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Progress in Invariant and Preserving Transforms for the Ratio of Collinear Points in a Desargues Affine Plane Skew Field

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ABSTRACT: This paper introduces invariant transforms that preserve the ratio of either two or three points collinear in the Desargues affine plane skew field. The results given here have a clean, geometric presentation based based Desargues affine plan axiomatics and definitions with skew field properties. The main results in this paper, are (1) the ratio of two and three points is *Invariant* under the following transforms: Inversion, Natural Translation, Natural Dilation, Mobiüs Transform, in a line of Desargues affine plane. (2) parallel projection of a pair of lines in a Desargues affine plane preserves the ratio of two and three points, (3) translations in the Desargues affine plane preserve the ratio of 2 and 3 points and (4) dilation in a Desargues affine plane preserves the ratio of 2 and 3 points.

Key Words: Desargues Affine Plane, Collinear, Dilation, Invariant Transform, Mobiüs Transform, Natural Translation, Preserving Transform, Skew-Field line.

Contents

1. Introduction and Preliminaries

In the advancement of our research in the connections of axiomatic geometry and algebraic structures, we have achieved some results which we have presented in this paper. More recently, results are given about the association of algebraic structures in affine planes and in Desargues affine plane, and vice versa in [15,16,4,13,14,12,23,24,22,21,17].

The foundations for the study of the connections between axiomatic geometry and algebraic structures were set forth by D. Hilbert [6]. And some classic research results in this context are given, for example, by E. Artin [1], D.R. Huges and F.C. Piper [7], H. S. M Coxeter [3]. Marcel Berger in [2], Robin Hartshorne in [5].

In this paper, we advanced the study regarding the ratio of two points and ratio of three points, in a line on Desargues affine plane. Specifically, we study some *Invariant-Transforms*, we introduce a numbe of transformations in a Desargues affine plane, and prove that for each transform, the ratio of two and three collinear points are *Invariant*.

Also we consider some transforms that *preserve ratio* of two and three points in a Desargues affine plane. Earlier, we have shown that for each line on a Desargues affine plane, we can construct a skew-field simply and constructively, using simple elements of elementary geometry, and only the basic axioms of Desargues affine plane (see [16,4,12,23]).

Results are given for the translations, parallel projections and dilation's that preserve the ratio of two and three points.

The novelty in this paper is that we achieve our results without the use coordinates. We make use of properties enjoyed by transformations in Desargues affine planes such as parallel projection, translation and dilation.

Desargues Affine Plane.

Let $\mathcal{P}$ be a nonempty space, $\mathcal{L}$ a nonempty subset of $\mathcal{P}$. The elements p of $\mathcal{P}$ are points and an element ℓ of $\mathcal{L}$ is a line.

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Definition 1.1 *The incidence structure $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, called affine plane, where satisfies the above axioms:*

1^o *For each points $\{P,Q\} \in \mathcal{P}$, there is exactly one line $\ell \in \mathcal{L}$ such that $\{P,Q\} \in \ell$.*

2^o *For each point $P \in \mathcal{P}, \ell \in \mathcal{L}, P \notin \ell$, there is exactly one line $\ell' \in \mathcal{L}$ such that $P \in \ell'$ and $\ell \cap \ell' = \emptyset$ (Playfair Parallel Axiom [9]). Put another way, if the point $P \notin \ell$, then there is a unique line ℓ' on P missing ℓ [10].*

3^o *There is a 3-subset of points $\{P,Q,R\} \in \mathcal{P}$, which is not a subset of any ℓ in the plane. Put another way, there exist three non-collinear points $\mathcal{P}$ [10].*

Desargues' Axiom, circa 1630 [8, §3.9, pp. 60-61] [11].
Let $A,B,C,A',B',C' \in \mathcal{P}$ and let pairwise distinct lines $\ell^{AA'}, \ell^{BB'}, \ell^{CC'}, \ell^{AC}, \ell^{A'C'} \in \mathcal{L}$ such that

$$\ell^{AA'} \parallel \ell^{BB'} \parallel \ell^{CC'} \text{ (Fig. 1(a)) } \textbf{or } \ell^{AA'} \cap \ell^{BB'} \cap \ell^{CC'} = P. \text{(Fig. 1(b))}$$
$$\text{and } \ell^{AB} \parallel \ell^{A'B'} \text{ and } \ell^{BC} \parallel \ell^{B'C'}.$$
$$A,B \in \ell^{AB}, A'B' \in \ell^{A'B'}, \text{ and } B,C \in \ell^{BC}, B'C' \in \ell^{B'C'}.$$
$$A \neq C, A' \neq C', \text{ and } \ell^{AB} \neq \ell^{A'B'}, \ell^{BC} \neq \ell^{B'C'}.$$

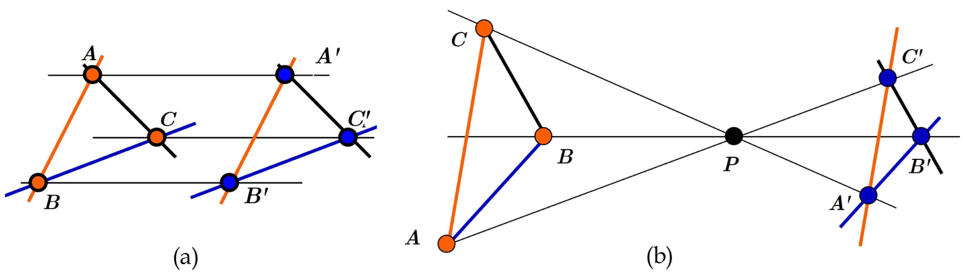

Figure 1: Desargues Axioms: (a) For parallel lines $\ell^{AA'} \parallel \ell^{BB'} \parallel \ell^{CC'}$; (b) For lines which are cutting in a single point P, $\ell^{AA'} \cap \ell^{BB'} \cap \ell^{CC'} = P$.

Then $\boldsymbol{\ell^{AC} \parallel \ell^{A'C'}}$.
A **Desargues affine plane** is an affine plane that satisfies Desargues' Axiom.

Note 1 *Three vertexes ABC and $A'B'C'$, which, fulfilling the conditions of the Desargues Axiom, we call* 'Desarguesian'.

We, earlier, have defined, the actions: 'addition of points' and 'multiplication of points' in a line of Desargues affine planes, presented in [12,4,16,14,17].

The process of construct the points C for adition and multiplication of points in $\ell^{OI}-$line in affine plane, is presented in the tow algorithm form

Addition Algorithm(Fig.2(a))	**Multiplication Algorithm**(Fig.2(b))
Step.1: $B_1 \notin \ell^{OI}$	**Step.1:** $B_1 \notin \ell^{OI}$
Step.2: $\ell_{OI}^{B_1} \cap \ell_{OB_1}^{A} = P_1$	**Step.2:** $\ell_{IB_1}^{A} \cap \ell^{OB_1} = P_1$
Step.3: $\ell_{BB_1}^{P_1} \cap \ell^{OI} = C(= A + B)$	**Step.3:** $\ell_{BB_1}^{P_1} \cap \ell^{OI} = C(= A \cdot B)$

In [12] and [4], we have prove that $(\ell^{OI}, +, \cdot)$ is a skew field in Desargues affine plane, and is field (commutative skew field) in the Papus affine plane.

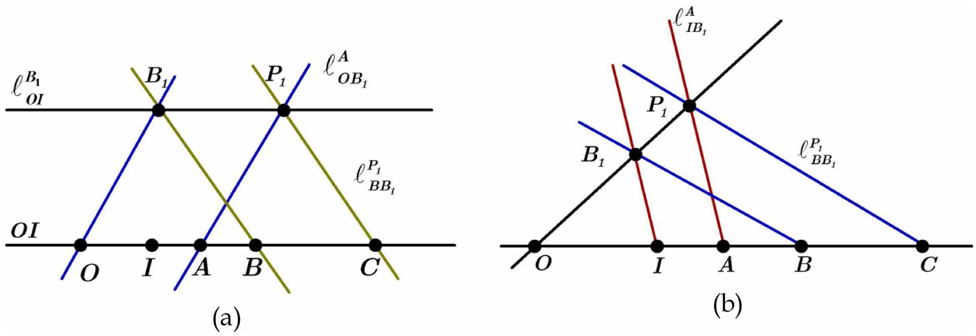

Figure 2: (a) Addition of points in a line in affine plane, (b) Multiplication of points in a line in affine plane

Definition 1.2 *The parallel projection between the two lines in the Desargues affine plane, will be called, a function,*

$$P_P : \ell_1 \to \ell_2, \quad \forall A, B \in \ell_1, \quad AP_P(A)||BP_P(B)$$

It is clear that this function is a bijection between any two lines in Desargues affine planes, for this reason, it can also be thought of as isomorphism between two lines.

Definition 1.3 *[13] Dilatation of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, called a its collineation δ such that: $\forall P \neq Q \in \mathcal{P}, \delta(PQ)||PQ$.*

Definition 1.4 *[13] Translation of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, called identical dilatation $id_{\mathcal{P}}$ his and every other of its dilatation, about which he affine plane has not fixed points.*

Some well-known results related to translations and dilation's in Desargues affine planes.

