Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–6.
ISSN-0037-8712
doi:10.5269/bspm.68851

A Matrix Summability Factor Theorem Involving Almost Increasing Sequences

H. NEDRET ÖZGEN

ABSTRACT: In [9], we have already proved a main theorem dealing with the $\mid A \mid_k$ summability factors of infinite series. In this paper, we have generalized this theorem to the $\mid A, \delta \mid_k$ summability methods under some suitable conditions.

Key Words: Summability factors, absolute matrix summability, infinite series, Minkowski inequality, Hölder inequality.

Contents

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exist a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \ \ n = 0, 1, \dots \ . \tag{1.1}$$

The series $\sum a_n$ is said to be summable $\mid A \mid_k$, $k \geq 1$, if (see [12])

$$\sum_{n=1}^{\infty} n^{k-1} \mid A_n(s) - A_{n-1}(s) \mid^k < \infty. \tag{1.2}$$

The series $\sum a_n$ is said to be summable $\mid A, \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} \mid A_n(s) - A_{n-1}(s) \mid^k < \infty. \tag{1.3}$$

In the special case for $\delta = 0$, the $\mid A, \delta \mid_k$ summability reduces to $\mid A \mid_k$ summability. If we take $a_{nv} = \frac{p_v}{P_n}$, then we have $\mid R, p_n; \delta \mid_k$ summability. Also if we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then we have $\mid R, p_n \mid_k$ summability (see [2]).

AMS Mathematics Subject Classification. 26D15, 40D15, 40F05, 40G99
Keywords and phrases. Summability factors, absolute matrix summability, infinite series, Minkowski's inequality, Hölder's inequality.

Let $A = (a_{nv})$ be a normal matrix, we associate two lower semi-matrices $\overline{A} = (\overline{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, n, v = 0, 1, \dots \tag{1.4}$$

2020 *Mathematics Subject Classification:* 26D15, 40D15, 40F05, 40G99.
Submitted July 07, 2023. Published December 05, 2025

and

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00}, \widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, n = 1, 2, ... \; . \tag{1.5}$$

$\overline{A}$ and $\widehat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \overline{a}_{nv} a_v \tag{1.6}$$

and

$$\overline{\Delta} A_n(s) = \sum_{v=0}^{n} \widehat{a}_{nv} a_v. \tag{1.7}$$

$A = (a_{nv})$ is said to be of class Ω if (see [9]) the following hold;
A is lower triangular

$$a_{nv} \geq 0, \quad n, v = 0, 1, ...; \tag{1.8}$$

$$a_{n-1,v} \geq a_{nv}, \quad for\ n \geq v + 1, \tag{1.9}$$

$$\overline{a}_{n0} = 1, \quad n = 0, 1, ... \quad . \tag{1.10}$$

A given by

$$A_1(x) = x_1 \quad and\ A_n(x) = \frac{x_{n-1} + x_n}{2} \quad for\ n > 1 \tag{1.11}$$

is an example of a matrix of class Ω.

2. Known result

Recently many papers have been done for absolute summability factors of infinite series ([4]- [11]). Among them, in [9], we have proved the following theorem.

Theorem 2.1 *Let $\sum a_n$ be a given infinite series with partial sums (s_n) and let $A \in \Omega$ satisfying*

$$na_{nn} = O(1), \tag{2.1}$$

$$\widehat{a}_{n,v+1} = O(v \mid \Delta_v \widehat{a}_{nv} \mid) \tag{2.2}$$

and let there be sequences $(\beta_n), (\lambda_n)$ and positive non-decreasing sequence (X_n), such the following conditions hold:

$$\mid \lambda_n \mid X_n = O(1) \quad as\ n \to \infty, \tag{2.3}$$

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{2.4}$$

$$\beta_n \to 0 \quad as\ n \to \infty, \tag{2.5}$$

$$\sum_{n=1}^{m} n \mid \Delta\beta_n \mid X_n < \infty \quad as\ m \to \infty, \tag{2.6}$$

$$\sum_{n=1}^{m} a_{nn} \mid s_n \mid^k = O(X_m) \quad as\ m \to \infty. \tag{2.7}$$

Then the series $\sum a_n \lambda_n$ is summable $\mid A \mid_k, k \geq 1$.

3. Main result

The purpose of this paper is to prove the following theorem dealing with $\mid A, \delta \mid_k$ summability.

Theorem 3.1 *Let A be of class Ω such that*

$$\sum_{n=v+1}^{m+1} n^{\delta k} \mid \Delta_v \widehat{a}_{nv} \mid = O(v^{\delta k} a_{vv}) \quad as\ m \to \infty. \tag{3.1}$$

$$\sum_{v=1}^{m} v^{\delta k} a_{vv} \mid s_v \mid^k = O(X_m) \quad as\ m \to \infty, \tag{3.2}$$

where (X_n) is an almost increasing sequence. If the conditions (2.1)-(2.6) of Theorem 2.1 are satisfied, then the series $\sum a_n \lambda_n$ is summable $\mid A, \delta \mid_k, k \geq 1$ and $0 \leq \delta < 1/k$.

