Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.68795

Connectedness in Mixed Fuzzy Topology

Hari Prasad Chetri and Gautam Chandra Ray*

ABSTRACT: In this article, connectedness in mixed fuzzy topology is introduced. Four notions of connectedness are investigated in mixed fuzzy topology. From the connectedness of parent fuzzy topology theories and concepts of connectedness in mixed fuzzy topology are studied and established and vice versa. Many exciting examples and counterexamples are also included to give a clear picture of the concept to the reader.

Key Words: Fuzzy sets, Fuzzy topology, Mixed fuzzy topology, Mixed topology, Connectedness in fuzzy topology, Connectedness in mixed fuzzy topology.

Contents

1. Introduction

In 1965, Zadeh introduced fuzzy sets and logic in his classical paper [11] which provided the foundation for fuzzy mathematics. The concept of fuzzy logic and fuzzy sets is applied in almost all disciplines of science and technology. In 1968, fuzzy sets and logic were applied by Chang [7] to study topology and topological properties. Thereafter many researchers have been studying fuzzy topology and fuzzified many concepts of general topology. In 1995, Baishya and Das mixed two fuzzy topologies on an underlying set with the help of quasi-coincidence in fuzzy sets and closure in one of the fuzzy topologies to get an open set for a fuzzy topology called mixed fuzzy topology [14]. Some of the contributions to the theory of mixed fuzzy topology are found in [3,4,5] by Tripathy and Ray. Vadivel and Vijayalakshmi had studied mixed e-fuzzy topological space (2017) [2]. Al-Omeri introduced and studied Mixed γ-Fuzzy in Mixed-fuzzy topological spaces and their application (2018) [22] and Mixed b-fuzzy topological spaces (2020) [23]. Ray and Chetri studied Mixed Fuzzy topological space its Hausdorff properties and base [10], the theory of Separation Axioms in Mixed Fuzzy Topological Spaces [8] and Countability in Mixed Fuzzy Topological Spaces [9]. Borah and Hazarika studied Soft ideal topological space and Mixed fuzzy soft ideal Topological space [12]. But no investigation is made by any researcher about the connectedness of mixed fuzzy topology. Lowen [16], Fatteh, and Bassan [21] studied connectedness in fuzzy topology but only for crisp sets in fts. Ajmal and Kohli [1] extended the notion of connectedness to an arbitrary fuzzy set using four notions of connectedness viz. c_1, c_2, c_3 and c_4 connectedness. Connectedness in fuzzy topological spaces much like in classical topology, helps to characterize the structure and behaviour of these spaces. In fuzzy topology connectedness is used to study the continuity of fuzzy mappings, fuzzy compactness and other properties. It plays a crusial role in understanding the relationship between different parts of the space, particularly when dealing with uncertainty or vagueness in the underlying structure. It is interesting to investigate whether the connectedness properties will remain the same or not in the mixed fuzzy topological spaces when the parent topologies have the properties.

In this paper, analogous four connectedness definitions and theories are formulated and established in a mixed fuzzy topology with these four notions of connectedness in fuzzy topology. Connectedness in mixed fuzzy topology is investigated in details, many interesting examples and counterexamples are also

* Corresponding author

Submitted July 03, 2023. Published April 09, 2024
2010 *Mathematics Subject Classification:* 35B40, 35L70.

included to give a clear insight into the concept. Basic definitions are included in the preliminary section though most of them are standard.

2. Preliminaries

Definition 2.1 *[15] Let $X \neq \emptyset$ and $I = [0,1]$be the unit interval. Let $\mu_A : X \to I$, where μ_A is the membership function of A characterize a function μ_A in X is called a fuzzy set. Here $\mu_A(x)$ represents the membership grade of $x\epsilon A$. The null fuzzy set is defined by $\mu_\varphi(x) = 0, \ \forall \ x\epsilon X$. The whole space X can be defined as $\mu_X(x) = 1, \ \forall \ x\epsilon X$. Sometimes fuzzy null set and fuzzy whole space are denoted X by $\bar{0}$ and $\bar{1}$, respectively. If the membership functions of two fuzzy sets A and B are equal i.e., $\mu_A(x) = \mu_B(x) \forall \ x\epsilon X$ then fuzzy sets are equal i.e. $A = B$. If A and B are two fuzzy sets with $\mu_A(x) \leq \mu_B(x) \ \forall \ x\epsilon X$, then it is written as $A \leq B$. The membership function $\mu_{A^/} = 1 - \mu_A$ defines the complement $A^/$ or A^c or $co(A)$ and is used to represent the fuzzy complement. For a collection $\{A_i : i\epsilon I \}$ of fuzzy sets $\bigvee_{i\epsilon I} A_i$ and $\bigwedge_i A_i$ represent fuzzy union and fuzzy intersections and are, respectively defined as follows:*

$$\mu_{\vee_i A_i}(x) = sup\{\mu_{A_i}(x) : i\epsilon I\}, \forall \ x\epsilon X.$$
$$\mu_{\wedge_i A_i}(x) = inf\{\mu_{A_i}(x) : i\epsilon I\}, \ \forall \ x\epsilon X.$$

Definition 2.2 *[3] Let I=[0,1] and $X \neq \emptyset$. Consider I^X be the collection of all mappings from X into I. A family τ of members of I^X such that*

(i) $\bar{1}$,$\bar{0}$ ϵ τ.

(ii) For any finite sub collection $\mathfrak{B} = \{B_i\}_{i=1}^n$ of members of τ, $\bigcap_{i=1}^n \{B_i\} \ \epsilon \ \tau$.

(iii) For any arbitrary collection Δ of members of τ, $\bigcup_{B\epsilon\Delta} \epsilon\tau$.

Then τ is called fuzzy topology and the pair (X, τ) is called a fuzzy topological space and members of τ are called $\tau -$ open fuzzy sets. Fuzzy topological space will be abbreviated as fts.