- The dilatation set $\mathbf{Dil}_{\mathcal{A}}$ of affine plane $\mathcal{A}$ forms a **group** with respect to composition $\circ$ ([13]).
- The translations set $\mathbf{Tr}_{\mathcal{A}}$ of affine plane $\mathcal{A}$ forms a **group** with respect to composition $\circ$; which is a sub-group of the dilation group $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ ([13]).
- In a affine plane: the group $(\mathbf{Tr}_{\mathcal{A}}, \circ)$ of translations is **normal sub-group** of the group of dilatations $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ ([13]).
- Every dilatation in Desargues affine plane $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ which leads a line in it, is an automorphism of skew-fields constructed on the same line $\ell \in \mathcal{L}$, of the plane $\mathcal{A}_{\mathcal{D}}$ ([15]).
- Every translations in Desargues affine plane $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ which leads a line in it, is an automorphism of skew-fields constructed on the same line $\ell \in \mathcal{L}$, of the plane $\mathcal{A}_{\mathcal{D}}$ ([15]).
- Each dilatation in a Desargues affine plane, $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an isomorphism between skew-fields constructed over isomorphic lines $\ell_1, \ell_2 \in \mathcal{L}$ of that plane ([23]).
- Each translations in a Desargues affine plane, $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an isomorphism between skew-fields constructed over isomorphic lines $\ell_1, \ell_2 \in \mathcal{L}$ of that plane ([23]).

Ratio of two points in a line on Desargues affine plane.

In the paper [17], we have done a detailed study, related to the ratio of two and three points in a line of Desargues affine plane. Below we are listing some of the results for ratio of two and three points.

Definition 1.5 *[17] Lets have two different points $A, B \in \ell^{OI}$−line, and $B \neq O$, in Desargues affine plane. We define as ratio of this tow points, a point $R \in \ell^{OI}$, such that,*

$$R = B^{-1}A, \qquad \textit{we mark this, with,} \qquad R = r(A : B) = B^{-1}A$$

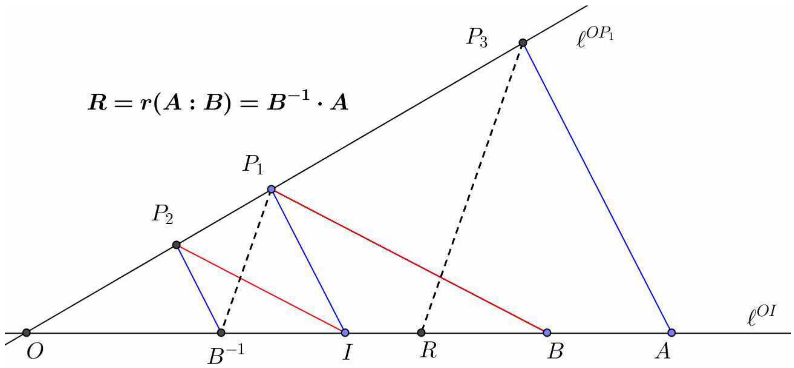

Figure 3: Ilustrate the Ratio-Point, of 2-Points in a line of Desargues affine plane $R = r(A : B) = B^{-1}A$.

For a 'ratio-point' $R \in \ell^{OI}$, and for point $B \neq O$ in line ℓ^{OI}, is a unique defined point, $A \in \ell^{OI}$, such that $R = B^{-1}A = r(A : B)$.

Some results for Ratio of 2-points in Desargues affine plane (see [17]).

- If have two different points $A, B \in \ell^{OI}-$line, and $B \neq O$, in Desargues affine plane, then,
$$r^{-1}(A : B) = r(B : A).$$
- For three collinear point A, B, C and $C \neq O$, in $\ell^{OI}-$line, have,
$$r(A + B : C) = r(A : C) + r(B : C).$$
- For three collinear point A, B, C and $C \neq O$, in $\ell^{OI}-$line, have,
  1. $r(A \cdot B : C) = r(A : C) \cdot B$.
  2. $r(A : B \cdot C) = C^{-1}r(A : C)$.
- Let's have the points $A, B \in \ell^{OI}-$line where $B \neq O$. Then have that,
$$r(A : B) = r(B : A) \Leftrightarrow A = B.$$
- This ratio-map, $r_B : \ell^{OI} \to \ell^{OI}$ is a bijection in $\ell^{OI}-$line in Desargues affine plane.
- The ratio-maps-set $\mathcal{R}_2 = \{r_B(X) | \forall X \in \ell^{OI}\}$, for a fixed point B in $\ell^{OI}-$line, forms a skew-field with 'addition and multiplication' of points. This, skew field $(\mathcal{R}_2, +, \cdot)$ is sub-skew field of the skew field $(\ell^{OI}, +, \cdot)$.

Ratio of three points in a line on Desargues affine plane. (see [17])

Definition 1.6 *If A, B, C are three points on a line ℓ^{OI} (collinear) in Desargues affine plane, then we define their **ratio** to be a point $R \in \ell^{OI}$, such that:*
$$(B - C) \cdot R = A - C, \quad \textit{concisely} \quad R = (B - C)^{-1}(A - C),$$
and we mark this with
$$r(A, B; C) = (B - C)^{-1}(A - C).$$

Some Results for Ratio of 3-points in Desargues affine plane ([17]).

- For 3-points A, B, C in a line ℓ^{OI} of Desargues affine plane, we have that,
$$r(-A, -B; -C) = r(A, B; C).$$

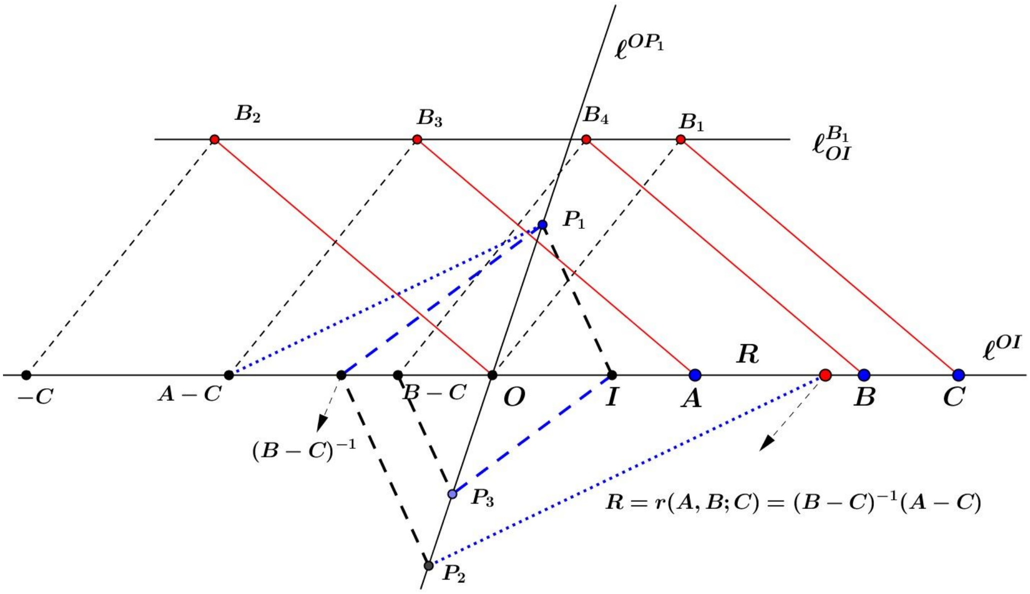

Figure 4: Ratio of 3-Points in a line of Desargues affine plane $R = r(A, B; C)$.

- For 3-points A, B, C in a line ℓ^{OI} in the Desargues affine plane, have
$$r^{-1}(A, B; C) = r(B, A; C).$$
- If A, B, C, are three different points, and different from point O, in a line ℓ^{OI} on Desargues affine plane, then
$$r(A^{-1}, B^{-1}; C^{-1}) = B[r(A, B; C)]A^{-1}.$$
- In the Pappus affine plane, for three point different from point O, in $\ell^{OI}-$line, we have
$$r(A^{-1}, B^{-1}; C^{-1}) = r(A, B; C) \cdot r(B, A; O).$$
- This ratio-map, $r_{BC} : \ell^{OI} \to \ell^{OI}$ is a bijection in $\ell^{OI}-$line in Desargues affine plane.
- The ratio-maps-set $\mathcal{R}_3 = \{r_{BC}(X) | \forall X \in \ell^{OI}\}$, for a different fixed points B, C in $\ell^{OI}-$line, forms a skew-field with 'addition and multiplication' of points in $\ell^{OI}-$line. This, skew field $(\mathcal{R}_3, +, \cdot)$ is sub-skew field of the skew field $(\ell^{OI}, +, \cdot)$.

