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On the stability of impulsive difference equations with variable coefficients *

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ABSTRACT: In this paper, we investigate the stability characteristics of non-autonomous impulsive difference systems, focusing on Ulam-Hyers, generalized Ulam-Hyers, and uniform exponential stabilities. We provide a comprehensive analysis of these stability concepts, establishing necessary conditions and results that extend existing theoretical frameworks. Additionally, we illustrate our findings through a series of examples that demonstrate the applicability and relevance of the proposed stability criteria. Our results contribute to the understanding of stability in impulsive systems, with potential implications for various applications in mathematical modeling and control theory.

Key Words: Non-autonomous Discrete Problems; Hyers-Ulam Stability, Exponential Stability.

Contents

1. Introduction

The concept of Ulam-Hyers stability was started in 1940, when Ulam put a question concerning the stability of a group homomorphism, before the mathematicians of that era, [1]. The positive answer to the question was partially given by Hyers [2] in 1941, in the context of Banach spaces for the additive mappings. The stability was then given the name Ulam-Hyers stability. In 1978, Rassias [3], extends the Hyers's stability in fact he genalized the Hyers-Ulam stability by replacing the error term by a continuous function. He was the first to prove the stability of the linear mapping. This result of Rassias attracted several mathematicians worldwide who began to be stimulated to investigate the stability problems of functional equations. The work done by Rassias is then known as the Ulam-Hyers-Rassias stability and it became an important part of mathematical analysis.

In many fields of sciences various schemes and models are explained by using differential system, [4,5,6,7]. But, the situation in some modeled phenomenon become different due to sudden changes in its states such as dynamical system with impacts, population dynamics, theoretical physics, biotechnology processes, pharmaceutics, chemistry, biological systems such as beating of heart, flowing blood, mathematical economy, medicine engineering, control theory and so on are different, that models are described by differential impulsive system because there is a quick change in their states. Their impulsive conditions are the combinations of short-term perturbations and conventional initial value problems whose duration can be small in comparing with the duration of the technique. Impulsive differential system have a lot of research article, like Samoilenko *et al.* [8], Lakshmikantham *et al.* [9], and Benchohra *et al.* [10].
The concept of differential impulsive system is important and a new branch of dynamical system. Initially this idea was presented by A. D. Mishkis and V. D. Mil'man [11]. Impulsive differential equation have three components:

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(i) Continuous-time differential equation, which is assume in the states of the system between impulses.
(ii) An impulse equation, described models of an impulsive jump define by a jump function.
(iii) A jump criterion: that discusses about jump events (see [12]).
Though the idea of continuous-time impulsive systems has developed quickly, see for example [13,14,15,16] and the references there in, but the corresponding concept developed very slowly. Anyhow, in many fields, such as control theory, numerical analysis, computer science and finite mathematics, we make the system in discrete form, see [17,18,19,20,21,22,23], and the references there in. In general, the analysis of stability about time-continuous systems is not valuable to the discrete-time systems. So, it is important to discuss the detailed analysis for discrete-time system in separate form. For more details of impulsive difference equations and its applications we refer to [24,25,26,27] and the references there in. For recent such type of stability of delay difference system we refer to, [31,29,30]. In this paper we present the Ulam-Hyers, Ulam-Hyers-Rassias and uniform exponential stabilities of the following impulsive difference equation,

$$\begin{cases} X_{n+1} = A_n X_n + F(n, X_n), & n \geq 0, \\ X_0 = b, & n = 0, \\ \Delta X_{n_k} = I_k(X_{n_k-1}), & k \in I, \end{cases} \tag{1.1}$$

where, the constant matrices $A_n \in \mathbb{R}^{m\times m}$, $f \in \mathbb{C}(\mathbb{Z}_+ \times \mathbb{X}, \mathbb{X})$ and $X_n \in B(\mathbb{Z}_+, \mathbb{X})$, $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and $\mathbb{X} = \mathbb{R}^m$, $I = \{0, 1, 2, \cdots, r\}$.

2. Notations, basic definitions and preliminaries

In this section we will discus some mathematical notation and basic definitions. Throughuout the paper we will denote by $\mathbb{R}^n$ the n-dimensional Euclidean space along with vector norm $\|\cdot\|$ and the space of $n \times n$ real valued matrices will be denoted by $\mathbb{R}^{n\times n}$, $\mathbb{C}(\mathbf{I}, \mathbf{X})$ will be the space of all convergent sequences with norm $\|v\|_C = \sup_{n\in\mathbf{I}} \|v_n\|$.