Definition 2.3 *[15] For a fuzzy set A in an fts (X, τ) the interior is defined as $A^0 = \vee\{B| \ B \leq A, \ B \ is \ an \ open \ fuzzy \ set\}$ and denoted by A^0. If there are more than one fuzzy topology then it is written as A^0_τ.*

Definition 2.4 *[15] Let A be fuzzy set in an fts (X, τ) then closure of A is defined as $\overline{A} = \wedge\{F|A \leq F, Fisaclosedfuzzyset\}$ and is denoted by $\bar{A}$. If there are more than one fuzzy topology then $\bar{A}_\tau$ will be used.*

Definition 2.5 *[15] A subfamily $\mathfrak{B}$ of τ in an fts (X, τ) is called a base for τ iff, $(iff \ is \ the \ abbreviation \ for \ if \ and \ only \ if)$ for every $A\epsilon \ \tau$, there exists $\mathfrak{B}_A \subseteq \mathfrak{B}$ such that $A = \vee\{B : B\epsilon \ \mathfrak{B}_A\}$.*

Definition 2.6 *[6] A fuzzy set is called a fuzzy point iff it takes the value $\lambda \, (0 < \lambda < 1)$ at one point say $x\epsilon \ X$ and 0 for all other $y\epsilon X$ and it is denoted by x_λ. Where X is the whole space, x is its support and λ its value.*

Definition 2.7 *[15] A fuzzy point $x_\lambda\epsilon \ A$ iff $\lambda \leq A(x)$.*

Definition 2.8 *[15] Let A and B be two fuzzy sets in X. They are said to be intersecting iff* $\exists$ *a point* $x\epsilon\ X$ *such that* $(A \wedge B)(x) \neq 0$. *Then it is called A and B intersects at x.*

Definition 2.9 *[15] If* $\lambda > A^{/}(x)$, *or* $\lambda + A(x) > 1$ *then* x_λ *is said to be quasi-coincident with a fuzzy set A. Which is denoted as* $x_\lambda qA$.

Definition 2.10 *[15] Two fuzzy sets A and B are called quasi-coincident (overlapping) iff* $\exists x\epsilon X$ *such that* $A(x) + B(x) > 1$ *and denote it by* AqB.

Definition 2.11 *[15] A fuzzy set A in* (X, τ) *is called a Q- neighborhood of* x_λ *iff* $\exists$ $B\epsilon\tau$ *such that* $B \leq A$ *and* $x_\lambda qB$.

Definition 2.12 *[14] Let* (X, τ_1) *and* (X, τ_2) *be two fts. Then the following collection of fuzzy sets* $\tau_1(\tau_2) = \left\{A\epsilon I^X : \text{ for every } x_\alpha qA, \exists\tau_2 - Q - \text{neighborhood } A_\alpha \text{ of } x_\alpha \text{ such that } (\bar{A_1})_{\tau_1} \leq A\right\}.$ *Then this collection of fuzzy sets is a fuzzy topology and it is called mixed fuzzy topology on X.*

Definition 2.13 *[1] A fuzzy set A in a fuzzy topological space* (X, τ) *has a fuzzy* c_i*-disconnection* $(i = 1,\ 2,\ 3,\ 4)$, *if there exist fuzzy open sets U and V in* (X, τ), *such that, respectively,*

$$c_1 : A \leq U \vee V,\ U \wedge V \leq \bar{1} - A, A \wedge U \neq \bar{0} \text{ and } A \wedge V \neq \bar{0}.$$
$$c_2 : A \leq U \vee V,\ A \wedge U \wedge V = \bar{0}, A \wedge U \neq \bar{0} \text{ and } A \wedge V \neq \bar{0}.$$
$$c_3 : A \leq U \vee V,\ U \wedge V \leq \bar{1} - A,\ U \nleq \bar{1} - A \text{ and } V \nleq \bar{1} - A.$$
$$c_4 : A \leq U \vee V,\ A \wedge U \wedge V = \bar{0},\ U \nleq \bar{1} - A \text{ and } V \nleq \bar{1} - A.$$

Definition 2.14 *[1] A fuzzy set A in a fts* (X, τ) *is said to be* c_i*-connected* $(i = 1,\ 2,\ 3,\ 4)$, *if there exist no* c_i*-disconnection* $(i = 1,\ 2,\ 3,\ 4)$, *of A in* (X, τ). *If the fuzzy set A=X, X whole space (i.e., constant function* $\bar{1}$*) in all the four forms of disconnections, we get the same disconnection of the fuzzy space X, which will be refer to as c-disconnection of the fuzzy topological space* (X, τ), *i.e., there exist nonzero fuzzy open sets U and Vsuch that* $U + V = \bar{1}$ *and* $U \wedge V = \bar{0}$.

Definition 2.15 *[1] A fuzzy topological space* (X, τ) *is said to be c-connected if there exist no c-disconnection of* $\bar{1}$.

Lemma 2.1 *[13] "Consider* (X, τ_1) *and* (X, τ_2) *be two fuzzy topological spaces such that every* τ_1 *- Q-neighborhood of* x_λ *is* τ_2 *- Q- neighborhood of* x_λ *for all fuzzy point* x_λ *then* τ_1 *is coarser than* τ_2*."*

Theorem 2.1 *[14] Let* τ_1 *and* τ_2 *be two fts on X. Then mixed fuzzy topology* $\tau_1(\tau_2)$ *is coarser than* τ_2 *. That is* $(\tau_2)\tau_2 \subseteq \tau_2$.

Theorem 2.2 *[14] If* τ_1 *is fuzzy regular and* $\tau_1 \subseteq \tau_2$ *then* $\tau_1 \subseteq \tau_1(\tau_2)$.