2. Invariant transforms for Ratio-2-points and Ratio-3-points

In this section we will see some transformations, for which the ratio of 2-points and ratio of 3-points, are invariant under their action. We define these transformations first,

Definition 2.1 *Inversion of points in $\ell^{OI}-$line, called the map*
$$j_P : \ell^{OI} \to \ell^{OI},$$
which satisfies the condition,
$$\forall A \in \ell^{OI} \quad j_P(A) = P \cdot A.$$

Definition 2.2 *A natural translation with point P, of points in $\ell^{OI}-$line, called the map*
$$\varphi_P : \ell^{OI} \to \ell^{OI},$$
for a fixed $P \in \ell^{OI}$ which satisfies the condition,
$$\forall A \in \ell^{OI} \quad \varphi_P(A) = P + A.$$

Definition 2.3 *A natural Dilation of points in $\ell^{OI}-$line, called the map*

$$\delta_n : \ell^{OI} \to \ell^{OI},$$

for a fixed natural number $n \in (N)$ which satisfies the condition,

$$\forall A \in \ell^{OI} \quad \delta_n(A) = nA = \underbrace{A + A + \cdots + A}_{n-times}.$$

Definition 2.4 **(a)** *For* Ratio of 2-points: *Lets have an fixed points $B \in \ell^{OI}$, which are different from point O. Mobiüs transform for ratio or two points in $\ell^{OI}-$line, we called the map,*

$$m : \ell^{OI} \to \ell^{OI},$$

which satisfies the condition,

$$\forall X \in \ell^{OI}, \quad m(X) = r(X : B).$$

(b) *For* Ratio of 3-points: *Lets have three fixed points $B, C \in \ell^{OI}$. Mobiüs transform for ratio, we called the map,*

$$\mu : \ell^{OI} \to \ell^{OI},$$

which satisfies the condition,

$$\forall X \in \ell^{OI}, \quad \mu(X) = r(X, B; C).$$

Theorem 2.1 *Ratio of 2-points is invariant under the natural dilation with a fixet $n \in \mathbb{N}$.*

Proof: For ratio definition 1.5 we have that, $r(A : B) = B^{-1}A$, and for natural dilation definition 2.2, we have, $\delta_n(A) = nA, \quad \text{and} \quad \delta_n(B) = nB, \forall A, B \in \ell^{OI}$, so,

$$\begin{aligned} r[\delta_n(A) : \delta_n(B)] &= r(nA : nB) \\ &= (nB)^{-1}(nA) \\ &= B^{-1}n^{-1}nA \\ &= B^{-1}A \\ &= r(A : B). \end{aligned}$$

Hence

$$r[\delta_n(A) : \delta_n(B)] = r(A : B).$$

□

Theorem 2.2 *Ratio of 2-points is invariant under inversion with a given point $P \in \ell^{OI}$.*

Proof: For ratio definition 1.5 we have that, $r(A : B) = B^{-1}A$, and for inversion definition 2.1 with a point P, we have, $j_P(A) = PA, \forall A \in \ell^{OI}$, so,

$$\begin{aligned} r[j_P(A) : j_P(B)] &= r(PA : PB) \\ &= (PB)^{-1}(PA) \\ &= B^{-1}P^{-1}PA \\ &= B^{-1}(P^{-1}P)A \\ &= B^{-1}IA \\ &= B^{-1}A \\ &= r(A : B). \end{aligned}$$

Hence, $r[j_P(A) : j_P(B)] = r(A : B)$. □

Theorem 2.3 *Ratio of 2-points is invariant under Mobiüs transform.*

Proof: For Mobiüs transform definition 2.4 we have $m(X) = r(X : B) = B^{-1}X$, so, for ratio of 2-point, the points $m(A), m(B) \in \ell^{OI}$ first, we calculate, this point, according to following the definition of $m-$map, and we have

- $m(A) = r(A : B)$,
- $m(B) = r(B : B)$, so $m(B) = B^{-1}B = I$.

Now, calculate,

$$\begin{aligned} r[m(A) : m(B)] &= r(\mu(A) : I) \\ &= I^{-1}[m(A)] \\ &= I \cdot m(A) \\ &= m(A) \\ &= r(A : B). \end{aligned}$$

Hence, $r[m(A) : m(B)] = r(A : B)$. □

Theorem 2.4 *Ratio of* $3-$*points, is invariant under the natural translation with a point* P.

Proof: For ratio definition 1.5 we have that, $r(A, B; C) = (B - C)^{-1}(A - C)$, and for natural translation with a point P definition 2.2, we have, $\varphi_P(A) = P + A, \forall A \in \ell^{OI}$, so, for ratio we have that,

$$\begin{aligned} r[\varphi_P(A), \varphi_P(B); \varphi_P(C)] &= r(A + P, B + P; C + P) \\ &= ([B + P] - [C + P])^{-1}([A + P] - [C + P]) \\ &= (B + P - C - P)^{-1}(A + P - C - P) \\ &= (B - C)^{-1}(A - C) \\ &= r(A, B; C). \end{aligned}$$

Hence,

$$r[\varphi_P(A), \varphi_P(B); \varphi_P(C)] = r(A, B; C).$$

□

Theorem 2.5 *Ratio of* $3-$*points is invariant under the natural dilation with a fixet* $n \in \mathbb{N}$.

Proof: For ratio definition 1.6 we have that, $r(A, B; C) = (B - C)^{-1}(A - C)$, and for natural dilation definition 2.3, we have, $\delta_n(A) = nA, \forall A \in \ell^{OI}$, so,

$$\begin{aligned} r[\delta_n(A), \delta_n(B); \delta_n(C)] &= r(nA, nB; nC) \\ &= (nB - nC)^{-1}(nA - nC) \\ &= (n[B - C])^{-1}(n[A - C]) \\ &= (B - C)^{-1}n^{-1}n(A - C) \\ &= (B - C)^{-1}(A - C) \\ &= r(A, B; C). \end{aligned}$$

Hence,

$$r[\delta_n(A), \delta_n(B); \delta_n(C)] = r(A, B; C).$$

□

Theorem 2.6 *Ratio of* $3-$*points is invariant under Inversion with a given point* $P \in \ell^{OI}$.

Proof: For ratio definition 1.6 we have that, $r(A, B; C) = (B - C)^{-1}(A - C)$, and for inversion definition 2.1 with a point P, we have, $j_V(A) = PA, \forall A \in \ell^{OI}$, so,

$$\begin{aligned} r[j_P(A), j_P(B); j_P(C)] &= r(PA, PB; PC) \\ &= (PB - PC)^{-1}(PA - PC) \\ &= (P[B - C])^{-1}(P[A - C]) \\ &= (B - C)^{-1}P^{-1}P(A - C) \\ &= (B - C)^{-1}I(A - C) \\ &= (B - C)^{-1}I(A - C) \\ &= r(A, B; C). \end{aligned}$$

Hence,

$$r[j_P(A), j_P(B); j_P(C)] = r(A, B; C).$$

□

Theorem 2.7 *Ratios of* $3-$*points, is invariant under Mobiüs transform.*

Proof: For Mobiüs transform definition 2.4 we have

$$\mu(X) = r(X, B; C) = [(B - C)^{-1}(X - C)]$$

so, for cross-ratio of the points $\mu(A), \mu(B), \mu(C) \in \ell^{OI}$ first, we calculate, this point, according to following the definition of $\mu-$map, and we have

- $\mu(A) = r(A, B; C)$,
- $\mu(B) = r(B, B; C) \Rightarrow \mu(B) = [(B - C)^{-1}(B - C)] = I \Rightarrow \mu(B) = I$.
- $\mu(C) = r(C, B; C) \Rightarrow \mu(C) = (B - C)^{-1}(C - C) = (B - C)^{-1}O = O \Rightarrow \mu(C) = O$.

Now, calculate,

$$\begin{aligned} r[\mu(A), \mu(B); \mu(C)] &= r(\mu(A), I; O) \\ &= (I - O)^{-1}(\mu(A) - O) \\ &= (I)^{-1}(\mu(A)) \\ &= \mu(A) \\ &= r(A, B; C). \end{aligned}$$

Hence, $r[\mu(A), \mu(B); \mu(C)] = r(A, B; C)$. □

Theorem 2.8 *For natural translation with a given point* P*, we have that the ratio of two points* A *and* B*, is equal with ratio of three points* $\varphi_P(A)$,$\varphi_P(B)$, P*, so*

$$r(A : B) = r\left(\varphi_P(A), \varphi_P(B); P\right)$$

Proof: For ratio definition 1.5 we have that $r(A : B) = B^{-1}A$, in this equation we substitute

$$B = [B + P] - P \qquad \text{and} \qquad A = [A + P] - P,$$

and have,

$$\begin{aligned} r(A:B) &= r\left([A+P]-P:[B+P]-P\right) \\ &= \left([B+P]-P\right)^{-1}\left([A+P]-P\right) \\ &= \left(\varphi_P(B)-P\right)^{-1}\left(\varphi_P(A)-P\right) \\ &= r\left(\varphi_P(A),\varphi_P(B);P\right). \end{aligned}$$

because from natural translation definition 2.2, we have that $B+P=\varphi(B)$ and $A+P=\varphi(A)$ and we consider the ratio of three points definition. Hence,

$$r(A:B) = r\left(\varphi_P(A),\varphi_P(B);P\right).$$

□

Theorems 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7,2.8, which we have proved in detail above, can all be summarized in below remark

Remark 2.1 *In a line on Desargues Affine Plane, the transforms: '**Natural Translation**, **Natural Dilation**, **Inversion** and **Mobiüs Transform**', are **Invariant-Transforms** related to the **ratio of 2 points** and **ratio of 3 points**, in a line on Desargues Affine Plane*

3. Transforms which Preserving Ratio

In this section we prove that the parallel projection, Translations and Dilation of a line in itself or in isomorphic line in Desargues affine plane, *preserving:* the ratio of of two and three points. The geometrical interpretations, even in the Euclidean view, are quite beautiful in the above theorems, regardless of the rather rendered figures. This is also the reason that we are giving the proofs in the algebraic sense. So we will always have in mind the close connection of skew field and a line in Desargues affine plane, and the properties of parallel projection, translations and dilation's. To prove our results, we refer to the parallel projection, translations and dilation's properties, which are studied in the papers [12,13,15,23,24,14] and our achieved results, for, ratio of two and ratio of three points in [17].

