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On Localization of Maximal and Minimal Open Sets

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ABSTRACT: The notions of maximal and minimal open and closed sets in topological spaces are introduced and studied in global sense. So we intend here to introduce and study the concepts of maximal and minimal open sets in topological spaces in local sense after the names locally maximal and minimal open sets.

Key Words: Maximal open set, minimal open set, locally maximal open set, locally minimal open set.

Contents

1. Introduction

Suppose that X is a nonempty set and $\mathscr{T}$ is a topology on X. As per usual convention, we write X to denote the topological space $(X, \mathscr{T})$. By a proper open set (resp. closed set) of X, we mean an open set $G \neq \emptyset, X$ (resp. closed set $E \neq \emptyset, X$).

The concepts of maximal and minimal open and closed sets in a topological space X are introduced and studied by Nakaoka and Oda [5,6,7]. The first author [2] of the present paper obtained some conditions on disconnectedness of topological spaces using maximal and minimal open sets. The paraopen and paraclosed sets in X are introduced and studied by Ittanagi and Benchalli [1] as an outcome of maximal and minimal open sets in X. A modification of paraopen and paraclosed sets in X are introduced and studied after the name mean open and closed sets [4] in X. All the concepts describe above are global in nature. Hence it is quite natural to focus on study all above notions on X in local sense. It instigates us to introduce and study local concepts of maximal and minimal open sets in a topological space after names locally maximal and minimal open sets.

Throughout the paper, we write R to denote the set of real numbers.

2. Locally Maximal Open Sets

We recall some notions and results which may require to use in the sequel to make the article self-contained as far as possible.

Definition 2.1 (Nakaoka and Oda [6]) *A proper open set U of X is called a maximal open set in X if no open set other than X and U contain the open set U.*

Theorem 2.2 (Nakaoka and Oda [6]) *If U is a maximal open set and V is an open set in X then either $U \cup V = X$ or $V \subset U$. If U and V both are maximal open sets in X then either $U \cup V = X$ or $U = V$.*

Definition 2.3 (Nakaoka and Oda [7]) *A proper open set U of X is called a minimal open set in X if no open set other than $\emptyset$ and U are contained in U.*

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Theorem 2.4 (Nakaoka and Oda [7]) *If U is a minimal open set and W is an open set in X then either $U \cap W = \emptyset$ or $U \subset W$. If U and W both are minimal open sets in X then either $U \cap W = \emptyset$ or $U = W$.*

Let $\mathscr{S}$ be the usual topology on the set R of real numbers. We see that for each $a \in R$, $R - \{a\}$ is maximal open in R.

Definition 2.5 *Let U be a proper open set in X with $x \in U$. The open set U is said to be locally maximal open in X at x if $V \subset U$ whenever $V(\neq X)$ is an open set in X with $x \in V$.*

We observe that if G is locally maximal open in X at $x \in X$, there exist no proper open set $H(\neq G)$ in X with $x \in H$ and $G \subset H$.

Theorem 2.6 *If G is locally maximal open in a topological space X at $x \in X$ and H is closed in X such that $x \notin H$ then $G \cup H = X$.*

Proof. We have $x \in X - H$. As G is locally maximal open at x, $X - H \subset G$ which implies $G \cup H = X$. ∎

Corollary 2.7 *If G is locally maximal open in a topological space X at $x \in X$ and H is a closed set in X such that $G \cap H = \emptyset$, then $G = X - H$.*

Proof. Since $G \cap H = \emptyset$, $x \notin H$. So by Theorem 2.6, $G \cup H = X$. Then $G \cap H = \emptyset$ and $G \cup H = X$ together imply that $G = X - H$. ∎

Corollary 2.8 *Let G be locally maximal open in X at $x_\alpha \in X$ for each $\alpha \in \Delta$ and $\{E_\alpha \mid \alpha \in \Delta\}$ be a collection of closed sets in X such that $x_\alpha \notin E_\alpha$. Then $\bigcap_{\alpha\in\Delta} E_\alpha \neq \emptyset$ with $x_\alpha \notin \bigcap_{\alpha\in\Delta} E_\alpha$ for all $\alpha \in \Delta$.*