3. Connectedness in Mixed fuzzy Topology

Definition 3.1 *[1] A fuzzy set A in a mixed fuzzy topological space* $(X, \tau_1(\tau_2))$ *has a fuzzy* c_i*-disconnection* $(i = 1,\ 2,\ 3,\ 4)$, *if there exist fuzzy open sets U and V in* $(X, \tau_1(\tau_2))$, *such that, respectively,*

$$c_1 : A \leq U \vee V,\ U \wedge V \leq \bar{1} - A, A \wedge U \neq \bar{0} \textit{ and } A \wedge V \neq \bar{0}.$$
$$c_2 : A \leq U \vee V,\ A \wedge U \wedge V = \bar{0}, A \wedge U \neq \bar{0} \textit{ and } A \wedge V \neq \bar{0}.$$
$$c_3 : A \leq U \vee V,\ U \wedge V \leq \bar{1} - A,\ U \nleq \bar{1} - A \textit{ and } V \nleq \bar{1} - A.$$
$$c_4 : A \leq U \vee V,\ A \wedge U \wedge V = \bar{0},\ U \nleq \bar{1} - A \textit{ and } V \nleq \bar{1} - A.$$

Definition 3.2 *A fuzzy set A in a fts $(X, \tau_1(\tau_2))$ is said to be c_i-connected $(i = 1,\ 2,\ 3,\ 4)$, if there exist no c_i-disconnection $(i = 1,\ 2,\ 3,\ 4)$, of A in $(X, \tau_1(\tau_2))$.*
On considering the fuzzy set A to be the fuzzy whole space X (i.e., constant function $\bar{1}$) in all the four forms of disconnections, we get the same disconnection of the fuzzy space X, which we shall refer to as c-disconnection of the mixed fuzzy topological space $(X, \tau_1(\tau_2))$, i.e., there exist nonzero fuzzy open sets U and Vsuch that $U + V = \bar{1}$ and $U \wedge V = \bar{0}$. Which is analogous to the definition of general topology.

Definition 3.3 *A mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is said to be c-connected if there exist no c-disconnection of $\bar{1}$.*

Theorem 3.1 *If a fuzzy set A is open in τ_2 and closed in τ_1 then A is open in $\tau_1(\tau_2)$.*

Proof: Let the fuzzy set A is open in τ_2 and closed in τ_1. Then for any fuzzy point x_α with $x_\alpha q A$ and $A \in \tau_2$ - Q- neighborhood of x_α and $\tau_1 - cl(A) = A \subseteq A$. Therefore, by definition 2.12. of $\tau_1(\tau_2), A \in \tau_1(\tau_2)$.

□

Example 3.1 *Let $X = [0,\ 1]$, define the following fuzzy sets*

$$A(x) = \begin{cases} 1 \textit{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} \textit{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.1}$$

$$B(x) = \begin{cases} \frac{1}{3} \textit{ if } 0 \leq x \leq \frac{1}{3} \\ 1 \textit{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.2}$$

$$C(x) = \begin{cases} \frac{1}{3} \textit{ if } 0 \leq x \leq \frac{1}{3} \\ 0 \textit{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.3}$$

$$D(x) = \begin{cases} 0 \textit{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} \textit{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.4}$$

$$E(x) = \begin{cases} \frac{2}{3} \textit{ if } 0 \leq x \leq \frac{1}{3} \\ 1 \textit{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.5}$$

$$F(x) = \begin{cases} 1 \textit{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{2}{3} \textit{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.6}$$

$\tau_1 = \{\ \bar{0},\ \bar{1}, C, D, C \vee D,\ A,\ B\ \}$, $\tau_2 = \{\ \bar{0},\ \bar{1}, E, F, E \wedge F, A, B, A \wedge B, E \wedge A, F \wedge B\}$. *Then mixed fuzzy topology is given by $\tau_1(\tau_2) = \{\ \bar{0},\ \bar{1}, E, F, E \wedge F\}$.*
Then the fuzzy set $P(x) = \frac{2}{3}$ if $0 \leq x \leq 1$ is c_1- disconnected in τ_1 and τ_2 but c_1- connected in $\tau_1(\tau_2)$. Because $P \leq A \vee B, A \wedge B \leq \bar{1} - P, P \wedge A \neq 0$ and $P \wedge B \neq 0$. Therefore, P is c_1-disconnected in τ_1. Also A and B are open sets in τ_2 therefore P is c_1- disconnected. But $P \leq E \vee F, E \wedge F \nleq \bar{1} - P$. So, E and F do not form a disconnection and no possible disconnection exists. Therefore, P is c_1- connected in $\tau_1(\tau_2)$.

Example 3.2 *Let $X = [0,\ 1]$, define the following fuzzy sets*

$$A(x) = \begin{cases} \frac{1}{3} \text{ if } 0 \leq x \leq \frac{1}{3} \\ 0 \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.7}$$

$$B(x) = \begin{cases} 0 \text{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.8}$$

$$C(x) = \begin{cases} \frac{1}{3} \text{ if } 0 \leq x \leq \frac{1}{3} \\ 1 \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.9}$$

$$D(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.10}$$

$$E(x) = \begin{cases} \frac{2}{3} \text{ if } 0 \leq x \leq \frac{1}{3} \\ 0 \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.11}$$

$$F(x) = \begin{cases} 0 \text{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{2}{3} \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.12}$$

Now $\tau_1 = \{\ \bar{0},\ \bar{1}, A, B, A \vee B, E, F, E \vee F, A \vee F, B \vee E\}$, $\tau_2 = \{\ \bar{0},\ \bar{1}, C, D, C \wedge D, E, F, E \vee F, E \wedge C, F \wedge D\}$ are two fuzzy topologies on X. Then mixed fuzzy topology $\tau_1 (\tau_2) = \{\ \bar{0},\ \bar{1}, C, D, C \wedge D, E \vee F\}$. The fuzzy set $P(x) = \frac{2}{3}$ if $0 \leq x \leq 1$ is c_2- disconnected in τ_1 and τ_2 but c_2- connected in $\tau_1 (\tau_2)$. Because $P \leq E \vee F, P \wedge E \wedge F = \bar{0}, P \wedge E \neq \bar{0}$ and $P \wedge F \neq \bar{0}$. Therefore, P is c_2-disconnected in τ_1. Also E and F are open sets in τ_2 therefore P is c_2- disconnected in τ_2. But $P \leq C \vee D, P \wedge C \wedge D \nleq \bar{0}$. So, C and D do not form a disconnection and no possible disconnection exists because $U \vee V \neq \bar{0}$ for $U \epsilon \tau_1 (\tau_2)$ and $V \epsilon \tau_1 (\tau_2)$ and $U \neq \bar{0} \neq V$. Therefore, P is c_2- connected in $\tau_1 (\tau_2)$.