Theorem 3.1 *The parallel projection between the two lines ℓ_1 and ℓ_2 in Desargues affine plane, **preserving the ratio** of $2-$points,*

$$P_P(r(A:B)) = r(P_P(A):P_P(B))$$

Proof: If $\ell_1||\ell_2$, we have that the parallel projection is a translation, and have true this theorem.
If lines ℓ_1 and ℓ_2 they are not parallel (so, they are cut at a single point), we have $A,B\in\ell_1$ and $P_P(A),P_P(B)\in\ell_2$. It is easily proved, with the help of Desargues' configuration (with Desargues axiom and affine plane axioms), that $P_P(r(A:B)) = r(P_P(A):P_P(B))$. In figure 5, we have market $\ell_1=\ell^{OI}$, $\ell_2=\ell^{OI'}$, $I'=P_P(I)$, $A'=P_P(A)$, $B'=P_P(A)$, $(A^{-1})'=P_P(A^{-1})$, $(B^{-1})'=P_P(B^{-1})$, $R'=P_P(R)$.

If we consider, two 'three-vertex' $B^{-1}P_1(B^{-1})'$ and $RP_3P_P(R)$ in Fig.5, they are Desarguesian, for this reason, and for the results for construction of points $R'=(B^{-1})'A'$ we have that,

$$R' = P_P(R)$$

because, $P_3R'||P_3P_P(R)||P_1P_P(B^{-1})$, and parallelism is equivalence relation. So the points, R' and $P_P(R)$ are the cutting points of lines $\ell^{OI'}$ and $\ell^{P_3}_{P_1(B^{-1})'}$, but this is a single point, for this, we have that $R'=P_P(R)$. □

Theorem 3.2 *Translations in Desargues affine plane, **preserving the ratio** of $2-$points A,B in a line ℓ^{OI} of this plane,*

$$\varphi(r(A:B)) = r(\varphi(A):\varphi(B))$$

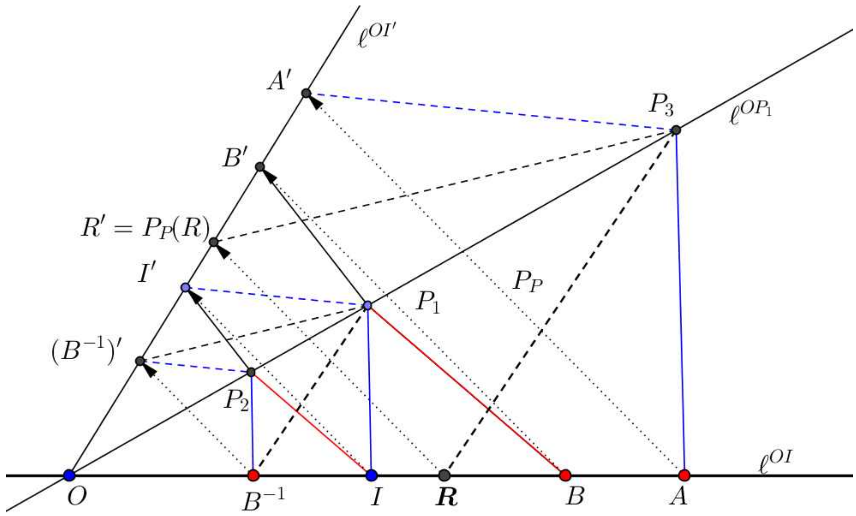

Figure 5: Ratio of 2-Points into parallel projection in Desargues affine plane.

Proof: Lets have a translation $\varphi : \ell^{OI} \to \ell^{O'I'}$, we know that translation preserves parallelism, therefore $\ell^{OI} \parallel \ell^{O'I'}$.

We mark with: $O' = \varphi(O)$, $I' = \varphi(I)$, $A' = \varphi(A)$, $B' = \varphi(B)$. Also, translation φ, we also apply it to auxiliary points, and mark $P_1' = \varphi(P_1)$, $P_2' = \varphi(P_2)$, $P_3' = \varphi(P_3)$. It is easily shown from the properties of the construction of the inverse point and from the Desargues condition, that $\varphi(B^{-1}) = [\varphi(B)]^{-1}$.

If we consider, two 'three-vertex' AP_3R and $A'P_3'\varphi(R)$, in Fig.6, we see that they are *Desarguesian*, for this reason, and for the results for construction of points $R' = (B^{-1})'A'$ we have that,

$$R' = \varphi(R)$$

because, $P_3R || P_1B^{-1} || P_1'(B^{-1})' = P_1'(B')^{-1} || P_3\varphi(R) || P_3R'$, and parallelism is equivalence relation. So the points, R' and $\varphi(R)$ are the cutting points of lines $\ell^{O'I'}$ and $\ell^{P_3'}_{P_1'(B^{-1})'}$, but this is a single point, for this, we have that $R' = \varphi(R)$.

The case when, $\varphi : \ell^{OI} \to \ell^{OI}$ (there are traces ℓ^{OI}), φ can be seen as a composition of two translations φ_1, φ_2, with different trace of the line ℓ^{OI}. □

Theorem 3.3 *Dilation δ with fixed point in the same line ℓ^{OI} or with fixed point out of line, of points A, B, preserve the ratio of this points,*

$$\delta(r(A : B)) = r(\delta(A) : \delta(B))$$

Proof: Lets have firstly, a dilation with an fixed point $V \notin \ell^{OI}$ in Desargues affine plane, which $\delta : \ell^{OI} \to \ell^{O'I'}$ we know that dilation preserves parallelism, therefore $\ell^{OI} \parallel \ell^{O'I'}$.

We mark with: $O' = \delta(O)$, $I' = \delta(I)$, $A' = \delta(A)$, $B' = \delta(B)$, $\delta(R)$ Fig.7. Also, dilation δ, we also apply it to auxiliary points, and mark $P_1' = \delta(P_1)$, $P_2' = \delta(P_2)$, $P_3' = \delta(P_3)$. It is easily shown from the properties of the construction of the inverse point and from the Desargu condition, that $\delta(B^{-1}) = [\delta(B)]^{-1} = (B^{-1})'$.

If we consider, two 'three-vertex' AP_3R and $A'P_3'\delta(R)$, in Fig.7, we see that they are Desarguesian, for this reason, and for the results for construction of points $R' = (B^{-1})'A'$ we have that,

$$R' = \delta(R)$$

because, $P_3R || P_1B^{-1} || P_1'(B^{-1})' = P_1'(B')^{-1} || P_3\delta(R) || P_3R'$, and parallelism is equivalence relation. So the points, R' and $\delta(R)$ are the cutting points of lines $\ell^{O'I'}$ and $\ell^{P_3'}_{P_1'(B^{-1})'}$ (the line which passes from

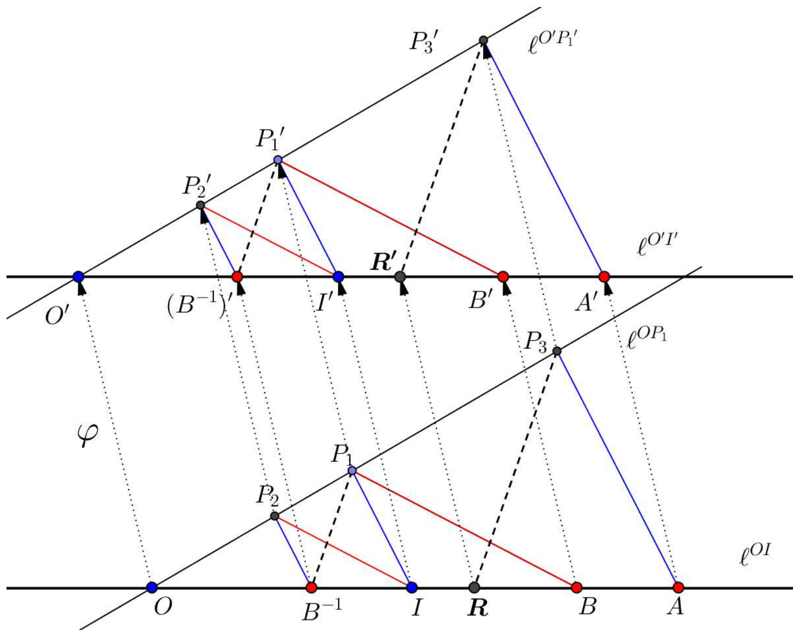

Figure 6: Ratio of 2-Points into translation in Desargues Affine Plane.

