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A note on Kähler spacetime

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ABSTRACT: In this paper we investigate the solitons on Kählerian spacetime manifolds admitting $\widetilde{m}$-projective curvature tensor and demonstrate the nature of solitons which depends on the relation between isotropic and anisotropic pressures, the cosmological constant, energy density, and gravitational constant.

Key Words: Kähler spacetime, solitons, $\widetilde{m}$-projective curvature tensor, perfect fluid, dust fluid and viscous fluid.

Contents

1. Background and Motivations

Relativistic fluid models serve an important role in many fields of physics, including astrophysics, plasma physics, nuclear physics, and cosmology. A 4-dimensional pseudo-Riemannian manifold endowed with the Lorentzian metric g is treated as a perfect fluid spacetime in general relativity. In cosmology, perfect fluids are employed to simulate idealized matter distributions such as the interior of a star or an isotropic pressure. The behavior of a perfect fluid within a spherical object is described by Einstein's equation. A perfect fluid has a mass density ρ and an isotropic pressure p. If $\rho = 3p$, then radiation fluid is the ideal space fluid.

In 1983, Neill [20] addressed the use of semi-Riemannian geometry in the theory of relativity. Kaigorodov investigates the curvature structure of spacetime in [14]. Several other differential geometers investigated the curvature structure and other properties of spacetime [21], [5], [27], [17], [26], [30]. We recall the notion of a $\widetilde{m}$-projective curvature tensor [23] as follows

$$\begin{aligned} \widetilde{\mathcal{M}}(\varrho_1,\varrho_2)\varrho_3 &= \mathcal{R}(\varrho_1,\varrho_2)\varrho_3 - \frac{1}{2(n-1)}\{\mathcal{S}(\varrho_2,\varrho_3)\varrho_1 - \mathcal{S}(\varrho_1,\varrho_3)\varrho_2 \\ &+ g(\varrho_2,\varrho_3)Q\varrho_1 - g(\varrho_1,\varrho_3)Q\varrho_2\}, \end{aligned} \tag{1.1}$$

where $\mathcal{R}$ and $\mathcal{S}$ are the curvature tensor and the Ricci tensor of (Ω^n, g). Several geometers have explored the $\widetilde{m}$-projective curvature tensor in different ways, including [37], [9], [19], [18], [25], [10], [31].
The study of Ricci solitons on Riemannian manifolds is an issue that is of great importance in the area of differential geometry and in physics as well. Hamilton in [13] was the first to introduce the latter

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notion, which generalises the notion of Einstein metric on a Riemannian manifold. A Ricci soliton is a Riemannian metric g on a manifold whenever the relation stated below is fulfilled:

$$(\mathfrak{L}_\xi g)(\varrho_1, \varrho_2) + 2\mathcal{S}(\varrho_1, \varrho_2) + 2\theta_1 g(\varrho_1, \varrho_2) = 0. \tag{1.2}$$

On a real hypersurface, an η-Ricci soliton is a pair (η, g) that fulfils the following relation [4]:

$$(\mathfrak{L}_\xi g)(\varrho_1, \varrho_2) + 2\mathcal{S}(\varrho_1, \varrho_2) + 2\theta_1 g(\varrho_1, \varrho_2) + 2\theta_2 \eta(\varrho_1)\eta(\varrho_2) = 0, \tag{1.3}$$

where θ_1, θ_2 are real constants, $\mathcal{S}(\varrho_1, \varrho_2)$=$g(\mathcal{Q}\varrho_1, \varrho_2)$, η is a 1-form on (Ω, g). η-Ricci solitons in (Ω, g) are structures $(g, \xi, \theta_1, \theta_2)$ that satisfy Eq.(1.3). If θ_2=0, (g, ξ, θ_1) is a Ricci soliton that fulfils Eq.(1.2) and is referred to as shrinking, steady, or expanding if $\theta_1 < 0$, $\theta_1 = 0$ or $\theta_1 > 0$, respectively [6]: The notion of such types solitons in different contexts of geometry is studied by [32], [33], [34], [35], [36] and many more.
Since the Lie derivative of $(\mathfrak{L}_\xi g)(\varrho_1, \varrho_2)$, is given by

$$(\mathfrak{L}_\xi g)(\varrho_1, \varrho_2) = g(\nabla_{\varrho_1}\xi, \varrho_2) + g(\varrho_1, \nabla_{\varrho_2}\xi). \tag{1.4}$$

It is feasible to write from (1.4) and (1.3) that

$$\mathcal{S}(\varrho_1, \varrho_2) = -\theta_1 g(\varrho_1, \varrho_2) - \theta_2\eta(\varrho_1)\eta(\varrho_2) - \frac{1}{2}[g(\nabla_{\varrho_1}\xi, \varrho_2) + g(\varrho_1, \nabla_{\varrho_2}\xi)], \tag{1.5}$$

for any ϱ_1, $\varrho_2 \in \chi(\Omega)$.
Now, we recall the notion of η-Einstein soliton in a perfect fluid spacetime [3]:

$$(\mathfrak{L}_\xi g)(\varrho_1, \varrho_2) + 2\mathcal{S}(\varrho_1, \varrho_2) + (2\theta_1 - \kappa)g(\varrho_1, \varrho_2) + 2\theta_2\eta(\varrho_1)\eta(\varrho_2) = 0, \tag{1.6}$$

where g, $\mathcal{S}$, ξ, η, θ_1 and θ_2 have the as usual meaning. In particular, if θ_2=0, (g, ξ, θ_1) is an Einstein soliton [7]. Again from (1.4) and (1.6), we get

$$\mathcal{S}(\varrho_1, \varrho_2) = -(\theta_1 - \frac{\kappa}{2})g(\varrho_1, \varrho_2) - \theta_2\eta(\varrho_1)\eta(\varrho_2) - \frac{1}{2}[g(\nabla_{\varrho_1}\xi, \varrho_2) + g(\varrho_1, \nabla_{\varrho_2}\xi)]. \tag{1.7}$$

The paper is organised as follows: After introduction, in section 2 basic results for Kähler spacetime and discuss solitons on perfect fluid Kähler spacetime (briefly, $(\mathrm{PFK})_4$ spcetime). In section 3 we investigate $\widetilde{m}$-projectively flat Kähler spacetime and deduce some interesting results. Beside this, $\widetilde{m}$-projectively flat $(\mathrm{PFK})_4$ spacetime are discuss in section 4. Section 5 is concerned with $(\mathrm{PFK})_4$ spacetime with divergence free $\widetilde{m}$-projective curvature tensor. Finally, in section 6 dust fluid Kähler spacetime (briefly, $(\mathrm{DFK})_4$ spacetime) with vanishing projective curvature tensor are included.