Example 3.3 *Let $X = [0,\ 1]$, define the following fuzzy sets*

$$A(x) = \begin{cases} \frac{1}{3} \text{ if } 0 \leq x \leq \frac{1}{3} \\ 1 \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.13}$$

$$B(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.14}$$

$$C(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{4} \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.15}$$

$$D(x) = \begin{cases} \frac{1}{4} \text{ if } 0 \leq x \leq \frac{1}{3} \\ 1 \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.16}$$

$$E(x) = \frac{1}{2}, \qquad x \epsilon [0, 1]. \tag{3.17}$$

$\tau_1 = \{\ \bar{0},\ \bar{1}, A, B, E, A \wedge B, A \wedge E, B \wedge E, A \vee E, B \vee E\}$,
$\tau_2 = \{\ \bar{0},\ \bar{1}, C, D, E, C \wedge D, C \wedge E, D \wedge E, C \vee E, D \vee E\}$,
$\tau_1 (\tau_2) = \{\ \bar{0},\ \bar{1}, E, C \vee E, D \vee E\}$.
The fuzzy set $P(x) = \frac{2}{3}$ if $0 \leq x \leq 1$ is c_3- disconnected in τ_1 and τ_2 but c_3- connected in $\tau_1 (\tau_2)$. Because $P \leq A \vee B,\ A \wedge B \leq \bar{1} - P, A \nleq \bar{1} - P$ and $B \nleq \bar{1} - P$. Therefore, P is c_3- disconnected in τ_1. Also $P \leq C \vee D, C \wedge D \leq \bar{1} - P, C \nleq \bar{1}$ -P and $B \nleq \bar{1} - P$. Therefore, P is c_3- disconnected in τ_2. But there is no possible c_3-disconnection for P exists in $\tau_1 (\tau_2)$ because $(C \vee E) \wedge (D \vee E) \nleq \bar{1} - P$. Therefore, P is c_3- connected in $\tau_1 (\tau_2)$.

Example 3.4 *Let $X = [0,\ 1]$, define the following fuzzy sets*

$$A(x) = \begin{cases} \frac{2}{3} \text{ if } 0 \leq x \leq \frac{1}{3} \\ 0 \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.18}$$

$$B(x) = \begin{cases} 0 \text{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{2}{3} \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.19}$$

$$C(x) = \begin{cases} \frac{3}{4} \text{ if } 0 \leq x \leq \frac{1}{3} \\ 0 \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.20}$$

$$D(x) = \begin{cases} 0 \text{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{3}{4} \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.21}$$

$$E(x) = \begin{cases} \frac{1}{3} \text{ if } 0 \leq x \leq \frac{1}{3} \\ 1 \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.22}$$

$$F(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} \text{ if } \frac{1}{3} < x \leq 1, \end{cases} \tag{3.23}$$

$$P(x) = \frac{2}{3}, \qquad x \epsilon [0, 1]. \tag{3.24}$$

$\tau_1 = \{\ \bar{0},\ \bar{1}, A, B, A \vee B\}$, $\tau_2 = \{\ \bar{0},\ \bar{1}, C, D, E, F, C \wedge D, C \wedge E, D \wedge E, C \vee E, D \vee E\}$.
Therefore, the mixed fuzzy topology is $\tau_1(\tau_2) = \{\ \bar{0},\ \bar{1}, E, F, E \wedge F\}$.
The fuzzy set $P(x) = \frac{2}{3}$ if $0 \leq x \leq 1$ is c_4- disconnected in τ_1 and τ_2 but c_4- connected in $\tau_1(\tau_2)$. Because $P \leq A \vee B$, $P \wedge A \wedge B = \bar{0}, A \nleq \bar{1} - P$ and $B \nleq \bar{1} - P$. Therefore, P is c_4- disconnected in τ_1. Also $P \leq C \vee D, P \wedge C \wedge D \leq \bar{0}, C \nleq \bar{1} - P$ and $D \nleq \bar{1} - P$ therefore, P is c_4- disconnected in τ_2. But there is no possible c_4-disconnection for P exists in $\tau_1(\tau_2)$ because $(P \wedge E) \wedge F \nleq \bar{0}$. Therefore, P is c_4-connected in $\tau_1(\tau_2)$.

Theorem 3.2 *If a fuzzy set A is c_i-connected $(i = 1,\ 2,\ 3,\ 4)$ in τ_2, then A is c_i-connected in $\tau_1(\tau_2)$.*

Proof: Let A is c_1- disconnected in $\tau_1(\tau_2)$. Then there exist open sets in U and V in $\tau_1(\tau_2)$ such that $A \leq U \vee V$, $U \wedge V \leq \bar{1} - A$, $A \wedge U \neq \bar{0}$ and $A \wedge V \neq 0$. But then U and V are in τ_2 as $\tau_1(\tau_2) \subseteq \tau_2$. Thus U and V forms a c_1- disconnection of A in τ_2. Contradicts the c_1- connectedness of A in τ_2. Therefore, A is c_1- connected in $\tau_1(\tau_2)$. Similarly other cases can be proved. □

Theorem 3.3 *If (X, τ_2) is fuzzy c-connected space then $(X, \tau_1(\tau_2))$ is also c-connected.*

Proof: Let (X, τ_1) and (X, τ_2) are fuzzy c-connected spaces. Let $(X, \tau_1(\tau_2))$ be fuzzy c- disconnected. Then there exist nonzero fuzzy open sets U and V in $(X, \tau_1(\tau_2))$ such that $U + V = \bar{1}$ and $U \wedge V = \bar{0}$. But $\tau_1(\tau_2) \subseteq \tau_2$. Therefore, U and V are in τ_2 such that $U + V = \bar{1}$ and $U \wedge V = \bar{0}$. This contradicts the fact that (X, τ_2) is fuzzy c-connected space. Therefore, $(X, \tau_1(\tau_2))$ is c-connected.