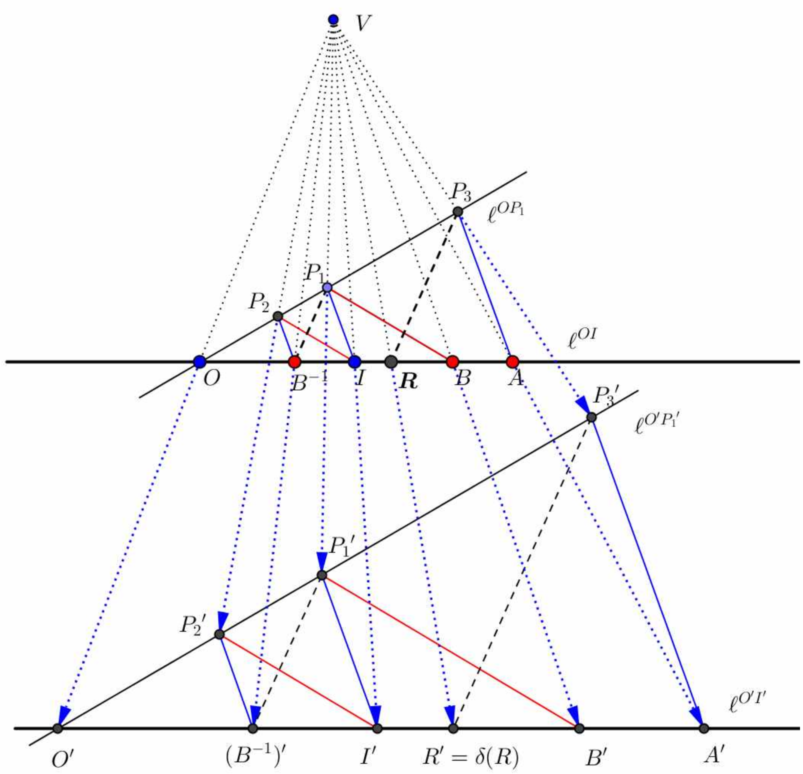

Figure 7: Ratio of 2-Points into translation in Desargues affine plane.

point P_3' and is parallel with line $\ell^{P_1'(B^{-1})'}$), but this is a single point, for this, we have that $R' = \delta(R)$.

$$R' = \ell^{OI} \cap \ell^{P_3'}_{P_1'(B^{-1})'} \quad \text{and} \quad \delta(R) = \ell^{OI} \cap \ell^{P_3'}_{P_1'(B^{-1})'},$$

so,

$$R' = \delta(R).$$

Hence,

$$\delta\left[r(A:B)\right] = r\left(\delta(A):\delta(B)\right).$$

Now let's look at the case where, $\delta : \ell^{OI} \to \ell^{OI}$ (is case where the fixed point $V \in \ell^{OI}$). For simplicity of interpretation, we are taking the fixed point $V = O$, (the case where the point $V \neq O$ will be discussed at the end of the proof.

As in the first case, we mark with: $O' = \delta(O) = O$, $I' = \delta(I)$, $A' = \delta(A)$, $B' = \delta(B)$, $\delta(R)$, see Fig.8 (a), (we are making these notes to simplify the symbolism a bit). Also, dilation δ, we also apply it to auxiliary points, and mark $P_1' = \delta(P_1)$, $P_2' = \delta(P_2)$, $P_3' = \delta(P_3)$. It is easily shown from the properties of the construction of the inverse point and from the Desargu condition, that $\delta(B^{-1}) = [\delta(B)]^{-1} = (B^{-1})'$.

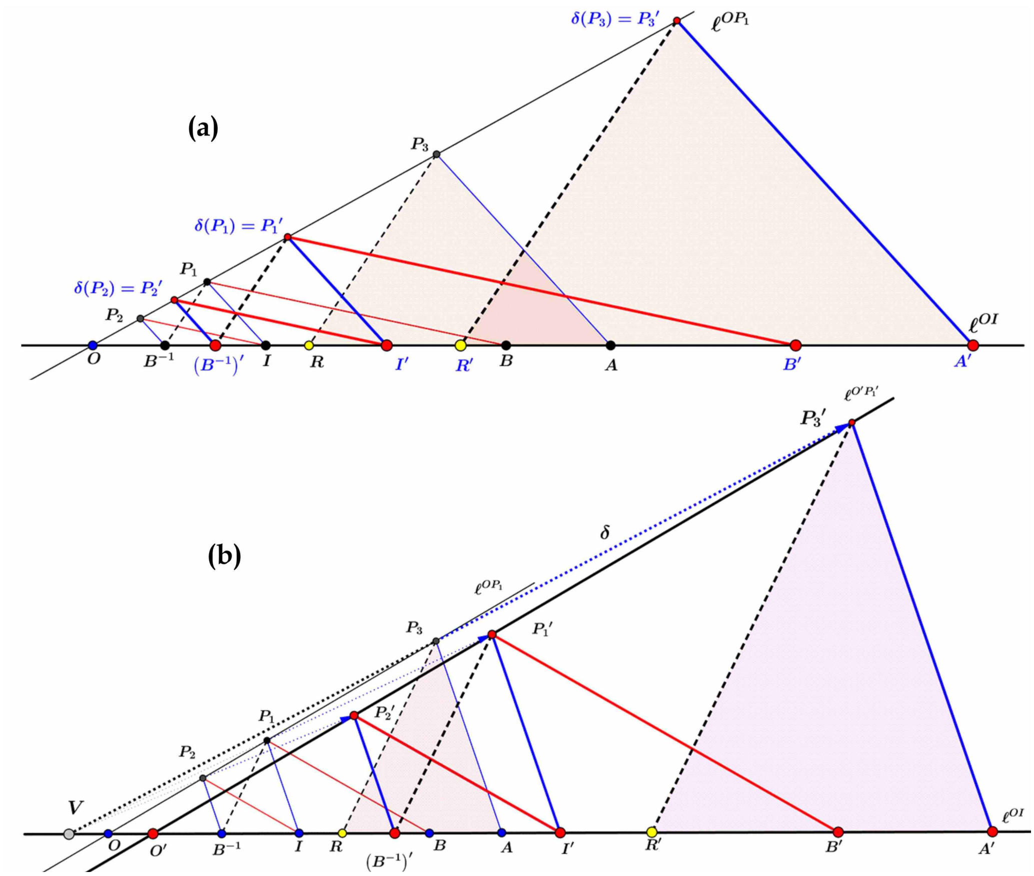

Figure 8: Ratio of 2-Points into Dilation with fixed point in $\ell^{OI}-$line, in Desargues affine plane; (a) case where $V = O$ and (b) case where $V \neq O$.

If we consider, two 'three-vertex' AP_3R and $A'P_3'\delta(R)$, in Fig.8 (a), we see that they are Desarguesian, for this reason, and for the results for construction of points $R' = (B^{-1})'A'$ we have that,

$$R' = \delta(R)$$

because, $P_3\delta(R)||P_3R||P_1B^{-1}||P_1'(B^{-1})' = P_1'(B')^{-1}||P_3R'$, and parallelism is equivalence relation, hence

$$P_3R' \parallel P_3\delta(R)$$

So the points, R' and $\delta(R)$ are the cutting points of lines ℓ^{OI} and $\ell^{P'_3}_{P'_1(B^{-1})'}$ (the line which passes from point P'_3 and is parallel with line $\ell^{P'_1(B^{-1})'}$), but this is a single point, for this, we have that $R' = \delta(R)$.

$$R' = \ell^{OI} \cap \ell^{P'_3}_{P'_1(B^{-1})'} \quad \text{and} \quad \delta(R) = \ell^{OI} \cap \ell^{P'_3}_{P'_1(B^{-1})'},$$

so,

$$R' = \delta(R).$$

Hence,

$$\delta\left[r(A:B)\right] = r\left(\delta(A):\delta(B)\right).$$

In the same way, the case where the fixed point $V \notin O$ will be proven. The proof can be done directly see Fig.8 (b), or this dilation can be expressed, as i composition of an translation which con point V to point O. □

Theorem 3.4 *Translations in Desargues affine plane,* ***preserving the ratio*** *of* $3-$*points* A, B, C *in a line* ℓ^{OI} *of this plane,*

$$\varphi(r(A,B;C)) = r(\varphi(A), \varphi(B); \varphi(C))$$

Proof: Lets have a translation with trace different from the $\ell^{OI}-$line, so $\varphi : \ell^{OI} \to \ell^{O'I'}$, we know that translation preserves parallelism, therefore $\ell^{OI} \parallel \ell^{O'I'}$.