Proof. Since $x_\alpha \notin E_\alpha$, we have $G \cup E_\alpha = X$ for each $\alpha \in \Delta$ by Theorem 2.6. Then

$$\begin{aligned} X &= \bigcap\nolimits_{\alpha\in\Delta} (G \cup E_\alpha) \\ &= G \cup \left(\bigcap\nolimits_{\alpha\in\Delta} E_\alpha\right). \end{aligned}$$

Since $G \neq X$, we conclude that $\bigcap_{\alpha\in\Delta} E_\alpha \neq \emptyset$. Since $x_\alpha \notin E_\alpha$ for at least one $\alpha \in \Delta$, we also find that $x_\alpha \notin \bigcap_{\alpha\in\Delta} E_\alpha$ for all $\alpha \in \Delta$. ∎

Theorem 2.9 (Uniqueness Theorem) *The locally maximal open set in X at a point $x \in X$, if exists is unique.*

Proof. If possible, let U and V be two locally maximal open sets at a point $x \in X$ of a topological space X. By local maximality of U and V at $x \in X$, we have $V \subset U$ and $U \subset V$. So we get $U = V$. ∎

Theorem 2.10 *If an open set G in X is locally maximal open at a point $x \in G$ then G is maximal open in X.*

Proof. If possible, suppose that G is not maximal open in X. Then there exists an open set $V(\neq G, X)$ in X such that $G \subset V$. As $x \in G$, $x \in V$. We see that V is an open set in X with $V \neq X$ and $V \not\subset G$. It means G is not locally maximal open in X at x, a contradiction. Therefore G is maximal open in X. ∎

Example 2.11 (Steen and Seebach [8]) *Let $X = [-1, 1]$. We define*

$$\mathscr{T} = \{\emptyset\} \cup \{G \subset X \mid 0 \notin G\} \cup \{G \mid (-1, 1) \subset G\}.$$

We see that $[-1, 0) \cup (0, 1]$ is maximal open in the topological space $(X, \mathscr{T})$ and it is locally maximal open at no $x \in [-1, 0) \cup (0, 1]$. So we see that a maximal open set G in a topological space need not be locally maximal open at any point of G. It means that the converse of Theorem 2.10 need not be true.

Corollary 2.12 *If U is locally maximal open at a point in X and V is open in X then either $U \cup V = X$ or $V \subset U$. If $V(\neq U)$ is also locally maximal open at a point in X then $U \cup V = X$.*

Proof. By Theorem 2.10, U is maximal open in X. Hence the result follows by Theorem 2.2. ∎

Theorem 2.13 *Let H be maximal open in X and $\{G_\gamma \mid \gamma \in \Gamma\}$ be a collection open sets in X such that no $G_\gamma, \gamma \in \Gamma$ is a proper subset of H and $\bigcap_{\gamma\in\Gamma} G_\gamma \subset H$. Then there exists a $\gamma \in \Gamma$ such that $H = G_\gamma$.*

Proof. We see that $H = H \cup \left(\bigcap_{\gamma\in\Gamma} G_\gamma\right) = \bigcap_{\gamma\in\Gamma}(H \cup G_\gamma)$. By Theorem 2.2, two cases may arise: $H \cup G_\gamma = X$ or $G_\gamma \subset H, \gamma \in \Gamma$. If $H \cup G_\gamma = X$ for each $\gamma \in \Gamma$, then $\bigcap_{\gamma\in\Gamma}(H \cup G_\gamma) = X = H$. As H is maximal open in X, $H = X$ is not possible. So $H \cup G_\gamma \neq X$ for some $\gamma \in \Gamma$ which means $G_\gamma \subset H$ for some $\gamma \in \Gamma$. Let $G_{\gamma'} \subset H$ for $\gamma = \gamma' \in \Gamma$. As $G_{\gamma'}$ is not a proper subset of H, we conclude that $H = G_{\gamma'}$. ∎

Remark 2.14 *It follows by Theorems 2.10 and 2.2 that Theorem 2.13 also hold if H is locally maximal open at some point in X.*

Theorem 2.15 *Let U be minimal open and locally maximal open in X at each $x \in U$ then U is the only proper open set in X containing x.*

Proof. Let V be a proper open set containing x. Since U is locally maximal open at x, $V \subset U$. Again we have $U \subset V$ by Theorem 2.4 as U is minimal open in X and $U \cap V \neq \emptyset$. So we get $U = V$. Therefore U is the only proper open set in X containing x. ∎

Theorem 2.16 *If G is locally maximal open in X at each $x \in G$ then no proper open subset H, if exists of G is locally maximal open at each point $x \in H$.*

Proof. If possible, let H be locally maximal open in X at $x \in H$. Then we have $G \subset H$ as $x \in H \subset G$. Hence $G = H$. ∎

3. Locally Minimal Open Sets

The minimal open sets [5,7] in a topological space X are global in nature. We give below the notion of minimality of open sets on topological spaces in local sense.