2. Kählerian spacetime manifold

On a 4-dimensional manifold, perfect fluid spacetime (briefly, $(\mathrm{PF})_4$-spacetime) according to general relativity exists. If the following conditions are met, such a manifold is known as a Kählerian spacetime manifold.

$$\mathcal{P}^2\varrho_1 = -\varrho_1, \tag{2.1}$$

$$g(\mathcal{P}\varrho_1, \mathcal{P}\varrho_2) = g(\varrho_1, \varrho_2), \tag{2.2}$$

$$(\nabla_{\varrho_1}\mathcal{P})\varrho_2 = 0, \tag{2.3}$$

$$\mathcal{S}(\mathcal{P}\varrho_1, \mathcal{P}\varrho_2) = \mathcal{S}(\varrho_1, \varrho_2), \tag{2.4}$$

$$\mathcal{S}(\varrho_1, \mathcal{P}\varrho_2) + \mathcal{S}(\mathcal{P}\varrho_1, \varrho_2) = 0, \tag{2.5}$$

where $\mathcal{P}$ is $(1,1)$-tensor field and g is Riemannian metric. We denote $\mathcal{P}(\varrho_1)$ by $\overline{\varrho}_1$. The Einstein's field equation with cosmological constant in 4-dimensional Kähler spacetime:

$$\mathcal{S}(\varrho_1, \varrho_2) - \frac{\kappa}{2}g(\varrho_1, \varrho_2) + \mu g(\varrho_1, \varrho_2) = \tau\mathcal{T}(\varrho_1, \varrho_2), \tag{2.6}$$

for all vector fields $\varrho_1,\ \varrho_2$, where μ is cosmological term, τ is the gravitational constant and $\mathcal{T}$ is the energy momentum tensor of type $(0,2)$.
The energy momentum tensor $\mathcal{T}$ for a perfect fluid [20]:

$$\mathcal{T}(\varrho_1,\varrho_2)=(\rho+p)\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)+pg(\varrho_1,\varrho_2), \tag{2.7}$$

where ρ is the energy density function, p is the isotropic pressure function of the fluid and $\mathcal{A}$ is a non-zero 1-form such that $g(\varrho_1,\mathcal{V})$=$\mathcal{A}(\varrho_1)$ for all ϱ_1; $\mathcal{V}$ being the flow vector field of the fluid and satisfies $g(\mathcal{V},\mathcal{V})$=-1. The energy density ρ and the pressure p can be described in the sense that if ρ vanishes, then the matter of the fluid is not pure, and if p vanishes, then the fluid is dust, respectively.
Using (2.7) in (2.6) we have

$$\mathcal{S}(\varrho_1,\varrho_2)+[\mu-\frac{\kappa}{2}-p\tau]g(\varrho_1,\varrho_2)=\tau(\rho+p)\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2). \tag{2.8}$$

Putting ϱ_1 by $\mathcal{P}\varrho_1$ and ϱ_2 by $\mathcal{P}\varrho_2$ in (2.8), using (2.2) and (2.4), we get

$$\mathcal{S}(\varrho_1,\varrho_2)+[\mu-\frac{\kappa}{2}-p\tau]g(\varrho_1,\varrho_2)=\tau(\rho+p)\mathcal{A}(\mathcal{P}\varrho_1)\mathcal{A}(\mathcal{P}\varrho_2). \tag{2.9}$$

Subtracting (2.8) from (2.9) and taking ϱ_2=$\mathcal{V}$, it yield

$$\tau(\rho+p)\mathcal{A}(\varrho_1)=0. \tag{2.10}$$

Since in Kähler spacetime $\tau\neq 0$ and $\mathcal{A}(\varrho_1)\neq 0$, we get

$$(\rho+p)=0. \tag{2.11}$$

In view of (2.11), Eq. (2.8) gives

$$\mathcal{S}(\varrho_1,\varrho_2)=\Big(\frac{\kappa}{2}-\mu+\tau p\Big)g(\varrho_1,\varrho_2). \tag{2.12}$$

Taking ϱ_1=ϱ_2=e_i, $1\leq\ i\leq 4$, in (2.12)) and taking summation over i we obtain

$$\mathcal{S}(\varrho_1,\varrho_2)=\frac{\kappa}{4}g(\varrho_1,\varrho_2). \tag{2.13}$$

Also from (1.5) and (2.8), we have

$$\begin{aligned}\Big[\theta_1-\mu+p\tau+\frac{\kappa}{2}\Big]g(\varrho_1,\varrho_2)\quad &+\quad [\tau(p+\rho)+\theta_2]\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)\\ &=\quad -\frac{1}{2}[g(\nabla_{\varrho_1}\xi,\varrho_2)+g(\varrho_1,\nabla_{\varrho_2}\xi)].\end{aligned} \tag{2.14}$$

Taking contraction over ϱ_1 and ϱ_1, we get

$$4\theta_1+\theta_2=4\mu-2\kappa-3p\tau+\mathrm{div}\xi. \tag{2.15}$$

Again putting ϱ_1=ϱ_2=$\mathcal{V}$ in (2.14), yields

$$-2\theta_1+2\theta_2=-2\mu+\kappa-2\rho\tau. \tag{2.16}$$

With the help of (2.15) and (2.16), we obtain

$$\theta_1=\mu-\frac{\kappa}{2}-\frac{1}{5}[\tau(3p-\rho)-div\xi],\ \ \theta_2=-\frac{1}{5}[\tau(3p+4\rho)-div\xi]. \tag{2.17}$$

If θ_2=0, then we get the Ricci soliton with θ_1=$\mu-\frac{\kappa}{2}+\tau\rho$ which is steady if ρ=$\frac{1}{\tau}[\frac{\kappa}{2}-\mu]$, expanding if $\rho>\frac{1}{\tau}[\frac{\kappa}{2}-\mu]$ and shrinking if $\rho<\frac{1}{\tau}[\frac{\kappa}{2}-\mu]$.
This bears the consequent results:

Proposition 2.1 *A (PFK)$_4$ spacetime is an Einstein spacetime.*

Theorem 2.1 *An η-Ricci soliton $(g, \xi, \theta_1, \theta_2)$ on a (PFK)$_4$ spacetime satisfying EFE with cosmological constant is steady if $\rho{=}\frac{1}{\tau}[\frac{\kappa}{2} - \mu]$, expanding if $\rho > \frac{1}{\tau}[\frac{\kappa}{2} - \mu]$ and shrinking if $\rho < \frac{1}{\tau}[\frac{\kappa}{2} - \mu]$.*