We mark with: $O' = \varphi(O)$, $I' = \varphi(I)$, $A' = \varphi(A)$, $B' = \varphi(B)$, $C' = \varphi(C)$. Also, translation φ, we also apply it to auxiliary points, and mark $B'_1 = \varphi(B_1)$, $B'_2 = \varphi(B_2)$, $B'_3 = \varphi(B_3)$, $B'_4 = \varphi(B_4)$ and $P'_1 = \varphi(P_1)$, $P'_2 = \varphi(P_2)$, $P'_3 = \varphi(P_3)$. From results in [15], [23], and from the properties of the construction of the reverse and inverse point, and sure from the Desargues condition, we have that,

$$\begin{aligned}
-C' &= -\varphi(C) = \varphi(-C),\\
(A-C)' &= \varphi(A-C) = \varphi(A) - \varphi(C) = A' - C',\\
(B-C)' &= \varphi(B-C) = \varphi(B) - \varphi(C) = B' - C',\\
[(B-C)^{-1}]' &= \varphi[(B-C)^{-1}] = [\varphi(B-C)]^{-1}\\
&= [\varphi(B) - \varphi(C)]^{-1}.
\end{aligned}$$

If we consider, two 'three-vertex' OP_2R and $O'P'_2\varphi(R)$, in Fig.9, we see that they are *Desarguesian*, for this reason, and for the results for construction of points $R' = ([B-C]^{-1})' \cdot A'$ we have that,

$$R' = \varphi(R)$$

because, $P_2R || P_1(A-C) || P'_1(A-C)' = P'_1(A'-C') || P'_2\varphi(R)$, and parallelism is equivalence relation. So the points, R' and $\varphi(R)$ are the cutting points of lines $\ell^{O'I'}$ and $\ell^{P'_2}_{P'_1(A-C)'}$, but this is a single point, for this, we have that $R' = \varphi(R)$.

The case when, $\varphi : \ell^{OI} \to \ell^{OI}$ (there are traces ℓ^{OI}), φ can be seen as a composition of two translations φ_1, φ_2, with different trace of the line ℓ^{OI}. □

Theorem 3.5 *The parallel projection between the two lines* ℓ_1 *and* ℓ_2 *in Desargues affine plane,* ***preserving the ratio*** *of* $3-$*points,*

$$P_P(r(A,B;C)) = r(P_P(A), P_P(B); P_P(C))$$

Proof: If $\ell_1 || \ell_2$, we have that the parallel projection is a translation, and have true this theorem.
If lines ℓ_1 and ℓ_2 they are not parallel (so, they are cut at a single point), we have $A, B, C \in \ell_1$ and $P_P(A), P_P(B), P_P(C) \in \ell_2$.

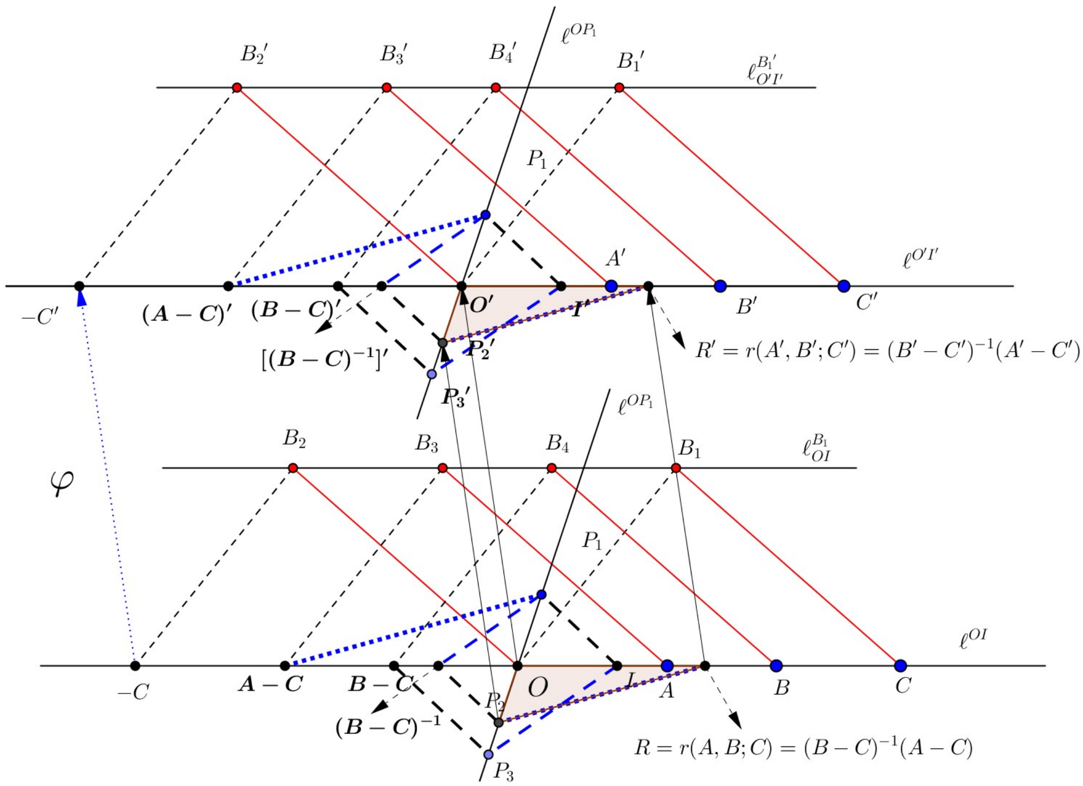

Figure 9: Ratio of 3-Points into translation in Desargues affine plane.

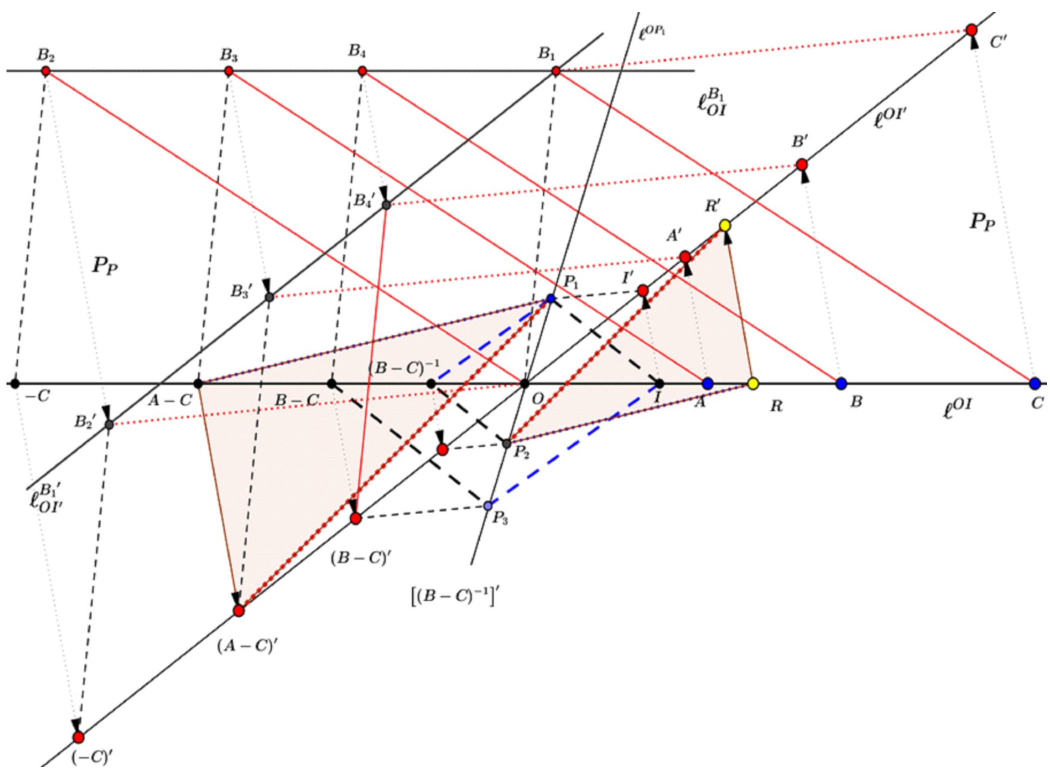

Figure 10: Ratio of 3-Points into parallel projection P_P in Desargues affine plane.

Lets have a parallel projection from the $\ell^{OI}-$line, to $\ell^{OI'}-$line (we are assuming that the lines intersect at the point O, if, the point of intersection would be another point, we can use a translation, to bring the point O to the point of intersection) Fig.10.

So, have

$$P_P : \ell^{OI} \to \ell^{OI'}.$$

and $\forall A, B \in \ell^{OI}$, we have that $AP_P(A) \parallel BP_P(B)$, and $O' = P_P(O) = O$.

We mark with: $I' = P_P(I)$, $A' = P_P(A)$, $B' = P_P(B)$, $C' = P_P(C)$, and $(-C)' = P_P(-C)$, $(A-C)' = P_P(A-C)$, $(B-C)' = P_P(B-C)$, $\left[(B-C)^{-1}\right]' = P_P\left[(B-C)^{-1}\right]$.

Since addition of points do not depend on the position of the auxiliary point (see [16], [12]), we are keeping the same auxiliary point B_1, and construct the line $\ell^{B_1}_{OI'}$, and use the parallel projection with which, we get the points: $P_P(B_1) = B_1$, $P_P(B_2) = B_2'$, $P_P(B_3) = B_3'$, $P_P(B_4) = B_4'$. Also, we know that the multiplication of points in a line of Desargues affine plane does not depend on the position of the auxiliary point (see [12], [4]), therefore for the multiplication of points in the line $\ell^{OI'}$, we we keep the same auxiliary points as in the case of multiplication of points, in the line ℓ^{OI}, so the points P_1, P_2, P_3.