Definition 3.1 *Let U be a proper open set in X with $x \in U$. The open set U is said to be locally minimal open in X at x if $x \notin V$ whenever $V(\neq \emptyset, U)$ is an open set in X such that $V \subset U$.*

It is easy to see that a minimal open set U in a topological space X is locally minimal open in X at each point $x \in U$.

Example 3.2 *Let $X = [-1, 1]$ and $\mathscr{T} = \{\emptyset\} \cup \{G \mid (-1, 1) \subset G\}$. In the topological space $(X, \mathscr{T})$, $(-1, 1]$ is not minimal open in X but it is locally minimal open at $1 \in X$.*

So we see that a locally minimal open set in X at a point $x \in X$ need not be a minimal open set in X.

In Example 3.2, the open set $(-1, 1)$ is not locally maximal open at any $x \in (-1, 1)$ but it is locally minimal open at each $x \in (-1, 1)$.

Example 3.3 (Mukharjee and Bagchi [3]) *Let $a, b \in R$ with $a < b$ and $\mathscr{T}$ be the topology on R generated by the base $\{\emptyset, (-\infty, a), (a, \infty), (-\infty, b), [b, \infty), \{b\}, (a, b)\}$. In the topological space $(R, \mathscr{T})$, $R - \{a\}$ is locally maximal open in R at any $c \in R$ with $c > b$ but $R - \{a\}$ is not locally minimal open in R at $c \in R$.*

So we conclude that locally maximal open and locally minimal open sets in a topological space are independent concepts.

Theorem 3.4 (Uniqueness Theorem) *The locally minimal open set in X at a point $x \in X$, if exists is unique.*

Proof. If possible, suppose that there exist two distinct locally minimal open sets G and H at a point $x \in X$. By the local minimality of G and H at x, $G \not\subset H$ as well as $H \not\subset G$. So $G \cap H \neq G, H$. Also we have $x \in G \cap H \subsetneq G$, which contradicts the fact that G is locally minimal open at $x \in X$. For the same reason, $x \in G \cap H \subsetneq H$ is not possible. It means that our assumption is wrong, i.e., locally minimal open set at a point $x \in X$ is unique. ■

Theorem 3.5 *If an open set U in X is locally minimal open at each point $x \in U$, then U is minimal open in X.*

Proof. If possible, suppose that U is not minimal open in X. Then there exist an open set $V(\neq \emptyset, U)$ in X such that $V \subset U$. As $V \neq \emptyset$, there exists $x \in X$ such that $x \in V \subset U$. It means that U is not locally minimal open at x, a contradiction to the hypothesis that U is locally minimal open at $x \in U$. ■

Corollary 3.6 *If U is locally minimal open at each point in U and V is an open set in X then either $U \cap V = \emptyset$ or $U \subset V$. If $V(\neq U)$ is also locally minimal open at each point in V then $U \cap V = \emptyset$.*

Proof. By Theorem 3.5, U is minimal open in X. Then result follows by Theorem 2.4. ■

Theorem 3.7 *Let H be minimal open in X and $\{G_\gamma \mid \gamma \in \Gamma\}$ be a collection open sets in X such that no $G_\gamma, \gamma \in \Gamma$ is a proper superset of H and $H \subset \bigcup_{\gamma\in\Gamma} G_\gamma$. Then there exists a $\gamma \in \Gamma$ such that $H = G_\gamma$.*