Lemma 3.2 *([11]) Let $A \in \Omega$. Then*

$$\sum_{v=1}^{n-1} \mid \Delta_v \widehat{a}_{nv} \mid \leq a_{nn},$$

$$\sum_{n=v+1}^{m+1} \mid \Delta_v \widehat{a}_{nv} \mid \leq a_{vv},$$

and

$$\sum_{n=v+1}^{m+1} \widehat{a}_{n,v+1} \leq 1$$

Remark 3.1 It should be noted that if we take (X_n) as a positive non-decreasing sequence, $\delta = 0$, then we get Theorem 2.1. In this case, condition (3.2) reduces to conditions (2.7). Also, noticed that the condition (3.1) is automatically satisfied.

Lemma 3.3 *([3]) Under the conditions of Theorem 2.1, we have the following*

$$n\beta_n X_n = O(1)\ as\ n \to \infty,$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

Proof of Theorem 3.1. Let (I_n) denotes A-transform of the series $\sum a_n \lambda_n$. Then, we have, by (1.6) and (1.7),

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \widehat{a}_{nv} a_v \lambda_v$$

Applying Abel's transformation, we have that

$$\begin{aligned} \bar{\Delta} I_n &= \sum_{v=1}^{n} \widehat{a}_{nv} a_v \lambda_v = \sum_{v=1}^{n-1} \Delta_v (\widehat{a}_{nv} \lambda_v) s_v + \widehat{a}_{nn} \lambda_n s_n \\ &= \sum_{v=1}^{n-1} \Delta_v (\widehat{a}_{nv}) \lambda_v s_v + \sum_{v=1}^{n-1} \widehat{a}_{n,v+1} s_v \Delta \lambda_v + a_{nn} \lambda_n s_n \\ &= I_{n,1} + I_{n,2} + I_{n,3}. \end{aligned}$$

Since

$$| I_{n,1} + I_{n,2} + I_{n,3} |^k \leq 3^k (| I_{n,1} |^k + | I_{n,2} |^k + | I_{n,3} |^k),$$

to complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} | I_{n,r} |^k < \infty \; for \; r = 1, 2, 3.$$

First, applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k+k-1} | I_{n,1} |^k &\leq \sum_{n=2}^{m+1} n^{\delta k+k-1} \left\{ \sum_{v=1}^{n-1} | \Delta_v(\widehat{a}_{nv}) || \lambda_v || s_v | \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} | \Delta_v(\widehat{a}_{nv}) || \lambda_v |^k | s_v |^k \right) \\
&\times \left(\sum_{v=1}^{n-1} | \Delta_v(\widehat{a}_{nv}) | \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} | \Delta_v(\widehat{a}_{nv}) || \lambda_v |^k | s_v |^k \right) \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k} \left(\sum_{v=1}^{n-1} | \Delta_v(\widehat{a}_{nv}) || \lambda_v |^k | s_v |^k \right) \\
&= O(1) \sum_{v=1}^{m} | \lambda_v |^{k-1} | \lambda_v || s_v |^k \sum_{n=v+1}^{m+1} n^{\delta k} | \Delta_v(\widehat{a}_{nv}) | \\
&= O(1) \sum_{v=1}^{m} v^{\delta k} a_{vv} | \lambda_v || s_v |^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta | \lambda_v | \sum_{r=1}^{v} r^{\delta k} a_{rr} | s_r |^k \\
&+ O(1) | \lambda_m | \sum_{v=1}^{m} v^{\delta k} a_{vv} | s_v |^k \\
&= O(1) \sum_{v=1}^{m} | \Delta\lambda_v | X_v + O(1) | \lambda_m | X_m \\
&= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) | \lambda_m | X_m \\
&= O(1) \; as \; m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.4.

Applying Hölder's inequality with the same indices above, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k+k-1} \mid I_{n,2} \mid^k &\leq \sum_{n=2}^{m+1} n^{\delta k+k-1} \left\{ \sum_{v=1}^{n-1} \mid \widehat{a}_{n,v+1} \mid\mid \Delta\lambda_v \mid\mid s_v \mid \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} \widehat{a}_{n,v+1}\beta_v \mid s_v \mid \right)^k \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} \widehat{a}_{n,v+1}\beta_v \mid s_v \mid^k \right) \\
&\times \left(\sum_{v=1}^{n-1} \widehat{a}_{n,v+1}\beta_v \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} v \mid \Delta_v \widehat{a}_{nv} \mid \beta_v \mid s_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k} \sum_{v=1}^{n-1} v \mid \Delta_v \widehat{a}_{nv} \mid \beta_v \mid s_v \mid^k \\
&= O(1) \sum_{v=1}^{m} v\beta_v \mid s_v \mid^k \sum_{n=v+1}^{m+1} n^{\delta k} \mid \Delta_v \widehat{a}_{nv} \mid \\
&= O(1) \sum_{v=1}^{m} v^{\delta k} a_{vv} v\beta_v \mid s_v \mid^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} r^{\delta k} a_{rr} \mid s_r \mid^k \\
&+ O(1) m\beta_m \sum_{v=1}^{m} v^{\delta k} a_{vv} \mid s_v \mid^k \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + \sum_{v=1}^{m-1} \beta_{v+1} X_{v+1} + O(1) m\beta_m X_m \\
&= O(1) \ as \ m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.4.