For parallel projection properties, we have the following parallelisms,

$$\ell^{(-C)(-C)'} \parallel \ell^{(A-C)(A-C)'} \parallel \ell^{(B-C)(B-C)'} \parallel \ell^{[(B-C)^{-1}][(B-C)^{-1}]'} \parallel \ell^{II'}$$

continue

$$\ell^{II'} \parallel \ell^{AA'} \parallel \ell^{RP_P(R)} \parallel \ell^{BB'} \parallel \ell^{CC'} \parallel \ell^{B_2B_2'} \parallel \ell^{B_3B_3'} \parallel \ell^{B_4B_4'}$$

Since parallelism is an equivalence relation, we, from the above parallelisms, can distinguish

$$\ell^{(A-C)(A-C)'} \parallel \ell^{RP_P(R)}.$$

For construction of ratio-point R, we have that, the point

$$R = \ell^{OI} \cap \ell^{P_2}_{P_1(A-C)}$$

and we have the follow parallelism

$$\ell^{(A-C)P_1} \parallel \ell^{P_2R}.$$

Also, we have triads of collinear points $(A-C), O, R$ and $(A-C)', O, P_P(R)$, and P_1, O, P_2. Hence we have that the two three-vertexes

$$(A-C)P_1(A-C)' \quad \text{and} \quad RP_P(R)P_2 \to \text{are Desarguesian.}$$

For this we have that

$$\ell^{(A-C)'P_1} \parallel \ell^{P_2P_P(R)}.$$

It is easy to prove that,

$$\begin{aligned}
(-C)' &= P_P(-C) = -P_P(C),\\
(A-C)' &= P_P(A-C) = P_P(A) - P_P(C) = A' - C',\\
(B-C)' &= P_P(B-C) = P_P(B) - P_P(C) = B' - C',\\
\left[(B-C)^{-1}\right]' &= P_P\left[(B-C)^{-1}\right]\\
&= \left[P_P(B-C)\right]^{-1}\\
&= \left[P_P(B) - P_P(C)\right]^{-1}\\
&= \left[B' - C'\right]^{-1}.
\end{aligned}$$

For construction of ratio point R' in $\ell^{OI'}-$line, we have following parallelisms:

$$\ell^{P_1[A'-C']} = \ell^{P_1(A-C)'} \parallel \ell^{P_2R'}$$

and, from construction of point R, we had the parallelisms

$$\ell^{P_1(A-C)} \parallel \ell^{P_2R}$$

Also, we have triads of collinear points $(A-C), O, R$ and $(A-C)', O, R'$, and P_1, O, P_2. So, we have that the two three-vertexes,

$$(A-C)P_1(A-C)' \quad \text{and} \quad RR'P_2 \to \text{are Desarguesian.}$$

For this we have that

$$\ell^{(A-C)'P_1} \parallel \ell^{P_2R'}.$$

Also we have that the points R' and $P_P(R)$ are in $\ell^{OI'}$−line. So, we have that,

$$\ell^{P_2P_P(R)} \parallel \ell^{P_2R} \quad \text{and points} \quad R', P_P(R) \in \ell^{OI'}$$

Hence, have

$$R' = P_P(R).$$

If, $P_P : \ell^{OI} \to \ell^{OI}$ we have the *identical-parallel projection*. The case where $P_P : \ell^{OI} \to \ell^{O'I'}$, and $\ell^{OI} \parallel \ell^{O'I'}$, the parallel projection is a translation. □

Remark 3.1 *If lines $\ell_1 = \ell_2$, then the parallel projection is a translation, or identical transform.*

Theorem 3.6 *Dilation δ preserve the ratio of this points, so*

$$\delta(r(A, B; C)) = r(\delta(A), \delta(B); \delta(C))$$

Proof: Lets have firstly, a dilation with an fixed point $V \notin \ell^{OI}$ in Desargues affine plane ($\delta(V) = V$), which $\delta : \ell^{OI} \to \ell^{O'I'}$ we know that dilation preserves parallelism, therefore $\ell^{OI} \parallel \ell^{O'I'}$.

We mark with: $O' = \delta(O)$, $I' = \delta(I)$, $A' = \delta(A)$, $B' = \delta(B)$, $C' = \delta(C)$, $\delta(R)$. Also, dilation δ, we apply it to auxiliary points, $B_1' = \delta(B_1)$, $B_2' = \delta(B_2)$, $B_3' = \delta(B_3)$, $B_4' = \delta(B_4)$ and mark $P_1' = \delta(P_1)$, $P_2' = \delta(P_2)$, $P_3' = \delta(P_3)$. With results in [15], [23], and from the properties of the construction of the inverse point, we have that,

$$\begin{aligned}
(-C)' &= \delta(-C) = -\delta(C) = -C' \\
(A-C)' &= \delta(A-C) = \delta(A) - \delta(C) = A' - C' \\
(B-C)' &= \delta(B-C) = \delta(B) - \delta(C) = B' - C' \\
[(B-C)^{-1}]' &= \delta[(B-C)^{-1}] = [\delta(B-C)]^{-1} = [\delta(B) - \delta(C)]^{-1} = [B' - C']^{-1}
\end{aligned}$$

Now construct the ratio point $R' = r(A', B'; C')$ in $\delta(\ell^{OI})$−line. From definition of ratio of three points and the above notes, we have that,

$$R' = (B' - C')^{-1}(A' - C')$$

is ratio point in $\delta(\ell^{OI})$−line.

If we consider, two 'three-vertex' OP_2R and $O'P_2'\delta(R)$, in Fig.11, we see that they are Desarguesian, for this reason, and for the results for construction of points $R' = (B' - C')^{-1}(A' - C')$ we have that,

$$R' = \delta(R)$$

because, $P_2R||P_1(A-C)||P_1'(A-C)' = P_1'(A' - C')||P_2'R'$, also from dilation properties we have that

$$P_2R||\delta(P_2)\delta(R) = P_2'\delta(R),$$

and parallelism is equivalence relation, hence we have

$$P_2'R' \parallel P_2'\delta(R),$$

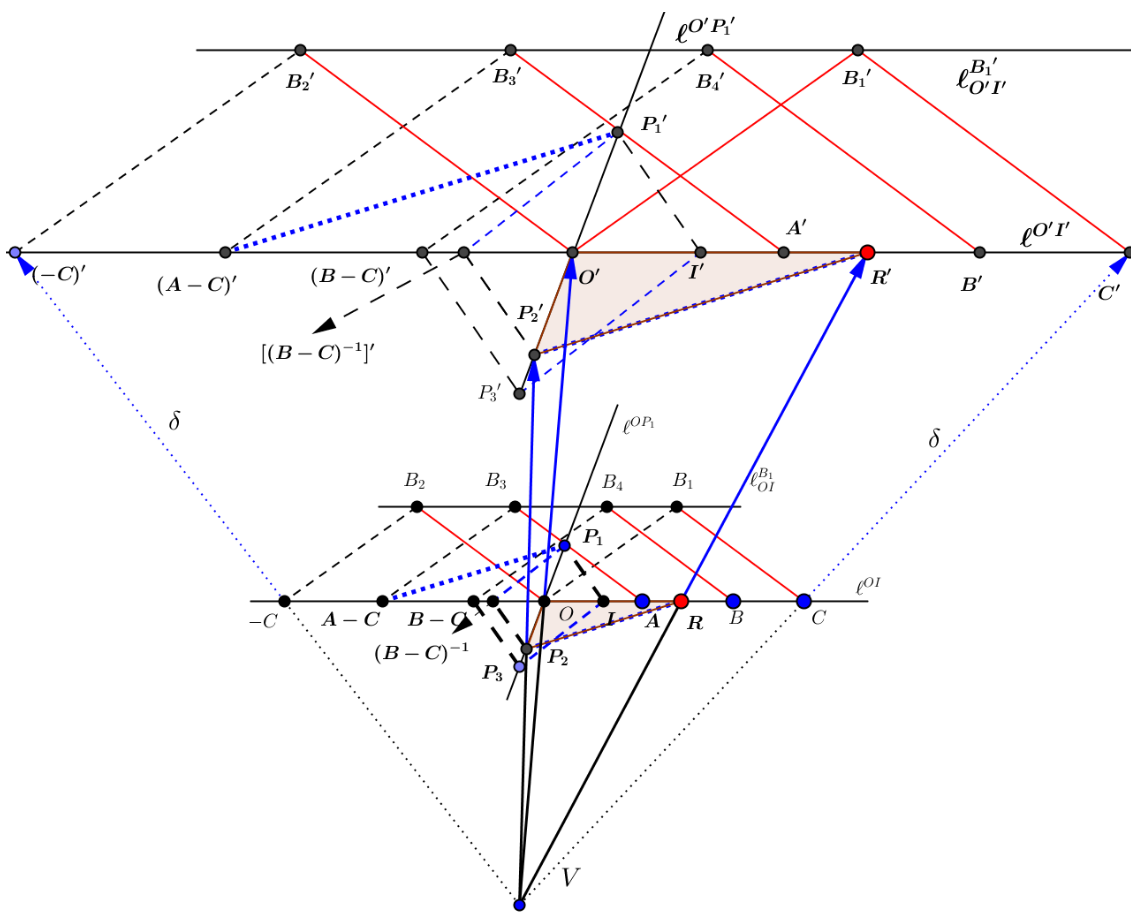

Figure 11: Ratio of 3-Points into a Dilation δ with fixed point $V \notin \ell^{OI}-$line, in Desargues affine plane.

and the points $R', \delta(R) \in \delta(\ell^{OI}) = \ell^{O'I'}$.