□

Note:If $\tau_1 = \tau_2 = \tau$ then we denote the mixed fuzzy topology as (X, τ^2).

Theorem 3.4 *If (X, τ) is fuzzy c-connected then the mixed fuzzy topology(X, τ^2) is also c-connected.*

Proof: Proof follows from Theorem 3.3. □

Theorem 3.5 *If a fuzzy set A is c_i -disconnected ($i = 1, 2, 3, 4$), in the mixed fuzzy topology $(X, \tau_1(\tau_2))$ then A is c_i- disconnected in (X, τ_2) but the converse is not true.*

Proof: According to hypothesis A is c_1-disconnected in $(X, \tau_1(\tau_2))$. Therefore, there exist open sets U and V in $\tau_1(\tau_2)$ such that $A \leq U \vee V, U \wedge V \leq \bar{1} - A$, $A \wedge U \neq \bar{0}$ and $A \wedge V \neq 0$. But $\tau_1(\tau_2) \subseteq \tau_2$ so, U and V are in τ_2. Therefore, $A \leq U \vee V$, $U \wedge V \leq \bar{1} - A$, $A \wedge U \neq \bar{0}$ and $A \wedge V \neq \bar{0}$ makes a disconnection of A in τ_2. Therefore, A is c_1- disconnected in (X, τ_2). Similarly, for c_2, c_3 and c_4- disconnection the results can be proved. For converse we consider Example 3.1. Where $P(x) = \frac{2}{3}$ if $0 \leq x \leq 1$ is c_1-disconnected in τ_1 and τ_2 but c_1-connected in $\tau_1(\tau_2)$. For c_2, c_3 and c_4- disconnection Example 3.2, Example 3.3 and Example 3.4 provides the required counter examples. □

Theorem 3.6 *Let (X, τ_1) and (X, τ_2) be two fuzzy topological spaces and $(X, \tau_1(\tau_2))$ is the mixed fuzzy topology. If (X, τ_2) is fuzzy c – disconnected then $(X, \tau_1(\tau_2))$ is c – disconnected.*

Proof: Let (X, τ_2) is fuzzy c-disconnected then there exists fuzzy open sets U and V such that $U + V = \bar{1}$ and $U \wedge V = \bar{0}$. But $\tau_1(\tau_2) \subseteq \tau_2$, therefore U and V are fuzzy open sets in τ_2 which constitutes a c-disconnection for $\tau_1(\tau_2)$. Therefore, $(X, \tau_1(\tau_2))$ is c-disconnected. □

Theorem 3.7 *Let (X, τ_1) and (X, τ_2) be two fuzzy topological spaces and $(X, \tau_1(\tau_2))$ is the mixed fuzzy topology. If (X, τ_2) is fuzzy c- connected then $(X, \tau_1(\tau_2))$ is c- connected.*

Proof: Let $(X, \tau_1(\tau_2))$ is c-disconnected. Therefore, there exists fuzzy open sets U and V in $\tau_1(\tau_2)$ such that $U + V = \bar{1}$ and $U \wedge V = \bar{0}$. But $\tau_1(\tau_2) \subseteq \tau_2$, therefore U and V are fuzzy open sets in τ_2. Therefore, (X, τ_2) is c-disconnected, which is a contradiction. Therefore, $(X, \tau_1(\tau_2))$ is c-connected. □

Theorem 3.8 *Let (X, τ_1) and (X, τ_2) be two fuzzy topological spaces. Let $\{A_i\}_{i\epsilon I}$ be a family of c_1-connected ($c_2 - connected$) fuzzy sets in (X, τ_2) such that for $i, j\epsilon I$ and $i \neq j, A_i$ and A_j are intersecting fuzzy sets. Then $\bigvee_{i\epsilon I} A_i$ is a c_1-connected ($c_2 - connected$) fuzzy set in $(X, \tau_1(\tau_2))$.*

Proof: Let $A = \bigvee_{i\epsilon I} A_i$ be c_1-disconnected ($c_2 - disconnected$) in $(X, \tau_1(\tau_2))$. Therefore, there exists open fuzzy sets U and V in in $(X, \tau_1(\tau_2))$ such that $A = \bigvee_{i\epsilon I} A_i \leq U \vee V$, $U \wedge V \leq \bar{1} - A$, $(U \wedge V \wedge A = 0 \; for \; c_2 - disconnected)$, $A \wedge U \neq 0$ and $A \wedge V \neq 0$.
Now $A \wedge U \neq 0$ so $\exists A_{i_0} \epsilon \{A_i\}_{i\epsilon I}$ such that $A_{i_0} \wedge U \neq 0$. Assume $U \wedge V = W$. Assume $U \vee V = W$. Therefore, we have open sets U and W in $\tau_1(\tau_2)$ such that $A_{i_0} \leq U \vee W$, $U \wedge W = U \wedge V \leq 1 - A \leq 1 - A_{i_0}, A_{i_0} \wedge U \neq 0$ and $A_{i_0} \wedge W \neq 0$. But $\tau_1(\tau_2) \subseteq \tau_2$, so U and W constitutes a c_1-disconnection of A_{i_0} in τ_2 which contradict the fact that $\{A_i\}_{i\epsilon I}$ is a family of c_1-connected ($c_2 - connected$) fuzzy sets in τ_2. Therefore, $\bigvee_{i\epsilon I} A_i$ is a c_1-connected ($c_2 - connected$) fuzzy set in $(X, \tau_1(\tau_2))$. □

If A and B are intersecting fuzzy c_3-connected in (X, τ_2) then $A \vee B$ need not be fuzzy c_3-connected set in $(X, \tau_1(\tau_2))$. The following is a counter example to illustrates it.