Proof. We see that $H = H \cap \left(\bigcup_{\gamma\in\Gamma} G_\gamma\right) = \bigcup_{\gamma\in\Gamma}(H \cap G_\gamma)$. By Theorem 2.4, we may have either $H \cap G_\gamma = \emptyset$ or $H \subset G_\gamma, \gamma \in \Gamma$. If $H \cap G_\gamma = \emptyset$ for each $\gamma \in \Gamma$, then $\bigcup_{\gamma\in\Gamma}(H \cap G_\gamma) = \emptyset = H$. As H is minimal open in X, $H = \emptyset$ is not possible. So $H \cap G_\gamma \neq \emptyset$ for some $\gamma \in \Gamma$. Let $H \cap G_{\gamma'} \neq \emptyset$ for $\gamma = \gamma' \in \Gamma$. It implies that $H \subset G_{\gamma'}$. As $G_{\gamma'}$ is not a proper super set of H, we conclude that $H = G_{\gamma'}$. ■

Remark 3.8 *By Theorems 3.5 and 2.4, we find that Corollary 3.7 also hold if H is locally minimal open at each point in H.*

Theorem 3.9 *Let U be a minimal open set in X with $x \in U$ and H be a proper open set with $x \in H$ and H is not locally minimal open in X at x then there exists an open set G in X such that $x \in U \subset G \subset H$.*

Proof. Since H is not locally minimal open in X at x, there exists an open set G such that $x \in G \subset H$. As U is minimal open in X, we have (i) $G \cap U = \emptyset$ or $U \subset G$ and (ii) $H \cap U = \emptyset$ or $U \subset H$ by Theorem 2.4. We see that $G \cap U = \emptyset$ and $H \cap U = \emptyset$ are absurd as $x \in G, H, U$. So only feasible result is $x \in U \subset G \subset H$. ■

Theorem 3.10 *Let U be a maximal open set and locally minimal open in X at each $x \in U$. Then $U \cup V = X$ for any open set $V(\neq U)$ in X with $x \in V$.*

Proof. As U is maximal open and V is open in X, we get $U \cup V = X$ or $V \subset U$ by Theorem 2.2. Since U is locally minimal open in X at x and $x \in V$, $V \not\subset U$. So we get $U \cup V = X$. ■

Theorem 3.11 *The union of two distinct open sets U, V which are locally minimal open in a topological space X at each of their points is not locally minimal open in X at any $x \in U \cup V$.*

Proof. We have nothing to prove if $U \cup V = X$. So we suppose that $U \cup V \neq X$. Since U, V are locally minimal open in X at each of their points, U, V are minimal open in X by Theorem 3.5. Hence we get $U \cap V = \emptyset$ by Theorem 2.4. Let $x \in U \cup V$. Then either $x \in U$ or $x \in V$. Let $x \in U$. Then $x \notin V$ as $U \cap V = \emptyset$. So $U \neq U \cup V$. Therefore $U \neq \emptyset, U \cup V$ is an open set such that $x \in U \subsetneq U \cup V$. So $U \cup V$ is not locally minimal open in X at x. Similarly, we can prove that $U \cup V$ is not locally minimal open in X at x if $x \in V$. Hence $U \cup V$ is not locally minimal open in X at each of its points. ■

Theorem 3.12 *Let X be a Hausdorff topological space and U be a proper open set in X with $|U| > 1$ then U is not locally minimal open set at each point $x \in U$.*

Proof. Let $x \in U$. Since $|U| > 1$, there exists $y \in U$ such that $x \neq y$. As X is Hausdorff, we have two open sets W_x and W_y containing x and y respectively such that $W_x \cap W_y = \emptyset$. We see that $x \in W_x \cap U \subset U$. If $W_x \cap U = U$ then $U \subset W_x$ which implies $y \in U \subset W_x$ which is not possible as $W_x \cap W_y = \emptyset$. So $W_x \cap U \neq \emptyset, U$ is an open set in X such that $x \in W_x \cap U \subsetneq U$ which implies that U is not locally minimal open in X at x. ■

Remark 3.13 Like the notions of locally maximal and minimal open sets in topological spaces, we may have the notions of locally maximal and minimal closed sets in topological spaces. Let E be a proper closed set in a topological space X with $x \in E$. The closed set E in X is said to be locally maximal closed at $x \in X$ if $F \subset E$ whenever $F(\neq X)$ is a closed set in X with $x \in F$. The closed set E in X is said to be locally minimal closed at $x \in X$ if $x \notin F$ whenever $F(\neq \emptyset, E)$ is a closed set in X such that $F \subset E$. The results on locally maximal and minimal closed sets on a topological space X may be obtain on dualization of results on locally maximal and minimal open sets respectively on X.

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