Definition 2.1 *A collection of bounded linear operators $\mathcal{P} = \{P(n, m) : n \geq m \geq 0\}$ is called an evolution family if it satisfy the following two conditions:*
(i) $P(n, n) = I$, and (ii) $P(n, r)P(r, m) = P(n, m)$, for each $n \geq r \geq m \geq 0$.

In the soultions of our systems we can take the evolution family as give below:

$$P(n, n) := \begin{cases} I, & \text{for all } n \in \mathbb{Z}_+; \\ X_{n-1}X_{n-2}X_{n-3}...X_m, & \text{for all } n \geq m \geq 0. \end{cases}$$

Lemma 2.1 *The impulsive difference system (1.1) has the solution*

$$X_n = P(n, 0)b + \sum_{i=1}^{n} P(n, i)F(i-1, X_{i-1}) + \sum_{k=0}^{r} P(n, n_k)I_k(X_{n_k-1}), \quad n \in I.$$

Lemma 2.2 *The impulsive difference system (1.1) has an approximate solution*

$$X'_n = P(n, 0)b + \sum_{i=1}^{n} P(n, i)F(i-1, X'_{i-1}) + \sum_{i=1}^{n} P(n, i)f_{i-1} + \sum_{n_k=0}^{n} P(n, n_k)I_k(X'_{n_k-1}) + \sum_{k=0}^{r} P(n, n_k)f_{n_k},$$

where $\|f_n\| \leq \epsilon$.

Consider the following inequalities:

$$\begin{cases} \|\mathbf{X'}_{n+1} - A_n\mathbf{X'}_n - F(n.\psi_n)\| \leq \epsilon, & n \geq 0, \\ \|\Delta(X'_{n_k}) - I_k(X'_{n_k-1})\| \leq \epsilon, & k \in I. \end{cases} \tag{2.1}$$

$$\begin{cases} \|\mathbf{X'}_{n+1} - A_n\mathbf{X'}_n - F(n.\psi_n)\| \leq \varphi_n, & n \geq 0, \\ \|\Delta(X'_{n_k}) - I_k(X'_{n_k-1})\| \leq \psi, & k \in I. \end{cases} \tag{2.2}$$

$$\begin{cases} \|\mathbf{X'}_{n+1} - A_n\mathbf{X'}_n - F(n.\psi_n)\| \leq \epsilon\varphi_n, & n \geq 0, \\ \|\Delta(X'_{n_k}) - I_k(X'_{n_k-1})\| \leq \epsilon\psi, & k \in I. \end{cases} \tag{2.3}$$

Definition 2.2 *The equation (1.1) is uniformly expoenentially stable if there are real numbers $L > 0$ and $\nu > 0$ such $\|X_n\| \leq Le^{-\nu n}$, for all $n \geq 0$.*

Definition 2.3 *The equation (1.1) is Hyers-Ulam stable if there is a real number $L > 0$ such that for every X' satisfying (2.1) there is a solution X of (1.1) such that*
$\|X'_n - X'_n\| \leq L\epsilon$, for all $n \geq 0$.

Definition 2.4 *The equation (1.1) is generelaized Hyers-Ulam stable if there is a bounded sequence $\phi_n > 0$ such that for every X' satisfying (2.2) there is a solution X of (1.1) such that*
$\|X'_n - X'_n\| \leq \phi_n\epsilon$, for all $n \geq 0$.

Definition 2.5 *The system (1.1) is said to be Hyers-Ulam-Rassias stable if for every solution of (2.1), there exists a solution $\mathbf{X}_n$ of (1.1) and a non-negative real number $\mathcal{C}_{\rho,M_1,\varphi}$ such that*

$$\|\mathbf{X}_n - \mathbf{X'}_n\| \leq \mathcal{C}_{\rho,M_1,\varphi}\epsilon(\varphi_n + \psi), \quad for \quad all \quad n \in I.$$

Definition 2.6 *The system (1.1) is said to have a generalized Hyers-Ulam-Rassias stability if for every solution of (2.3), there is a solution $\mathbf{X}_n$ of (1.1) and a non-negative real scalar $\mathcal{C}_{N,\Psi,\varphi}$ such that $\|\mathbf{X}_n - \mathbf{X'}_n\| \leq \mathcal{C}_{N,\Psi,\varphi}(\varphi_n + \psi)$, for all $n \in I$.*

