting the terms involving q^{2n} from both sides of (3.137) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(4n+1)q^n \equiv g_1^2 \pmod{3}. \tag{3.140}$$

Employing (2.7) in (3.140), we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(4n+1)q^n \equiv \frac{g_2 g_8^5}{g_4^2 g_{16}^2} - 2q\frac{g_2 g_{16}^2}{g_8} \pmod{3}. \tag{3.141}$$

Extracting the terms involving q^{2n+1} from both sides of (3.141), dividing by q and replacing q^2 by q and then using (2.2), we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(8n+5)q^n \equiv g_1\psi(q^4) \pmod{3}. \tag{3.142}$$

The remaining proof of the identity (3.139) is similar to the proof of (3.56) and the desired result can be obtained by appealing to Lemmas 2.4 and 2.5. So, the details are omited. □

Theorem 3.12 *For any integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$M_{9,12}\left(\frac{5^{2\alpha+2}(12n+1)+2}{3}\right) \equiv 2^{\alpha+1}M_{9,12}(4n+1) \pmod{3}, \tag{3.143}$$

$$M_{9,12}\left(4\cdot 7^{2\alpha+2}n + \frac{7^{2\alpha+2}+2}{3}\right) \equiv M_{9,12}(4n+1) \pmod{3}. \tag{3.144}$$

Proof: Employing (2.4) in (3.140) and then extracting the terms involving q^{5n+2} from both sides of the resultant equation, dividing by q^2 and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(20n+9)q^n \equiv 2g_5^2 \pmod{3}. \tag{3.145}$$

Extracting the terms involving q^{5n} from both sides of (3.145) and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(100n+9)q^n \equiv 2g_1^2 \pmod{3}. \tag{3.146}$$

Combining (3.140) and (3.146), we obtain

$$M_{9,12}(100n+9) \equiv 2M_{9,12}(4n+1) \pmod{3}. \tag{3.147}$$

Using (3.147) and by induction on α, we arrive at (3.143). Employing (2.5) in (3.140) and then extracting the terms involving q^{7n+4} from both sides of the resultant equation, dividing by q^4 and replacing q^7 by q, we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(28n+17)q^n \equiv g_7^2 \pmod{3}. \tag{3.148}$$

Extracting the terms involving q^{7n} from both sides of (3.148) and replacing q^7 by q, we obtain

$$\sum_{n=0}^{\infty} M_{9,12}(196n+17)q^n \equiv g_1^2 \pmod{3}. \tag{3.149}$$

Combining (3.140) and (3.149), we obtain

$$M_{9,12}(196n+17) \equiv M_{9,12}(4n+1) \pmod{3}. \tag{3.150}$$

Using (3.150) and by induction on α, we arrive at (3.144).

□

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