Remark 2.1 *For radiation fluid $\rho = 3p$, then $\theta_1 = \mu + \frac{div\xi}{5}$, $\theta_2 = -3p\tau - \frac{div\xi}{5}$.*

Again from (1.7) and (2.8), we obtain

$$\begin{aligned}[\theta_1 - \mu + p\tau]\, g(\varrho_1, \varrho_2) &+ [\tau(p+\rho) - \theta_2]\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2) \\ &= -\frac{1}{2}[g(\nabla_{\varrho_1}\xi, \varrho_2) + g(\varrho_1, \nabla_{\varrho_2}\xi)]. \end{aligned} \tag{2.18}$$

On contracting over ϱ_1 and ϱ_1, we obtain

$$4\theta_1 - \theta_2 = 4\mu - 3p\tau + \tau\rho - \text{div}\xi. \tag{2.19}$$

Putting $\varrho_1{=}\varrho_2{=}\mathcal{V}$ in (2.18), yields

$$-\theta_1 + \theta_2 = -\mu - \rho\tau. \tag{2.20}$$

With the help of (2.19) and (2.20), we have

$$\theta_1 = \frac{1}{5}[5\mu - \tau(3p + 3\rho) - div\xi],\ \theta_2 = \frac{1}{5}[3\tau(p+\rho) + div\xi]. \tag{2.21}$$

If $\theta_2{=}0$ then we get the Einstein soliton with $\theta_1{=}\mu + \tau\rho$, which is steady if $\rho{=}-\frac{\mu}{\tau}$, expanding if $\rho > -\frac{\mu}{\tau}$ and shrinking if $\rho < -\frac{\mu}{\tau}$.

We therefore identify the consequence:

Theorem 2.2 *An η-Einstein soliton $(g, \xi, \theta_1, \theta_2)$ on a (PFK)$_4$ spacetime satisfying EFE with cosmological constant is steady if $\rho{=}-\frac{\mu}{\tau}$, expanding if $\rho > -\frac{\mu}{\tau}$ and shrinking if $\rho < -\frac{\mu}{\tau}$.*

Remark 2.2 *Considering the radiation fluid $\rho = p$, then $\theta_1 = \frac{1}{5}[5\mu - 9p\tau - div\xi]$, $\theta_2 = \frac{1}{5}[12p\tau + div\xi]$.*

3. $\widetilde{m}$-projectively flat Kähler spacetime

Let Ω_4 be the spacetime of general relativity. Next, based on (1.1), we conclude

$$\begin{aligned}\widetilde{\mathcal{M}}(\varrho_1, \varrho_2, \varrho_3, \varrho_4) &= \mathcal{R}(\varrho_1, \varrho_2, \varrho_3, \varrho_4) - \frac{1}{6}\{\mathcal{S}(\varrho_2, \varrho_3)g(\varrho_1, \varrho_4) - \mathcal{S}(\varrho_1, \varrho_3)g(\varrho_2, \varrho_4) \\ &+ g(\varrho_2, \varrho_3)\mathcal{S}(\varrho_1, \varrho_4) - g(\varrho_1, \varrho_3)\mathcal{S}(\varrho_2, \varrho_4)\}. \end{aligned} \tag{3.1}$$

If $\widetilde{\mathcal{M}}(\varrho_1, \varrho_2, \varrho_3, \varrho_4){=}0$, then (3.1) exhibits the form

$$\begin{aligned}\mathcal{R}(\varrho_1, \varrho_2, \varrho_3, \varrho_4) &= \frac{1}{6}\{\mathcal{S}(\varrho_2, \varrho_3)g(\varrho_1, \varrho_4) - \mathcal{S}(\varrho_1, \varrho_3)g(\varrho_2, \varrho_4) \\ &+ g(\varrho_2, \varrho_3)\mathcal{S}(\varrho_1, \varrho_4) - g(\varrho_1, \varrho_3)\mathcal{S}(\varrho_2, \varrho_4)\}. \end{aligned} \tag{3.2}$$

Taking $\varrho_1 = \varrho_4{=}e_i,\ 1 \leq i \leq 4$, in (3.2) is our case

$$\mathcal{S}(\varrho_2, \varrho_3) = \frac{\kappa}{4}g(\varrho_2, \varrho_3). \tag{3.3}$$

Replacing ϱ_2 by $\overline{\varrho}_2$ and ϱ_3 by $\overline{\varrho}_3$ in (3.3), using (2.2) and (2.4), we get (3.3) that is, Kähler spacetime is an Einstein manifold. Consequently, we assert:

Theorem 3.1 *$\widetilde{m}$-projectively flat Kähler spacetime is an Einstein manifold.*

Corollary 3.1 *$\widetilde{m}$-projectively flat Kähler spacetime is a (PFK)$_4$ spacetime.*

According to Zengin [37] $\widetilde{m}$-projectively flat Riemannian manifold is Ricci symmetric. Therefore, we state:

Corollary 3.2 *$\widetilde{m}$-projectively flat (PFK)$_4$ spacetime is Ricci symmetric.*

It is well acknowledged that Ricci symmetry describes a general relativistic spacetime that has a covariant constant energy momentum tensor [5] So, we conclude the following:

Corollary 3.3 *$\widetilde{m}$-projectively flat (PFK)$_4$ spacetime is of covariant constant energy momentum.*

Again, from (3.2) and (3.3) it yield

$$\mathcal{R}(\varrho_1, \varrho_2, \varrho_3, \varrho_4) = \frac{\kappa}{12}\left\{g(\varrho_2, \varrho_3)g(\varrho_1, \varrho_4) - g(\varrho_1, \varrho_3)g(\varrho_2, \varrho_4)\right\}. \tag{3.4}$$

We therefore assert:

Theorem 3.2 *$\widetilde{m}$-projectively flat (PFK)$_4$ spacetime is of constant curvature.*

It is generally known that a Riemannian manifold (Ω^n, g),$(n > 3)$ is conformally flat if its curvature is constant. As a result, we obtain:

Corollary 3.4 *$\widetilde{m}$-projectively flat (PFK)$_4$ spacetime is conformally flat.*

According to Petrov [24], a spacetime is said to be of type O if it is conformally flat. Thus, we declare the results:

Theorem 3.3 *$\widetilde{m}$-projectively flat (PFK)$_4$ spacetime is of type O.*