Lemma 2.3 *(see [31] lemma 2) If z_n and g_n are nonnegative sequences and $a \geq 0$, which satisfies the inequality*

$$\|z_n\| \leq a + \sum_{i=0}^{n} \|g_i\|\|z_i\|, \quad n \geq 0,$$

then

$$\|z_n\| \leq a \exp\left(\sum_{i=0}^{n} \|z_i\|\right)$$

Lemma 2.4 *Let $< Y_n >$ and $< G_n >$ be nonnegative sequences and c a nonnegative constant. If*

$$Y_n \leq c + \sum_{0 \leq k < n} G_k Y_k \quad for \quad n \geq 0,$$

then

$$Y_n \leq c \prod_{j=0}^{n} (1 + G_j) \leq c \exp \sum_{0 \leq j < n} G_j \quad for \quad n \geq 0,$$

(see Proposition: special Grownwall Inequality [31]).

Remark 2.1 *[32] A semigroup is always an evolution family but every evolution need not be a semigroup in generel.*

Remark 2.2 *[32] An evolution family is a semigroups if it is periodic of period one.*

3. Uniqueness and Existence of solutions

To described the uniqueness and existence of the solution of the system (1.1) we will put the following conditions:
G_1 : The system $\mathbf{X}_{n+1} = A_n\mathbf{X}_n$ is well posed.
G_2 : $F : I \times X \to X$ and $I_k : X \to X$, $k \in I$, are such that there exist constants $L > 0$ and $h > 0$, with

$$\|F(n,\zeta) - F(n,\eta)\| \leq L\|\zeta - \eta\|,$$

$$\|I_k(\mathrm{w}) - I_k(\mathrm{w}')\| \leq h\|\mathrm{w} - \mathrm{w}'\|.$$

G_3 : There is a positive number q such that $\|P(n,m)\| \leq Me^{-q(n-m)} \quad for \quad all \quad n \geq m.$
G_4 : $\frac{Me^q(L+h)}{1+e^q} < 1$.

Theorem 3.1 *If assumptions G_1, G_2, G_3 and G_4 are holds, then the system (1.1) has a unique solution.*

Proof: Define $\mathcal{A} : \mathbb{B}(\mathbf{I}, \mathbb{X}) \to \mathbb{B}(\mathbf{I}, \mathbf{X})$ by

$$\mathcal{A}X_n = P(n,0)b + \sum_{i=1}^{n} P(n,i)F(i-1, X_{i-1}) + \sum_{i=0}^{n} K(n,i)h_i + \sum_{k=0}^{r} P(n,n_k)I_k(X_{n_k-1}).$$

Now, for $X, X' \in \mathrm{B}(\mathbf{I}, \mathbf{X})$, we have

$$\begin{aligned}\|\mathcal{A}X_n - \mathcal{A}X'_n\| &\le \sum_{i=1}^{n} \|P(n,i)\| \|(F(i-1, X_{i-1}) - F(i-1, X'_{i-1}))\| + \\ \sum_{k=0}^{r} &\|P(n,n_k)\| \|(I_k(X_{nk-1}) - I_k(X'_{nk-1}))\| \\ &\le \sum_{i=1}^{n} Me^{-q(n-i)} L \|X_{i-1} - X'_{i-1}\| + \sum_{k=0}^{r} hMe^{-q(n-n_k)} \|X_{nk-1} - X'_{nk-1}\|\end{aligned} \tag{3.1}$$

Taking supremum over n we have,

$$\begin{aligned}\|\mathcal{A}X - \mathcal{A}X'\| &\le \sum_{i=1}^{n} Me^{-q(n-i)} L \|X - X'\| + \sum_{k=0}^{r} hMe^{-q(n-n_k)} \|X - X'\| \\ &\le \frac{Me^q(L+h)}{1+e^q} \|X - X'\|.\end{aligned} \tag{3.2}$$

Hence, by G_4 the opertaor $\mathcal{A}$ become a contraction, using the Banach contraction principle the operator $\mathcal{A}$ has a unique fixed point which is in fact the unique solution of the system (1.1). □

4. Main Results

In this section we will discuss the Hyers-Ulam, generalized Ulam-Hyers and unifrom exponential stablities of our system.