Example 3.5 *Let $X = [0, 1]$ and define fuzzy sets A, B, C, D and E as follows:*

$$A(x) = \begin{cases} \frac{6}{7} \; if \; \frac{4}{7} < x \leq 1 \\ \frac{2}{7} \; if \; 0 \leq x \leq \frac{4}{7}, \end{cases} \tag{3.25}$$

$$B(x) = \begin{cases} \frac{2}{7} \; if \; \frac{4}{7} < x \leq 1 \\ \frac{6}{7} \; if \; 0 \leq x \leq \frac{4}{7}, \end{cases} \tag{3.26}$$

$$C(x) = \begin{cases} 0 \; if \; \frac{4}{7} < x \leq 1 \\ \frac{2}{7} \; if \; 0 \leq x \leq \frac{4}{7}, \end{cases} \tag{3.27}$$

$$D(x) = \begin{cases} \frac{1}{7} \ if \ \frac{4}{7} < x \leq 1 \\ \frac{2}{7} \ if \ 0 \leq x \leq \frac{4}{7}, \end{cases} \tag{3.28}$$

$$E(x) = \begin{cases} \frac{5}{7} \ if \ \frac{4}{7} < x \leq 1 \\ \frac{1}{7} \ if \ 0 \leq x \leq \frac{4}{7}, \end{cases} \tag{3.29}$$

$$F(x) = \begin{cases} \frac{1}{7} \ if \ \frac{4}{7} < x \leq 1 \\ \frac{5}{7} \ if \ 0 \leq x \leq \frac{4}{7}, \end{cases} \tag{3.30}$$

$$P(x) = \begin{cases} \frac{1}{7} \ if \ \frac{4}{7} < x \leq 1 \\ \frac{2}{7} \ if \ 0 \leq x \leq \frac{4}{7}, \end{cases} \tag{3.31}$$

$$Q(x) = \begin{cases} \frac{2}{7} \ if \ \frac{4}{7} < x \leq 1 \\ \frac{1}{7} \ if \ 0 \leq x \leq \frac{4}{7}, \end{cases} \tag{3.32}$$

Now $\tau_1 = \{\ \bar{0},\ \bar{1}, D, E, F, D \wedge E, D \vee V, E \vee F\}$, *and* $\tau_2 = \{\ \bar{0},\ \bar{1}, A, B, C, A \wedge B, A \vee B\}$, *are two fuzzy topologies and* $\tau_1(\tau_2) = \{\ \bar{0},\ \bar{1}, A, B, A \wedge B, A \vee B\}$ *is mixed fuzzy topology. Here* P *and* Q *are* c_3*-connected fuzzy sets in* τ_2 *because no pair of fuzzy set from* τ_2 *makes a disconnection of* P *or* Q*. But* $P \vee Q$ *is not* c_3*-connected fuzzy set in* $\tau_1(\tau_2)$*. Since,* $P \vee Q \leq A \vee B, A \wedge B \leq \bar{1} - P \vee Q, A \nleq \bar{1} - P \vee Q, B \nleq \bar{1} - P \vee Q$*. If* A *and* B *are intersecting fuzzy* c_4*-connected in* (X, τ_2) *then* $A \vee B$ *need not be fuzzy* c_4*-connected set in* $(X,\ \tau_1(\tau_2))$*. The following is a counter example to illustrates it.*

Example 3.6 *Let* $X = [0, 1]$ *and define fuzzy sets* A*,* B*,* C*,* D *and* E *as follows:*

$$A(x) = \begin{cases} \frac{4}{5} \ if \ \frac{4}{5} < x \leq 1 \\ 0 \ if \ 0 \leq x \leq \frac{4}{5}, \end{cases} \tag{3.33}$$

$$B(x) = \begin{cases} 0 \ if \ \frac{4}{5} < x \leq 1 \\ \frac{4}{5} \ if \ 0 \leq x \leq \frac{4}{5}, \end{cases} \tag{3.34}$$

$$C(x) = \begin{cases} 0 \ if \ \frac{4}{5} < x \leq 1 \\ \frac{2}{5} \ if \ 0 \leq x \leq \frac{4}{5}, \end{cases} \tag{3.35}$$

$$D(x) = \begin{cases} \frac{1}{5} \ if \ \frac{4}{5} < x \leq 1 \\ 1 \ if \ 0 \leq x \leq \frac{4}{5}, \end{cases} \tag{3.36}$$

$$E(x) = \begin{cases} 1 \ if \ \frac{4}{5} < x \leq 1 \\ \frac{1}{5} \ if \ 0 \leq x \leq \frac{4}{5}, \end{cases} \tag{3.37}$$

$$P(x) = \begin{cases} \frac{3}{5} \ if \ \frac{4}{5} < x \leq 1 \\ 0 \ if \ 0 \leq x \leq \frac{4}{5}, \end{cases} \tag{3.38}$$

$$Q(x) = \begin{cases} 0 \ if \ \frac{4}{5} < x \leq 1 \\ \frac{3}{5} \ if \ 0 \leq x \leq \frac{4}{5}, \end{cases} \tag{3.39}$$

Now $\tau_1 = \{\ \bar{0},\ \bar{1}, D, E, D \wedge E\}$, *and* $\tau_2 = \{\ \bar{0},\ \bar{1}, A, B, C, A \vee B, A \vee C\}$, *are two fuzzy topologies and* $\tau_1(\tau_2) = \{\ \bar{0},\ \bar{1}, A, B, A \vee B\}$ *is the mixed fuzzy topology. Now* $P \leq A \vee B, P \wedge A \wedge B = \bar{0}, A \nleq \bar{1} - PandB \leq \bar{1} - P$*. That is* A *and* B *does not forms a* c_4*- disconnection of* P *and no other possible* c_4*-disconnection exists for* P*. Similarly,* Q *is* c_4*-connected. But* $P \wedge Q$ *is* c_4*-disconnected.* $P \wedge Q \leq A \vee B, (P \vee Q) \wedge A \wedge B = \bar{0}, A \nleq \bar{1} - (P \vee Q)$ *and* $B \nleq \bar{1} - (P \vee Q)$*.*