Now, the geometrical symmetries of a spacetime are given by [16]:

$$\mathfrak{L}_\xi \Lambda - 2\omega\Lambda = 0, \tag{3.5}$$

where Λ denote a geometrical (or physical) quantity and ω is a scalar. One of the most simple and widely used example is the metric inheritance symmetry for $\Lambda = g$ in (2.10) and for this case, ξ is the Killing vector field if ω=0. Therefore,

$$(\mathfrak{L}_\xi g)(\varrho_1, \varrho_2) = 2\omega g(\varrho_1, \varrho_2). \tag{3.6}$$

A symmetry known as a curvature collineation (CC) is said to be admissible in a spacetime [8], [11] if a vector field ξ exists such that

$$(\mathfrak{L}_\xi \mathcal{R})(\varrho_1, \varrho_2, \varrho_3) = 0. \tag{3.7}$$

We consider (PFK)$_4$ spacetime admitting $\widetilde{m}$-projective curvature tensor with a Killing vector field ξ is a curvature collineation. Next, we have

$$(\mathfrak{L}_\xi g)(\varrho_1, \varrho_2) = 0. \tag{3.8}$$

Taking Lie derivative of (3.1), using (3.7) and (3.8), we get $(\mathfrak{L}_\xi \mathcal{M})(\varrho_1, \varrho_2, \varrho_3)$=0. We reach a conclusion:

Theorem 3.4 *If (PFK)$_4$ spacetime bearing the $\widetilde{m}$-projective curvature tensor with ξ is a Killing vector field is curvature collineation, then the Lie derivative of the $\widetilde{m}$-projective curvature tensor vanishes along the vector field ξ.*

Again, if the matter collineation is represented by the energy-momentum tensor $\mathcal{T}$, then

$$(\mathfrak{L}_\xi \mathcal{T})(\varrho_1, \varrho_2) = 0. \tag{3.9}$$

Also, from (2.6) and (3.3), we have

$$\left(\mu - \frac{\kappa}{2}\right) g(\varrho_1, \varrho_2) = \tau\mathcal{T}(\varrho_1, \varrho_2). \tag{3.10}$$

Taking Lie derivatives of (3.10) with respect to ξ and using (3.8), we get

$$(\mathfrak{L}_\xi \mathcal{T})(\varrho_1, \varrho_2) = 0. \tag{3.11}$$

It means (PFK)$_4$ spacetime admits matter collineation. Therefore we state:

Theorem 3.5 *$\widetilde{m}$-projectively flat $(PFK)_4$ spacetime EFE with a cosmological term then matter collineation with respect to a vector field ξ iff ξ is a Killing vector field.*

Especially if the field ξ is a conformal Killing vector. Next, we have

$$(\mathfrak{L}_\xi g)(\varrho_1, \varrho_2) = 2\omega g(\varrho_1, \varrho_2). \tag{3.12}$$

Using (2.17) and the Lie derivatives on both sides of (2.15) with regard to ξ, we acquire

$$\left(\mu - \frac{\kappa}{2}\right) 2\omega g(\varrho_1, \varrho_2) = \tau(\mathfrak{L}_\xi \mathcal{T})(\varrho_1, \varrho_2). \tag{3.13}$$

In view of (3.10) and (3.13) reduces to

$$(\mathfrak{L}_\xi \mathcal{T})(\varrho_1, \varrho_2) = 2\omega \mathcal{T}(\varrho_1, \varrho_2). \tag{3.14}$$

As a result, the energy-momentum tensor possesses Lie inheritance ability along ξ. As a result, we have:

Theorem 3.6 *$\widetilde{m}$-projectively flat $(PFK)_4$ spacetime with EFE admits a conformal killing vector field iff the energy momentum tensor has the symmetric inheritance property.*

4. Perfect fluid Kähler spacetime with $\widetilde{m}$-projectively flat

In this part, we take into consideration a perfect fluid Kähler spacetime (briefly, $(\mathrm{PFK})_4$ spacetime) with vanishing $\widetilde{m}$-projective curvature tensor and Einstein field equation that is not subject to the cosmological constant. Then from (2.8) and (3.3), we have

$$-\left(\frac{\kappa}{4} + p\tau\right) g(\varrho_1, \varrho_2) = \tau(\rho + p)\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2). \tag{4.1}$$

Putting ϱ_1 by $\overline{\varrho}_1$ and ϱ_2 by $\overline{\varrho}_2$ in (4.1), using (2.2), we get

$$-\left(\frac{\kappa}{4} + p\tau\right) g(\varrho_1, \varrho_2) = \tau(\rho + p)\mathcal{A}(\overline{\varrho}_1)\mathcal{A}(\overline{\varrho}_2). \tag{4.2}$$

Subtracting (4.1) from (4.2) , it yield

$$(\rho + p) = 0, \; \tau \neq 0, \; \mathcal{A}(\varrho_1) \neq 0. \tag{4.3}$$

That is to say the fluid exhibits cosmological constant behavior and from (4.3), we get $\rho = -p$, that is, spacetime is rapidly expanding in cosmology which is called inflation. Since $\widetilde{m}$-projectively flat perfect fluid spacetime is an Einstein manifold, that is, the manifold of constant scalar curvature κ. Then from (4.1) implies that the pressure p is constant and hence from (4.3), ρ is also constant.

Now, the energy and force equation for a perfect fluid [20]:

$$\mathcal{V}\rho = -(\rho + p) div \mathcal{V}, \tag{4.4}$$

$$(\rho + p)\nabla_{\mathcal{V}}\mathcal{V} = -grad(p) - (\mathcal{V}p)\mathcal{V}, \tag{4.5}$$

respectively. Since both the parameter ρ and p are constant. Then from (4.4) and (4.5) we get $div\mathcal{V}$=0 and $\nabla_{\mathcal{V}}\mathcal{V}$=0. This means that the expansion scalar and acceleration vector are both zero. As a result, we have:

Theorem 4.1 *$\widetilde{m}$-projectively flat $(PFK)_4$ spacetime obeying Einsteins equation without cosmological term, then it has vanishing acceleration vector and expansion scalar and the perfect fluid always behaves as a cosmological constant.*

Further, from (2.6) we have

$$\kappa = -\tau\psi, \tag{4.6}$$

where ψ=trace $\mathcal{T}$.
Eq. (2.6) adopts the form in light of (4.6) that is

$$\mathcal{S}(\varrho_1, \varrho_2) = \tau[\mathcal{T}(\varrho_1, \varrho_2) - \frac{\psi}{2} g(\varrho_1, \varrho_2)]. \tag{4.7}$$