Theorem 4.1 *System (1.1) is Ulam-Hyers stable, if the assumptions G_1 to G_4 are satisfied.*

Proof: Using Lemmas 2.1 and 2.2 we have,

$$\begin{aligned}\|\psi_n - \psi'_n\| &= \Bigg\| \sum_{i=1}^{n} P(n,i)(F(i-1,\psi_{i-1}) - F(i-1,\psi'_{i-1})) + \sum_{k=0}^{r} P(n,n_k)(I_k(\psi_{n_k-1}) - I_k(\psi'_{n_k-1})) \\ &+ \sum_{i=1}^{n} P(n,i) f_{i-1} + \sum_{k=0}^{r} P(n,n_k) f_{n_k} \Bigg\| \\ &\le \sum_{i=1}^{n} Me^{-V(n-i)} L \|\psi_{i-1} - \psi'_{i-1}\| + \sum_{k=0}^{r} Me^{-q(n-n_k)} h \|\psi_{n_k-1} - \psi'_{n_k-1}\| \\ &+ \sum_{i=1}^{n} Me^{-q(n-i)}\epsilon + \sum_{k=0}^{r} Me^{-q(n-n_k)}\epsilon.\end{aligned} \tag{4.1}$$

After calculation and taking supremum over the solution we will get the following:

$$\begin{aligned}\|\psi - \psi\| &\le ML\frac{e^q}{e^q-1}\|\psi - \psi'\| + Mh\|\psi - \psi'\|\frac{e^q}{e^q-1} \\ &+ M\epsilon\frac{e^q}{e^q-1} + \epsilon\frac{e^q}{e^q-1} \\ &= \frac{Me^q(L+h)}{e^q-1}\|\psi - \psi'\| + \frac{e^q(M+1)}{e^q-1}\epsilon.\end{aligned} \tag{4.2}$$

This implies that

$$\|\psi-\psi\| \leq \frac{e^q(M+1)}{e^q-1-Me^q(L+h)}\epsilon.$$

Hence the system (1.1) is Ulam-Hyers stable. □

Before going to the next theorem we need the following assumption:
G_5 : The sequence f_n is such that $\|f_n\| \leq \phi_n$ and $\sum_{i=1}^{n} Me^{-q(n-i)}\phi_{n-i} \leq \Phi_n\epsilon$, for all n, where ϕ_n and Φ_n are bounded and monotionic sequnces.

Theorem 4.2 *If the assumptions G_1 to G_5 are satisfied, then the system (1.1) is generalized Ulam-Hyers stable.*

Proof: Using Lemmas 2.1 and 2.2 we have,

$$\begin{aligned}
&\|\psi_n-\psi'_n\| = \Bigg\| \sum_{i=1}^{n} P(n,i)(F(i-1,\psi_{i-1})-F(i-1,\psi'_{i-1})) + \sum_{k=0}^{r} P(n,n_k)(I_k(\psi_{n_k-1})-I_k(\psi'_{n_k-1}))\\
&+\sum_{i=1}^{n} P(n,i)f_{i-1} + \sum_{k=0}^{r} P(n,n_k)f_{n_k}\Bigg\|\\
&\leq \sum_{i=1}^{n} Me^{-q(n-i)}L\|\psi_{i-1}-\psi'_{i-1}\| + \sum_{k=0}^{r} Me^{-q(n-n_k)}h\|\psi_{n_k-1}-\psi'_{n_k-1}\|\\
&+\sum_{i=1}^{n} Me^{-q(n-i)}\phi_{n-i} + \sum_{k=0}^{r} Me^{-q(n-n_k)}\phi_{nk}\\
&\leq \sum_{i=1}^{n} Me^{-q(n-i)}L\|\psi_{i-1}-\psi'_{i-1}\| + \sum_{k=0}^{r} Me^{-q(n-n_k)}h\|\psi_{n_k-1}-\psi'_{n_k-1}\|\\
&+\Phi_n\epsilon+\Phi_r\epsilon.
\end{aligned} \tag{4.3}$$

Clearly Φ_n is increasing and taking supremum over the solution we will get the following:

$$\begin{aligned}
\|\psi-\psi\| &\leq ML\frac{e^q}{e^q-1}\|\psi-\psi'\| + Mh\|\psi-\psi'\|\frac{e^q}{e^q-1} + 2\Phi_n\epsilon\\
&= \frac{Me^q(L+h)}{e^q-1}\|\psi-\psi'\| + 2\Phi_n\epsilon.
\end{aligned} \tag{4.4}$$

This implies that

$$\|\psi-\psi\| \leq \frac{e^q(M+1)}{e^q-1-Me^q(L+h)}2\Phi\epsilon.$$

Hence the system (1.1) is generalized Hyers-Ulam stable. □

Next we have to present our last result about the uniform exponential stability for which we need the follwoing assumption: $G_6 : \|F(i-1,X_{i-1})\| \leq K$.