So the points, R' and $\delta(R)$ are the cutting points of lines $\ell^{O'I'}$ and $\ell^{P_2'}_{P_1'(A-C)'}$ (the line which passes from point P_2' and is parallel with line $\ell^{P_1'(A-C)'}$), so

$$R' = \ell^{O'I'} \cap \ell^{P_2'}_{P_1'(A-C)'} \quad \text{and} \quad \delta(R) = \ell^{O'I'} \cap \ell^{P_2'}_{P_1'(A-C)'}$$

thus, this is a single point, for this, we have that $R' = \delta(R)$. Hence

$$\delta[r(A, B; C)] = r[\delta(A), \delta(B); \delta(C)].$$

Let's now consider the case, when $\delta : \ell^{OI} \to \ell^{OI}$ (is case where the fixed point $V \in \ell^{OI}$), Fig.12.

We mark (same as in the first case) with: $O' = \delta(O)$, $I' = \delta(I)$, $A' = \delta(A)$, $B' = \delta(B)$, $C' = \delta(C)$, $\delta(R)$, all this points are in $\ell^{OI}-$line. Also, dilation δ, we apply it to auxiliary points, $B_1' = \delta(B_1)$, $B_2' = \delta(B_2)$, $B_3' = \delta(B_3)$, $B_4' = \delta(B_4)$ (from dilation properties have that, $\ell^{B_1}_{OI} \parallel \delta\left[\ell^{B_1}_{OI}\right] = \ell^{B_1'}_{OI}$), also marked $P_1' = \delta(P_1)$, $P_2' = \delta(P_2)$, $P_3' = \delta(P_3)$. With results in [15], [23], and from the properties of the construction of the inverse point, we have that,

$$\begin{aligned}
(-C)' &= \delta(-C) = -\delta(C) = -C' \\
(A-C)' &= \delta(A-C) = \delta(A) - \delta(C) = A' - C' \\
(B-C)' &= \delta(B-C) = \delta(B) - \delta(C) = B' - C' \\
[(B-C)^{-1}]' &= \delta[(B-C)^{-1}] = [\delta(B-C)]^{-1} = [\delta(B) - \delta(C)]^{-1} = [B' - C']^{-1}
\end{aligned}$$

Now construct the ratio point $R' = r(A', B'; C')$ in $\delta(\ell^{OI}) \equiv \ell^{OI}-$line. From definition of ratio of three points and the above notes, we have that,

$$R' = (B' - C')^{-1}(A' - C')$$

is ratio point in $\ell^{OI} = \delta(\ell^{OI})-$line.

If we consider, two 'three-vertex' OP_2R and $O'P_2'\delta(R)$, in Fig.11, we see that they are Desarguesian, for this reason, and for the results for construction of points $R' = (B' - C')^{-1}(A' - C')$ we have that,

$$R' = \delta(R)$$

because, $P_2R||P_1(A-C)||\delta(P_1)\delta(A-C) = P_1'(A-C)' = P_1'(A'-C')||P_2'R'$, also from dilation properties we have that

$$P_2R||\delta(P_2)\delta(R) = P_2'\delta(R),$$

and parallelism is equivalence relation, hence we have

$$P_2'R' \parallel P_2'\delta(R),$$

and we have that the points $R', \delta(R) \in \delta(\ell^{OI}) = \ell^{OI}$.

So the points, R' and $\delta(R)$ are the cutting points of lines ℓ^{OI} and $\ell^{P_2'}_{P_1'(A-C)'}$ (the line which passes from point P_2' and is parallel with line $\ell^{P_1'(A-C)'}$), so

$$R' = \ell^{O'I'} \cap \ell^{P_2'}_{P_1'(A-C)'} \quad \text{and} \quad \delta(R) = \ell^{O'I'} \cap \ell^{P_2'}_{P_1'(A-C)'}$$

thus, this is a single point, for this, we have that $R' = \delta(R)$. Hence

$$\delta[r(A, B; C)] = r[\delta(A), \delta(B); \delta(C)].$$

□

Theorems 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, which we have proved in detail above, can all be summarized in

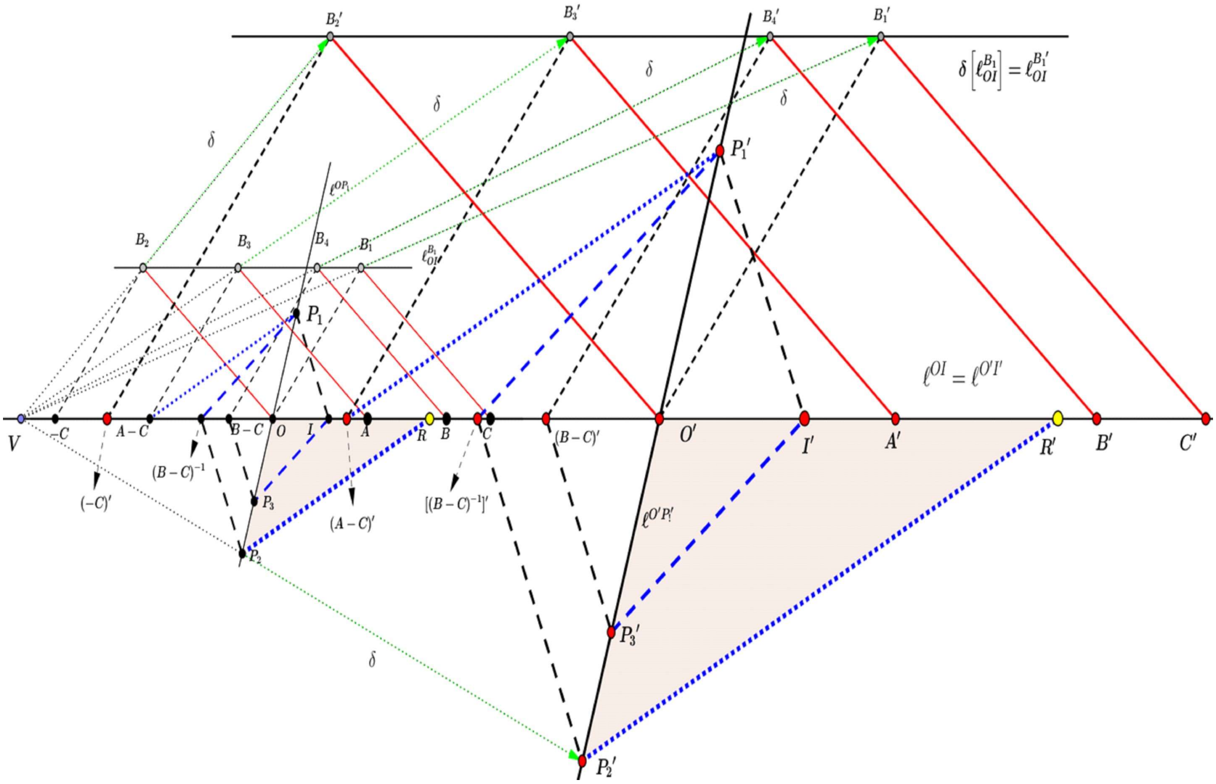

Figure 12: Ratio of 3-Points into a Dilation δ with fixed point $V \in \ell^{OI}-$line, in Desargues affine plane.

Remark 3.2 *In Desargues Affine Plane, the transforms:*

- *Parallel Projection,*
- *Translation, and*
- *Dilation,*

are ***Preserving-Transforms*** *related to the* ***ratio of 2 points*** *and* ***ratio of 3 points****, in a line on Desargues Affine Plane.*

After presenting the new idea, related to the ratio of 2 and 3 points in a line on Desargues affine plane (see [17]), in this paper we studied some transforms in Desargues affine plane, which we divided into two categories: **Invariant Transforms** and **Preserving Transforms**, this are summarized in In Remarks 2.1 and 3.2, respectively in Remark 2.1 Invariant-Transforms and in Remark 3.2 Preserving Transforms. In paper [18], we extended the understanding related to ratio-points for 4-points, also in paper [19], we studied some invariant transforms and preserving transforms for ratio of 4-points. Also a very nice interpretation is presented in papers [17], [20] related to Dyck Free Group presentation and Dyck Fundamental Group.

Declaration of Interests:

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Conflict of Interest Statement

There is no conflict of interest with any funder.

Authors' contributions

The authors' contribution is as follows: O. Zaka introduced the geometry of ratio for two and for three points in a line on Desargues affine plane, introduces invariant transforms that preserve the ratio of either two or three points collinear, discovered and proved the results in this paper, related to the transforms of ratio for two and for three points and their connections with "line-skew field" on a Desargues affine plane. J.F. Peters reviewd and refined and clarified various aspects of the presented geometry and results in this paper.

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