According to [1], the Einstein field equation for a purely electromagnetic distribution in Kähler spacetime is as follows:

$$\mathcal{S}(\varrho_1, \varrho_2) = \tau\mathcal{T}(\varrho_1, \varrho_2). \tag{4.8}$$

With the help of (4.2) and (4.3) we get ψ=0. So from (4.1) we get κ=0. Thus, from (3.4) we obtain $\mathcal{R}(\varrho_1, \varrho_2, \varrho_3, \varrho_4)$=0 it confirms the flatness of spacetime. As a consequence, we get

Theorem 4.2 *$\widetilde{m}$-projectively flat (PFK)$_4$ spacetime obeying Einstein field equation without cosmological term for a purely electromagnetic distribution is an Euclidean space.*

5. (PFK)$_4$ spacetime with divergence-free $\widetilde{m}$-projective curvature

Here we study (PFK)$_4$ spacetime obeying Einstein field equation without cosmological term admitting divergence-free m-projective curvature tensor and deduce some result.

Definition 5.1 A (PFK)$_4$ spacetime is said to be $\widetilde{m}$-projective conservative if

$$(div\widetilde{\mathcal{M}})(\varrho_1, \varrho_2, \varrho_3) = 0, \tag{5.1}$$

where div stands for divergence.

Definition 5.2 If a symmetric $(0, 2)$-type tensor field $\mathcal{E}$ on a semi-Riemannian manifold (Ω^n, g) meets the following criteria, it is said to be of Codazzi type.

$$(\nabla_{\varrho_1}\mathcal{E})(\varrho_2, \varrho_3) = (\nabla_{\varrho_2}\mathcal{E})(\varrho_1, \varrho_3), \tag{5.2}$$

for every smooth vector fields ϱ_1,ϱ_2 and ϱ_3 on (Ω^n, g).
Now, Eq.(3.1) can be written as

$$\begin{aligned}(div\widetilde{\mathcal{M}})(\varrho_1, \varrho_2, \varrho_3) &= -\frac{1}{6}\{(\nabla_{\varrho_1}\mathcal{S})(\varrho_2, \varrho_3) - (\nabla_{\varrho_2}\mathcal{S})(\varrho_1, \varrho_3)\} \\ &- \frac{1}{12}\{g(\varrho_2, \varrho_3)dr(\varrho_1) + g(\varrho_1, \varrho_3)dr(\varrho_2)\}.\end{aligned} \tag{5.3}$$

In view of (2.13) and (5.1), Eq.(5.3) reduces to

$$(\nabla_{\varrho_1}\mathcal{S})(\varrho_2, \varrho_3) - (\nabla_{\varrho_2}\mathcal{S})(\varrho_1, \varrho_3) = 0. \tag{5.4}$$

From (5.4) and (2.6), we get

$$(\nabla_{\varrho_1}\mathcal{T})(\varrho_2, \varrho_3) = (\nabla_{\varrho_2}\mathcal{T})(\varrho_1, \varrho_3). \tag{5.5}$$

Thus we assert:

Theorem 5.1 *A (PFK)$_4$ spacetime obeying Einstein's equation without cosmological term, admitting divergence free $\widetilde{m}$-projective curvature then the energy-momentum tensor $\mathcal{T}$ is of Codazzi type.*

It is well-known that in a perfect fluid spacetime the energy-momentum tensor is of Codazzi type, then each of the shear and vorticity of the fluid vanishes and its velocity vector field is hypersurface orthogonal, which means that it is proportional to the gradient vector field of the energy density [14], [15], [12]. Thus we sate the results.

Theorem 5.2 *A (PFK)$_4$ spacetime obeying Einstein equation without cosmological term, admitting divergence free m-projective curvature then its velocity vector field is proportional to the gradient vector field of the energy density.*

According to Barnes [2] if a perfect fluid spacetime is shear-free and vorticity-free and the velocity vector field V is hypersurface orthogonal and the energy density is constant over a hypersurface orthogonal to V, then the possible local cosmological structures of the spacetime are of Petrov type I, D or O. As per this result we have:

Theorem 5.3 *A (PFK)$_4$ spacetime obeying Einstein equation without cosmological term, with divergence free $\widetilde{m}$-projective curvature then the possible local cosmological structure of such a spacetime is of type I, D or O.*

6. Dust fluid Kähler spacetime with $\widetilde{m}$-projectively flat

The energy momentum tensor in a dust fluid Kähler spacetime (briefly, (DFK)$_4$ spacetime) [28]:

$$\mathcal{T}(\varrho_1, \varrho_2) = \rho\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2). \tag{6.1}$$

With the help of (3.10) and (6.1) we obtain

$$[\mu - \frac{\kappa}{2}]g(\varrho_1, \varrho_2) = \tau\rho.\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2), \tag{6.2}$$

If we replace ϱ_1=ϱ_2=$\mathcal{V}$ in (6.2) we have

$$\mu = \frac{\kappa}{4} - \tau\rho. \tag{6.3}$$

Again taking contraction over ϱ_1 and ϱ_2, Eq. (6.2) leads to

$$\mu = \frac{\kappa}{4} - \frac{\tau\rho}{4}. \tag{6.4}$$

In view of (6.3) and (6.4) we finally get ρ=0, then from (6.1), we have $\mathcal{T}(\varrho_1, \varrho_2)$=0. As a result, we can state:

Theorem 6.1 *$\widetilde{m}$-projectively flat (DFK)$_4$ spacetime satisfying Einstein field equation with cosmological constant is vacuum.*

Also from (2.6) and (6.1), we have

$$\mathcal{S}(\varrho_1, \varrho_2) = [-\mu + \frac{\kappa}{2}]g(\varrho_1, \varrho_2) + \tau\rho\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2). \tag{6.5}$$

By virtue of (6.5), Eq. (1.5) takes the form

$$[\mu + \frac{\kappa}{2} + \theta_1]g(\varrho_1, \varrho_2) + [\tau + \theta_2]\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2) = -\frac{1}{2}[g(\nabla_{\varrho_1}\xi, \varrho_2) + g(\varrho_1, \nabla_{\varrho_2}\xi)]. \tag{6.6}$$