Theorem 4.3 *Assume that $G_1 - G_4$ and G_6 are holds, then the system (1.1) is uniformly exponentially stable.*

Proof: The solution of system (1.1) is

$$X_n = P(n,0)b + \sum_{i=1}^{n} P(n,i)F(i-1,X_{i-1}) + \sum_{n_k=0}^{n} P(n,n_k)I_k(X_{n_k-1}).$$

Now we have,

$$\begin{aligned} \|X_n\| &= \left\| P(n,0)b + \sum_{i=1}^{n} P(n,i)F(i-1,X_{i-1}) + \sum_{k=0}^{r} P(n,n_k)I_k(X_{n_k-1}) \right\| \\ &\le Me^{-qn}b + \sum_{i=1}^{n} Me^{-q(n-i)}K + \sum_{k=0}^{r} Me^{-q(n-n_k)}H \\ &\le Me^{-qn}b + MK\frac{e^q}{e^q-1} + MH\frac{e^q}{e^q-1}. \end{aligned} \tag{4.5}$$

Now usnig the Grownwall inequality we have

$$\|X_n\| \le B(M,K,H)e^{-qn},$$

thus, the system (1.1) is uniformly exponentially stable. □

5. Examples

In this section we will present few examples to illustrate our results.

Example 5.1 *Consider the following impulsive difference system:*

$$\begin{cases} X_{n+1} = 3^{-\mu}X_n + f(n,X_n), & n \ge 0, \\ X_0 = b, & n = 0, \\ \Delta X_{n_k} = I_k(X_{n_k-1}), & k \in I, \end{cases} \tag{5.1}$$

Then by Lemma 2.1 its solution will be as given below:

$$X_n \;=\; P(n,0)b + \sum_{i=1}^{n} P(n,i)F(i-1,X_{i-1}) + \sum_{k=0}^{r} P(n,n_k)I_k(X_{n_k-1}), \;\; n \in I,$$

where $P(\mu,r) = 3^{-\mu-1}3^{-\mu-2}3^{-\mu-3}...3^{-r}$*. Now for a sequence* $f_n = 0.5$ *the solution of the problem*

$$\begin{cases} X_{n+1} = 3^{-n}X_n + F(n,X_n) + 0.5, & n \ge 0, \\ X_0 = b, & n = 0, \\ \Delta X_{n_k} = I_k(X_{n_k-1}) + f_{n_k}, & k \in I, \end{cases} \tag{5.2}$$

will be

$$X'_n = P(n,0)b + \sum_{i=1}^{n} P(n,i)F(i-1,X'_{i-1}) + \sum_{i=1}^{n} P(n,i)f_{i-1} + \sum_{n_k=0}^{n} P(n,n_k)I_k(X'_{n_k-1}) + \sum_{k=0}^{r} P(n,n_k)f_{n_k},$$

where $\|f_n\| \le 0.5$.
Clearly all the axioms $G_1 - G_5$ *are holds therefore the system (1.1) is Ulam-Hyers stable.*

Example 5.2 *Consider again the same impulsive difference system as in the above example and consider:*

$$\begin{aligned} \|X_n\| &\le \|P(n,0)\| + \sum_{i=0}^{n} \|P(n,i)\|\|F(i-1,X_{i-1})\| + \sum_{k=0}^{r} \|P(n,n_k)\|\|I_k(X_{n_k-1})\| \\ &\le e^{-n} + K\sum_{i=0}^{n} e^{-(n-i)} + h\sum_{k=0}^{r} e^{-(n-n_k)} \\ &\le e^{-n} + (K+h)\frac{e}{e-1} \end{aligned}$$

The fulfilment of the axioms $G_1 - G_4$ *and* G_6 *and by using the Grownwal lemma, the system is uniformly exponentially stable.*

6. Conclusion

The paper presents a thorough investigation into the stability characteristics of non-autonomous impulsive difference systems, focusing on Ulam-Hyers, generalized Ulam-Hyers, and uniform exponential stability. Our analysis establishes necessary conditions that extend existing theoretical frameworks, providing a deeper understanding of these stability concepts. The examples we provided illustrate the practical applicability of our findings, highlighting their relevance to mathematical modeling and control theory. These results not only enhance the theoretical landscape of stability in impulsive systems but also open avenues for further exploration in related areas. Future research could build upon our findings by exploring additional stability criteria or applying the established principles to more complex systems. Overall, our work contributes to the broader field of stability analysis, offering valuable insights for both theoretical advancements and practical applications.

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