Theorem 3.9 *Let* (X, τ_1) *and* (X, τ_2) *be two fuzzy topological spaces and* $(X,\ \tau_1(\tau_2))$ *the mixed fuzzy topology. If* A *and* B *are overlapping* c_3*-connected* $(c_4 - connected)$ *fuzzy sets in* (X, τ_2) *then* $A \vee B$ *is a* c_3*-connected* $(c_4 - connected)$ *fuzzy set in* $(X, \tau_1(\tau_2))$*.*

Proof: Let A and B be overlapping c_3-disconnected ($c_4 - disconnected$) fuzzy set in $(X, \tau_1(\tau_2))$. Then there exist fuzzy open sets U and V in $\tau_1(\tau_2)$ such that $A \vee B \leq U \vee V, U \wedge V \leq \bar{1} - A \vee B$, $[U \wedge V \wedge (A \wedge B) = \bar{0},\ for\ c_2 - disconnected]$, $U \nleq \bar{1} - A \vee B and V \nleq \bar{1} - A \vee B$. But $\tau_1(\tau_2) \subseteq \tau_2$. So, U and V constitutes a c_3-disconnection of $A \vee B$ in τ_2 contradiction to Theorem 3.7. [1].Therefore, $A \vee B$ is a c_3-connected ($c_4 - connected$) fuzzy set in $(X,\ \tau_1(\tau_2))$. □

Theorem 3.10 *Let (X, τ_1) and (X, τ_2) be two fuzzy topological spaces and $(X, \tau_1(\tau_2))$ is the mixed fuzzy topological space. If A is c_3-connected ($c_4 - connected$) fuzzy set in (X, τ_2) and $A \leq B \leq \bar{A}_{\tau_1(\tau_2)}$ then B is c_3-connected ($c_4 - connected$) fuzzy set in $(X,\ \tau_1(\tau_2))$.*

Proof: Let U and V be fuzzy open sets in $(X, \tau_1(\tau_2))$ such that $B \leq U \vee V$ and $U \wedge V \leq \bar{1} - B$ $[For\ c_4 - connected\ U \wedge V \wedge A = \bar{0}]$. Since A is c_3-connected ($c_4 - connected$) *in* τ_2 therefore, $A \leq \bar{1} - U or A \leq \bar{1} - V$. If $A \leq \bar{1}\ - U$ then $\bar{A}_{\tau_1(\tau_2)} \leq \overline{(\bar{1} - U)}_{\tau_1(\tau_2)}$. So $\bar{A}_{\tau_1(\tau_2)} \leq\ \bar{1} - U^0{}_{\tau_1(\tau_2)}$. Therefore, $\bar{A}_{\tau_1(\tau_2)} \leq \bar{1} - U$, since U is open in$\tau_1(\tau_2)$. Similarly for $A \leq \bar{1} - V$ we have $\bar{A}_{\tau_1(\tau_2)} \leq \bar{1} - V$. Therefore, $B \leq \bar{A}_{\tau_1(\tau_2)} \leq \bar{1} - U$ or $B \leq \bar{A}_{\tau_1(\tau_2)} \leq \bar{1} - V$. So B is c_3-connected ($c_4 - connected$). □

The last Theorem 3.10. fails in case of c_1-connected ($c_2 - connected$) fuzzy set in τ_2. The following is a counter example for that.

Example 3.7 *Let $X = [0, 1]$ and define fuzzy sets A, B, C and D as follows:*

$$A(x) = \begin{cases} 1\ \textit{if}\ \frac{2}{5} < x \leq 1 \\ 0\ \textit{if}\ 0 \leq x \leq \frac{2}{5}, \end{cases} \tag{3.40}$$

$$B(x) = \begin{cases} 0\ \textit{if}\ \frac{2}{5} < x \leq 1 \\ \frac{3}{5}\ \textit{if}\ 0 \leq x \leq \frac{2}{5}, \end{cases} \tag{3.41}$$

$$C(x) = \begin{cases} 1\ \textit{if}\ \frac{2}{5} < x \leq 1 \\ \frac{2}{5}\ \textit{if}\ 0 \leq x \leq \frac{2}{5}, \end{cases} \tag{3.42}$$

$$D(x) = \begin{cases} 0\ \textit{if}\ \frac{2}{5} < x \leq 1 \\ 1\ \textit{if}\ 0 \leq x \leq \frac{2}{5}, \end{cases} \tag{3.43}$$

Now $\tau_1 = \{\ \bar{0},\ \bar{1}, C, C \wedge D\}$, and $\tau_2 = \{\ \bar{0},\ \bar{1}, A, B, C, A \vee B, A \vee C, B \wedge C\}$. are two fuzzy topologies then by Theorem 3.1. and Definition 2.12. $\tau_1(\tau_2) = \{\ \bar{0},\ \bar{1}, A, B, C, A \vee B\}$. Here A is c_1-connected fuzzy set in τ_2 as it has no c_1-disconnection in τ_2. But $\bar{A}_{\tau_1(\tau_2)} = C$ and C is not c_1-connected in $\tau_1(\tau_2)$. Because $\bar{A}_{\tau_1(\tau_2)} = C \leq A \vee B, A \wedge B \leq 1 - \bar{A}_{\tau_1(\tau_2)}$ also, $\bar{A}_{\tau_1(\tau_2)} \wedge A \neq 0$ and $\bar{A}_{\tau_1(\tau_2)} \wedge B \neq 0$. Therefore, $\bar{A}_{\tau_1(\tau_2)}$ is c_1-disconnected fuzzy set in $\tau_1(\tau_2)$.
Note: The same example shows $\bar{A}_{\tau_1(\tau_2)}$ is c_2-disconnected fuzzy set in $\tau_1(\tau_2)$ as$\bar{A}_{\tau_1(\tau_2)} \wedge (A \wedge B) = 0$.