After contracting over ϱ_1 and ϱ_1 leads to

$$4\theta_1 - \theta_2 = \tau - 4\mu - 2\kappa - \mathrm{div}\xi. \tag{6.7}$$

Again, if we put ϱ_1=ϱ_2=$\mathcal{V}$ in (6.6), we get

$$\theta_1 - \theta_2 = -\mu - \frac{\kappa}{2} + \tau. \tag{6.8}$$

With the help of (6.7) and (6.8) we yields

$$\theta_2 = -\mu - \frac{\tau(\rho - 3p)}{2} - \frac{div\xi}{2} \quad \text{and} \quad \theta_2 = -\tau - \frac{div\xi}{3}. \tag{6.9}$$

It is obvious that if θ_2=0, we get the Ricci soliton with θ_1=-$\mu - \frac{\tau(\rho-3p)}{2} + \tau$, which is steady if p=$\frac{2}{3} - \frac{3}{2}(\frac{\mu}{\tau}) + \frac{\varrho}{3}$, expanding if $p > \frac{2}{3} - \frac{3}{2}(\frac{\mu}{\tau}) + \frac{\varrho}{3}$ and shrinking if $p < \frac{2}{3} - \frac{3}{2}(\frac{\mu}{\tau}) + \frac{\varrho}{3}$.
The consequence is as follows:

Theorem 6.2 *An η-Ricci soliton $(g,\xi,\theta_1,\theta_2)$ in a $(PFK)_4$ spacetime satisfying Einstein field equation with cosmological constant is steady if p=$\frac{2}{3}-\frac{3}{2}\frac{\mu}{\tau}+\frac{\varrho}{3}$, expanding if $p>\frac{2}{3}-\frac{3}{2}(\frac{\mu}{\tau})+\frac{\varrho}{3}$ and shrinking if $p<\frac{2}{3}-\frac{3}{2}(\frac{\mu}{\tau})+\frac{\varrho}{3}$.*

Remark 6.1 *About radiation fluid, that is, ρ=3p then θ_1=-μ-$\frac{div\xi}{2}$ and θ_2=-τ-$\frac{div\xi}{3}$.*

Also taking (6.5) and (1.7) we have

$$[-\mu+\theta_1]g(\varrho_1,\varrho_2)+[\tau\rho+\theta_2]\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)=-\frac{1}{2}[g(\nabla_{\varrho_1}\xi,\varrho_2)+g(\varrho_1,\nabla_{\varrho_2}\xi)]. \tag{6.10}$$

On contracting over ϱ_1 and ϱ_1, Eq.(6.10) leads to

$$4\theta_1-\theta_2=\tau\rho+4\mu-\mathrm{div}\xi. \tag{6.11}$$

Again, if we put ϱ_1=ϱ_2=$\mathcal{V}$ in (6.11), we get

$$\theta_1-\theta_2=\mu+\tau\rho. \tag{6.12}$$

With the help of (6.11) and (6.12) we yields

$$\theta_1=\mu-\frac{div\xi}{3} \quad \text{and} \quad \theta_2=-\tau\rho-\frac{div\xi}{3}. \tag{6.13}$$

Now, if θ_2=0 then we obtain the Einstein soliton with θ_1=$\mu+\tau\rho$, which is steady if μ=-$\tau\rho$, expanding if $\mu>\tau\rho$ and shrinking if $\mu<\tau\rho$.
Thus we have:

Theorem 6.3 *An η-Einstein soliton $(g,\xi,\theta_1,\theta_2)$ in a $(DFK)_4$ spacetime satisfying Einstein field equation with cosmological constant is steady if $\mu=-\tau\rho$, expanding if $\mu>\tau\rho$ and shrinking if $\mu<\tau\rho$.*

Again we consider the energy momentum tensor $\mathcal{T}$ of a viscous fluid Kähler spacetime(briefly, $(\text{VFK})_4$ spacetime) [20], [22]:

$$\mathcal{T}(\varrho_1,\varrho_2)=(\rho+p)\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)+pg(\varrho_1,\varrho_2)+\mathcal{H}(\varrho_1,\varrho_2), \tag{6.14}$$

where ρ, p are the energy density and isotropic pressure respectively and $\mathcal{H}$ denotes the anisotropic pressure of the fluid.
Using (3.10) in (6.14), we get

$$[\mu-\frac{\kappa}{2}]g(\varrho_1,\varrho_2)=\tau[(\rho+p)\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)+pg(\varrho_1,\varrho_2)+\mathcal{H}(\varrho_1,\varrho_2)]. \tag{6.15}$$

For fix,ϱ_1=ϱ_2=$\mathcal{V}$ in (6.15), yields

$$\rho=\frac{1}{\tau}[\frac{\kappa}{2}-\mu-\tau\mathcal{I}], \tag{6.16}$$

where $\mathcal{I}$=$\mathcal{H}(\mathcal{V},\mathcal{V})$.
Again contracting (6.15) over ϱ_1 and ϱ_2 , we obtain

$$p=\frac{4}{3\tau}[\mu-\frac{\kappa}{2}+\frac{\tau(\rho-\mathcal{N})}{4}], \tag{6.17}$$

where $\mathcal{N}$=Trace of $\mathcal{H}$. Thus we can state:

Theorem 6.4 *In a viscous fluid $\widetilde{m}$-projectively flat Kähler spacetime satisfying Einstein field equation with cosmological constant, the energy density and the isotropic pressure are given by the relations (6.16) and (6.17), respectively.*

Again from (2.6) and (6.14), we have

$$\mathcal{S}(\varrho_1,\varrho_2)=[\frac{\kappa}{2}-\mu+p\tau]g(\varrho_1,\varrho_2)+\tau[(\rho+p)\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)+\mathcal{H}(\varrho_1,\varrho_2)]. \tag{6.18}$$

Keeping in mind (6.18), Eq. (1.5) can be written as

$$\begin{aligned}[\frac{\kappa}{2}-\mu+p\tau+\theta_1]g(\varrho_1,\varrho_2) &+ [\tau(\rho+p)+\theta_2]\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)+\tau\mathcal{H}(\varrho_1,\varrho_2)\\ &= -\frac{1}{2}[g(\nabla_{\varrho_1}\xi,\varrho_2)+g(\varrho_1,\nabla_{\varrho_2}\xi)]. \end{aligned}\tag{6.19}$$

A frame field after contraction over ϱ_1 and ϱ_1, where $\mathcal{N}$=Trace of $\mathcal{H}$ leads to

$$4\theta_1-\theta_2=4\mu-3p\tau-2\kappa+\tau\rho-\tau\mathcal{N}-\text{div}\xi. \tag{6.20}$$