Finally, by the similar process in $I_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\delta k+k-1} \mid I_{n,3} \mid^k &= \sum_{n=1}^{m} n^{\delta k+k-1} \mid a_{nn}\lambda_n s_n \mid^k \\
&= \sum_{n=1}^{m} n^{\delta k+k-1} a_{nn}^k \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid\mid s_n \mid^k \\
&= O(1) \sum_{n=1}^{m} n^{\delta k} a_{nn} \mid \lambda_n \mid\mid s_n \mid^k = O(1) \ as \ m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.4.
So we get

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \mid I_{n,r} \mid^k < \infty, \ for \ r = 1, 2, 3.$$

This completes the proof of Theorem 3.1.

Corollary 3.4 *Let* $\sum a_n$ *be a given infinite series with partial sums* (s_n) *and let* (p_n) *be a sequence of positive real constants such that*

$$P_n = \sum_{v=0}^{n} p_v \to \infty,$$

$$np_n = O(P_n), \tag{3.3}$$

$$P_n = O(np_n), \tag{3.4}$$

and

$$\sum_{n=v+1}^{m+1} n^{\delta k} \mid \frac{p_n}{P_n P_{n-1}} \mid = O\left(\frac{v^{\delta k}}{P_v}\right). \tag{3.5}$$

Let (X_n) *be an almost increasing sequence, satisfying conditions (2.3)-(2.6) of Theorem 2.1. If*

$$\sum_{n=1}^{m} n^{\delta k} \frac{p_n}{P_n} \mid s_n \mid^k = O(X_m) \text{ as } m \to \infty, \tag{3.6}$$

then the series $\sum a_n \lambda_n$ *is summable* $\mid R, p_n; \delta \mid_k, k \geq 1$ *and* $0 \leq \delta < 1/k$.

Proof: Conditions (2.1) and (2.2) of Theorem 2.1 are automatically satisfied for any weighted mean method. Condition (3.1) and (3.2) of Theorem 3.1 becomes condition (3.5) and (3.6) of Corollary 3.5. □

Disclosure statement. The author declares that there is no conflict of interest regarding the publication of this paper.

References

1. N. K. Bari, S. B. Stečkin, *Best approximation and differential properties of two conjugate functions, Tr. Mosk. Mat. Obŝ.*, **5** *(1956), pp. 483-522 (in Russian).*
2. *H. Bor, On the relative strength of two absolute summability methods, Proc. Amer. Math. Soc.,* **113** *(1991) 1009–1012.*
3. *K. N. Mishra, R. S. L. Strivastava, On* $\mid \bar{N}, p_n \mid$ *summability factors of infinite series, Indian J. Pure Appl. Math.,* **15** *(1984), 651–656.*
4. *H. S. Özarslan, On almost increasing sequences and its applications, Int. J. Math. Math. Sci.,* **25** *(5) (2001), 293-298.*
5. *H. S. Özarslan, A note on generalized absolute Riesz summability, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.),* **60** *(1) (2014), 51-56.*
6. *H. S. Özarslan, Generalized almost increasing sequences, Lobachevskii J. Math.,* **42** *(2021), 167-172.*
7. *H. S. Özarslan, B. Kartal, Absolute matrix summability via almost increasing sequence, Quaest. Math.* **43** *(10) (2020), 1477–1485.*
8. *H. S. Özarslan and H. N. Özgen, Necessary conditions for absolute matrix summability methods, Boll. Unione Mat. Ital.,* **8** *(3) (2015), 223-228.*
9. *H. N. Özgen, A note on generalized absolute summability, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.),* **59** *(1) (2013), 185-190.*
10. *H. N. Özgen, On two absolute matrix summability methods, Boll. Unione Mat. Ital.,* **9** *(2016), 391-397.*
11. *B. E. Rhoades, Inclusion theorems for absolute matrix summability methods, J. Math. Anal. Appl.,* **238** *(1999), 82-90.*
12. *N. Tanovic-Miller, On strong summability, Glas. Mat. Ser. III,* **14** *(1979), 87-97.*

H. NEDRET ÖZGEN

Department of Mathematics and Science Education, Faculty of Education, Mersin University, TR-33169 Mersin, Turkey

E-mail address: nogduk@gmail.com