4. Conclusions.

In Mixed fuzzy topology connectedness is not studied by any researcher before. In this paper connectedness in mixed fuzzy topology is investigated using four notions of connectedness due to Ajmal and Kohli [1]. There is huge scope for research in mixed fuzzy topology.

Acknowledgments

We would like to thank the referee for giving constructive comments and suggestions.

References

1. Naseem, A. and Jogender, K.K., *Connectedness in fuzzy topological spaces.* Fuzzy Sets and System, 31(3), 369-388, July (1989). https://doi.org/10.1016/0165-0114(89)90207-8.

2. Vadivel, A. and Vijayalakshmi, B., *Mixed e-fuzzy Topological Spaces.* International Journal of Pure and Applied Mathematics, 113 (12), 115 – 122 (2017). https://doi.org/10.5391/IJFIS.2020.20.3.242.

3. Tripathy, B.C., and Ray, G.C., *On Mixed Fuzzy Topological Spaces and countability.* Soft Computing, 16(10), 1691-1695 (2012). https://do i.org/10.1007/s00500-012-0853-1.

4. B.C. Tripathy and G.C. Ray, *Fuzzy δ -I- continuity in mixed fuzzy ideal topological spaces.* Journal of applied analysis, 24(2), 233-239 (2018). https://doi.org/10.1515/jaa-2018-0022.

5. Tripathy, B.C., and Ray, G.C., *Fuzzy δ^* almost continuous and fuzzy δ^* -continuous functions in mixed fuzzy ideal topological spaces.* Proyecciones (Antofagasta), 39(2), 435- 449(2020). https://doi.org/10.22199/issn.0717-6279-2020-02-0027.

6. Wong, C. K., *Fuzzy points and local properties of fuzzy topology.* Journal of Mathematical Analysis and Applications, 46, 316-328 (1974). https://doi.org/10.1016/0022-247X(74)90242-X.

7. Chang, C.L., *Fuzzy topological spaces* J. Math. Anal. Appl., 24, 182-190 (1968). https://doi.org/10.1016/0022-247X(68)90057-7.

8. Ray, G. C. and Chetri, H. P., *Separation Axioms in Mixed Fuzzy Topological Spaces.* International Journal of Fuzzy Logic and Intelligent Systems, 21(4), 423-430 (2021). https://doi.org/10.5391/IJFIS.2021.21.4.423.

9. Ray, G. C. and Chetri, H. P., *On Countability in Mixed Fuzzy Topological Spaces.* Advances and Applications in Mathematical Sciences, 21(6), 2949-2960, April (2022).

10. Chetri, H. P. and Ray,G.C., *Mixed Fuzzy topological space its Hausdorff properties and base.* Proyecciones Journal of Mathematics, 42(2), 393-405, April (2023). https://doi.org/10.22199/issn.0717-6279-5427.

11. L.A. Zadeh, *Fuzzy sets.* Information and Control, 8, 338-353(1965). https://doi.org/10.1016/S0019-9958(65)90241-X.

12. Borah, M., and Hazarika, B., *Soft ideal topological space and Mixed fuzzy soft ideal topological space.* Boletim da Sociedade Paranaense de Matemática, 37(1), 141-151(2019). https://doi.org/10.5269/bspm.v37i1.29647.

13. Ghanim,M.H., Kerre, E. E. and Mashhour, A. S.,. Separation axioms, subspaces and sums in fuzzy topology. Journal of mathematical analysis and applications, 102(1), 189-202(1984) . https://doi.org/10.1016/0022-247x(84)90212-9.

14. Das, N.R. and Baishya, P.C., *Mixed Fuzzy Topological Spaces.* J. Fuzzy Math., 3(4), 777-784 (1995).

15. Ming, P.P. and Ming, L.Y., *Fuzzy Topology I Neighborhood Structure of a Fuzzy Point and Moore Smith Convergence.* J. Math. Anal. Appl., 76, 571-599(1980). https://doi.org/10.1016/0022-247X(80)90048-7.

16. Lowen,R. *Connectedness in fuzzy topological spaces.* The Rocky Mountain Journal of Mathematics, 11(3), 427-433(1981). https://www.jstor.org/stable/44236610.

17. Shravan, K. and Tripathy,B.C., *Multiset ideal topological spaces and Kuratowski closureoperator.* Bulletin of the Universuty Transilvania Brasov, Series III: Mathematics, Informatics, Physics, 13(62)(1), 273-284(2020).

18. Shravan, K. and Tripathy,B.C., *Metrizability of multiset topological spaces.* Bull. Univ.Transilvania Brasov, Series III: Mathematics, Informatics, Physics, 13(62)(2), 683-696(2020). https://doi.org/10.31926/but.mif.2020.13.62.2.24.

19. Shravan, K. and Tripathy,B.C., *Multiset ideal topological spaces and local functions.* Proyecciones Jour Math., 37(4), 699-711(2018).

20. Shravan, K. and Tripathy,B.C., *Generalised closed sets in multiset topological spaces.* Proyecciones Jour Math., 37(2), 223-237(2018).

21. Fatteh, U. V. and Bassan, D.S., *Fuzzy connectedness and its stronger forms.* Journal of mathematical analysis and applications, 111(2), 449-464(1985).

22. AL-Omeri, W.F.,*Mixed γ-Fuzzy in Mixed Fuzzy Topological Spaces and its Application.* International Journal of Pure and Applied Mathematics, 120(4) 547-553(2018).

23. AL-Omeri, W.F., *On Mixed b-Fuzzy Topological Spaces.* International Journal of Fuzzy Logic and Intelligent Systems, 20(3), 242-246, September (2020). https://doi.org/10.5391/IJFIS.2020.20.3.242.

Hari Prasad Chetri,
Department of Mathematics,
Central Institute of Technology, Kokrajhar-783370,
Assam, India.
E-mail address: hariprasadchetri1234@gmail.com

and

Gautam Chandra Ray,
Department of Mathematics,
Central Institute of Technology, Kokrajhar-783370,
Assam, India.
E-mail address: gautamofcit@gmail.com