Again, if we put ϱ_1=ϱ_2=$\mathcal{V}$ in (6.19), where $\mathcal{I}$=$\mathcal{H}(\mathcal{V},\mathcal{V})$ have

$$\theta_1-\theta_2=\mu-\frac{\kappa}{2}+\tau(\rho+\mathcal{I}). \tag{6.21}$$

With the help of (6.20) and (6.21) we yields

$$\theta_1=\mu+\frac{\tau(\rho-3p)}{2}+\tau\left[\frac{\mathcal{I}}{4}-p-\frac{\mathcal{N}}{3}\right]-\frac{div\xi}{3} \text{ and } \theta_2=-\tau\left[p+\rho+\frac{3\mathcal{I}}{4}+\frac{\mathcal{N}}{3}\right]-\frac{div\xi}{3}. \tag{6.22}$$

In particular if $\theta_2=0$ then we get the Ricci soliton with $a=\mu-\frac{\tau(\rho-3p)}{2}+\tau(\rho+\mathcal{I})$ which is steady if $p=\rho+(\frac{2}{3})\mathcal{I}-\frac{2}{3}(\frac{\mu}{\tau})$, expanding if $p>\rho+(\frac{2}{3})\mathcal{I}-\frac{2}{3}(\frac{\mu}{\tau})$ and shrinking if $p<\rho+(\frac{2}{3})\mathcal{I}-\frac{2}{3}(\frac{\mu}{\tau})$.
We state the result:

Theorem 6.5 *An η-Ricci soliton $(g,\xi,\theta_1,\theta_1)$ in a (VFK)$_4$ spacetime satisfying Einstein field equation with cosmological constant steady if p=$\rho+(\frac{2}{3})\mathcal{I}-\frac{2}{3}(\frac{\mu}{\tau})$, expanding if $p>\rho+(\frac{2}{3})\mathcal{I}-\frac{2}{3}(\frac{\mu}{\tau})$ and shrinking if $p<\rho+(\frac{2}{3})\mathcal{I}-\frac{2}{3}(\frac{\mu}{\tau})$.*

Remark 6.2 *For radiation fluid ρ=$3p$ then a=$\mu+\tau[\frac{\mathcal{I}}{4}-p-\frac{\mathcal{N}}{3}]$-$\frac{div\xi}{3}$ and b=-$\tau(4p+\frac{3\mathcal{I}}{4}+\frac{\mathcal{N}}{3})$-$\frac{div\xi}{3}$.*

Also, with the help of (6.18), Eq. (1.7) can be written as

$$\begin{aligned}[\mu-\theta_1+p\tau]g(\varrho_1,\varrho_2) &- [\tau(\rho+p)+\theta_2]\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)-\tau\mathcal{H}(\varrho_1,\varrho_2)\\ &= \frac{1}{2}[g(\nabla_{\varrho_1}\xi,\varrho_2)+g(\varrho_1,\nabla_{\varrho_2}\xi)]. \end{aligned}\tag{6.23}$$

A frame field after contraction over ϱ_1 and ϱ_1, where $\mathcal{N}$=Trace of $\mathcal{H}$ leads to

$$-4\theta_1+\theta_2=-4\mu+3p\tau-\tau\rho+\tau\mathcal{N}+\text{div}\xi. \tag{6.24}$$

Again, if we put ϱ_1=ϱ_2=$\mathcal{V}$ in (6.23), where $\mathcal{I}$=$\mathcal{H}(\mathcal{V},\mathcal{V})$ reduces to

$$\theta_1-\theta_2=\mu+\tau\rho+\tau\mathcal{I}. \tag{6.25}$$

With the help of (6.24) and (6.25) we obtain

$$\theta_1=\mu-\tau p-\frac{\mathcal{I}\tau}{3}-\frac{\mathcal{N}\tau}{3}-\frac{div\xi}{3} \quad\text{and}\quad \theta_2=-\tau\left[p+\rho+\frac{4\mathcal{I}}{3}+\frac{\mathcal{N}}{3}\right]-\frac{div\xi}{3}. \tag{6.26}$$

It is clear that if θ_2=0 then we obtain the Einstein soliton with θ_1=$\mu-\tau p-\frac{\mathcal{I}\tau}{3}-\frac{\mathcal{N}\tau}{3}-\frac{div\xi}{3}$ which is steady if p=$\frac{\mu}{\tau}-\frac{\mathcal{I}}{3}+\frac{\mathcal{N}}{3}-\frac{div\xi}{3\tau}$, expanding if $p>\frac{\mu}{\tau}-\frac{\mathcal{I}}{3}+\frac{\mathcal{N}}{3}-\frac{div\xi}{3\tau}$ and shrinking if $p<\frac{\mu}{\tau}-\frac{\mathcal{I}}{3}+\frac{\mathcal{N}}{3}-\frac{div\xi}{3\tau}$.
As a consequence, we get:

Theorem 6.6 *An η-Einstein soliton $(g,\xi,\theta_1,\theta_2)$ in a $(VFK)_4$ spacetime satisfying Einstein field equation with cosmological constant steady if $p=\frac{\mu}{\tau}-\frac{\mathcal{I}}{3}+\frac{\mathcal{N}}{3}-\frac{div\xi}{3\tau}$, expanding if $p>\frac{\mu}{\tau}-\frac{\mathcal{I}}{3}+\frac{\mathcal{N}}{3}-\frac{div\xi}{3\tau}$ and shrinking if $p<\frac{\mu}{\tau}-\frac{\mathcal{I}}{3}+\frac{\mathcal{N}}{3}-\frac{div\xi}{3\tau}$.*

Remark 6.3 . *If there is radiation fluid $\rho=3p$ then $\theta_1=\mu-\tau p-\frac{\mathcal{I}\tau}{3}$-$\frac{\mathcal{N}\tau}{3}$-$\frac{div\xi}{3}$ and θ_2=-$\tau\left[4p+\frac{4\mathcal{I}}{3}+\frac{\mathcal{N}}{3}\right]-\frac{div\xi}{3}$.*

Finally, we explore whether or not a viscous fluid in a Kähler spacetime that is $\widetilde{m}$-projectively flat may permit heat flux. In order to do this, we take into account the energy momentum tensor, $\mathcal{T}$ [20], [22]:

$$\mathcal{T}(\varrho_1,\varrho_2)=(\rho+p)\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)+pg(\varrho_1,\varrho_2)+\mathcal{A}(\varrho_1)\mathcal{F}(\varrho_2)+\mathcal{A}(\varrho_2)\mathcal{F}(\varrho_1), \tag{6.27}$$

where $\mathcal{F}(\varrho_1)$=$g(\varrho_1,\xi)$ for all vector fields ϱ_1; ξ being the heat flux vector field and $g(\mathcal{V},\xi)$=0, that is, $\mathcal{F}(\mathcal{V})$=0.
Using (3.10) in (6.27), we have

$$\left(\mu-\frac{\kappa}{2}\right)g(\varrho_1,\varrho_2)=\tau[(\rho+p)\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)+pg(\varrho_1,\varrho_2)+\mathcal{A}(\varrho_1)\mathcal{F}(\varrho_2)+\mathcal{A}(\varrho_2)\mathcal{F}(\varrho_1)]. \tag{6.28}$$

Putting ϱ_2=$\mathcal{V}$ in (6.10), yields

$$\mathcal{F}(\varrho_1)=-\frac{1}{\tau}[\mu-\frac{\kappa}{2}+\tau p]\mathcal{A}(\varrho_1). \tag{6.29}$$

Therefore, we have:

Theorem 6.7 *A viscous fluid in m-projectively flat Kähler spacetime satisfying Einstein's field equation with cosmological constant, admits heat flux, provided $\mu-\frac{\kappa}{2}+\tau p\neq 0$.*

Also using (6.27) in (2.6), we have

$$\mathcal{S}(\varrho_1,\varrho_2)=[p-\mu+\frac{\kappa}{2}]g(\varrho_1,\varrho_2)+\tau[(\rho+p)\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)+\mathcal{A}(\varrho_1)\mathcal{F}(\varrho_2)+\mathcal{A}(\varrho_2)\mathcal{F}(\varrho_1)]. \tag{6.30}$$

By virtue of (6.30), Eq.(1.4) can be written as

$$\begin{aligned}[p+\frac{\kappa}{2}-\mu+\theta_1]g(\varrho_1,\varrho_2) \quad &+\quad [\tau(\rho+p)+\theta_2]\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2)+\tau[\mathcal{A}(\varrho_1)\mathcal{F}(\varrho_2)+\mathcal{A}(\varrho_2)\mathcal{F}(\varrho_1)]\\ &=\quad -\frac{1}{2}[g(\nabla_{\varrho_1}\xi,\varrho_2)+g(\varrho_1,\nabla_{\varrho_2}\xi)].\end{aligned} \tag{6.31}$$

Taking contraction over ϱ_1 and ϱ_1, we get

$$4\theta_1-b\theta_2=4\mu-4p-2\kappa+\tau(p+\rho)-\mathrm{div}\xi. \tag{6.32}$$

Putting ϱ_1=ϱ_2=$\mathcal{V}$ in (6.31) , yields

$$-2\theta_2+2\theta_2=2p+\kappa-2\mu-\tau(p+\rho). \tag{6.33}$$

From (6.32) and (6.33), we obtain

$$\theta_1=\mu-p+\frac{1}{6}\tau(p+\rho)-\frac{\tau(\rho-3p)}{2}-\frac{div\xi}{3}\quad\text{and }\theta_2=\frac{\tau(p+\rho)}{3}-\frac{div\xi}{3}. \tag{6.34}$$

If θ_2=0 then we get the Ricci soliton with θ_1=$-p+\mu-\frac{\kappa}{2}+\frac{\tau(\rho+p)}{2}$ which is steady if p=$\frac{\mu}{1-\tau}$, expanding if $p>\frac{\mu}{1-\tau}$ and shrinking if $p<\frac{\mu}{1-\tau}$. Thus we state:

Theorem 6.8 *An η-Ricci soliton $(g,\xi,\theta_1,\theta_2)$ in a $(VFK)_4$ spacetime satisfying Einstein field equation with cosmological constant steady if $p=\frac{\mu}{1-\tau}$, expanding if $p>\frac{\mu}{1-\tau}$ and shrinking if $p<\frac{\mu}{1-\tau}$.*

Remark 6.4 . *For radiation fluid $\rho=3p$ then $\theta_1=\mu-p+\frac{2p\tau}{3}$-$\frac{div\xi}{3}$ and θ_2=$\frac{4p\tau}{3}$-$\frac{div\xi}{3}$.*

Finally, we use (6.30) in (1.7) we obtain

$$[p-\mu+\theta_1]g(\varrho_1,\varrho_2)+[\tau(\rho+p)+\theta_2]\mathcal{A}(\varrho_1)\mathcal{A}(\varrho_2) + \tau[\mathcal{A}(\varrho_1)\mathcal{F}(\varrho_2)+\mathcal{A}(\varrho_2)\mathcal{F}(\varrho_1)] = -\frac{1}{2}[g(\nabla_{\varrho_1}\xi,\varrho_2)+g(\varrho_1,\nabla_{\varrho_2}\xi)]. \quad (6.35)$$

Now, contracting over ϱ_1 and ϱ_1, we have

$$4\theta_1+\theta_2=4\mu-4p+\tau(p+\rho)-\mathrm{div}\xi. \quad (6.36)$$

Putting ϱ_1=ϱ_2=$\mathcal{V}$ in (6.35) , yields

$$\theta_1-\theta_2=\mu-p+\tau(p+\rho). \quad (6.37)$$

From (6.36) and (6.37) , we obtain

$$\theta_1=\mu-p+\frac{1}{8}[5\tau(p+\rho)-div\xi],\ \ \theta_2=\frac{1}{8}[-3\tau(p+\rho)-div\xi]. \quad (6.38)$$

If $\theta_2=0$ then we get the Eistein soliton with $\theta_1=\mu-p+\tau(p+\rho)$, which is steady if p=$\frac{1}{1-\tau}(\mu+\tau\rho)$, expanding if $p>\frac{1}{1-\tau}(\mu+\tau\rho)$ and shrinking if $p<\frac{1}{1-\tau}(\mu+\tau\rho)$.

Theorem 6.9 *An η-Einstein soliton $(g,\xi,\theta_1,\theta_2)$ in a (VFK)$_4$ spacetime satisfying Einstein field equation with cosmological constant steady if $p=\frac{1}{1-\tau}(\mu+\tau\rho)$, expanding if $p>\frac{1}{1-\tau}(\mu+\tau\rho)$ and shrinking if $p<\frac{1}{1-\tau}(\mu+\tau\rho)$.*

Remark 6.5 . *In case of radiation fluid $\rho=3p$, then $\theta_1=\mu-p+\frac{1}{8}[20p\tau-div\xi]$, and $\theta_2=\frac{1}{8}[-12p\tau-div\xi]$.*

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