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Complete lifts from a Sasakian manifold concerning the quarter symmetric metric connection to its tangent bundle

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ABSTRACT: The present paper aims to study the complete lifts of the quarter symmetric metric connection and we establish the interrelation between a Levi-Civita connection and a quarter symmetric metric connection on a Sasakian manifold to its tangent bundle. The curvature and the Ricci tensors are formulated in the form of lifts concerning the quarter symmetric metric connection on a Sasakian manifold to its tangent bundle. The symmetric property of the Ricci tensor on the tangent bundle is deduced. Finally, we establish necessary and sufficient conditions for the tangent bundle of the Sasakian manifold to be quasi-conharmonically flat, ϕ^C-conharmonically flat and ξ^C-conharmonically flat concerning the quarter symmetric metric connection.

Key Words: Complete lift, vertical lift, tangent bundle, partial differential equations, sasakian manifold, quarter symmetric metric connection, curvature tensor, mathematical operators, Ricci tensor.

Contents

1. Introduction

One of the principal contributions of differential geometry of tangent bundles is to give an effective domain of differential geometry. It provides the study of complete, vertical, and horizontal lifts of tensor fields and connections to tangent bundles introduced by Yano and Kobayashi [28]. Tani [25] introduced the notion of such lifts of a Riemannian connection to tangent bundles. Pandey and Chaturvedi [22] studied the complete and vertical lifts of the quarter symmetric nonmetric connection on a Kähler manifold. Akpinar [1] determined the complete lift of Weyl connection to the tangent bundle of hypersurface. The author [15,18] studied the quarter symmetric nonmetric connection on an almost Hermitian manifold and a Kähler manifold to tangent bundles. Numerous investigators [14,19,28,16,17] have studied several connections such as quarter symmetric semimetric connection, quarter symmetric nonmetric connection, semisymmetric nonmetric connection on the tangent bundle and provided their theories.

On the other hand, the study of semisymmetric metric and linear connections on differential manifold were started in early 1930 by Friedman and Schouton [8,23] and Hayden [12]. In 1975, Golab [9] introduced the notion of quarter symmetric metric connection and studied the various properties on it. De and Sengupta in [5] studied the quarter symmetric metric connection on a Sasakian manifold and established the necessary and sufficient conditions for conformal curvature tensor in 2000. Various types of quarter symmetric metric connection have been recently discussed in [2,6,10,20,21,24,26,29].

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Let M be an n-dimensional differentiable manifold of class C^∞ with the Levi-Civita connection ∇. Let λ_1 and λ_2 be vector fields on M then a linear connection $\tilde{\nabla}$ is known as symmetric connection on M if the torsion tensor T of $\tilde{\nabla}$ defined by

$$T(\lambda_1, \lambda_2) = \tilde{\nabla}_{\lambda_1}\lambda_2 - \tilde{\nabla}_{\lambda_2}\lambda_1 - [\lambda_1, \lambda_2] \tag{1.1}$$

is zero, or else it is non-symmetric. If the torsion tensor T satisfies

$$T(\lambda_1, \lambda_2) = \pi(\lambda_2)\phi\lambda_1 - \pi(\lambda_1)\phi\lambda_2 \tag{1.2}$$

where

$$\pi(\lambda_1) = g(P, \lambda_1), \forall P \in \Im_0^1(M), \tag{1.3}$$

π is 1-form, ϕ is a tensor field of type (1,1) and g is the Riemannian metric, then $\tilde{\nabla}$ is called a quarter symmetric connection. Also, if the Riemannian metric g satisfies $\tilde{\nabla}g = 0$, then $\tilde{\nabla}$ is said to be a quarter symmetric metric connection.

The section-wise paper is organized as: Section 2 devotes a brief account of the tangent bundle, vertical and complete lifts, quarter symmetric metric connection and the Sasakian manifold. Section 3 discusses the complete lift of quarter symmetric metric connection on a Sasakian manifold to its tangent bundle TM. In Section 4, the relationship between ∇^C and $\tilde{\nabla}^C$ on a Sasakian manifold to its the tangent bundle TM is established. Also, the curvature and the Ricci tensors of $\tilde{\nabla}^C$ are determined. The symmetric property of Ricci tensor on the tangent bundle TM is investigated. Finally, we establish necessary and sufficient conditions for the tangent bundle of the Sasakian manifold to be quasi-conharmonically flat, ϕ^C-conharmonically flat and ξ^C-conharmonically flat concerning the quarter symmetric metric connection.

2. Preliminaries

Let M be an n-dimensonal differentiable manifold and TM be the tangent bundle over M. Suppose TM be the tangent bundle and $\lambda_1 = {\lambda_1}^i \frac{\partial}{\partial x^i}$ be a local vector field on M, then its vertical and complete lifts in the term of partial differential equations are

$$\lambda_1{}^V = {\lambda_1}^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

$$\lambda_1{}^C = {\lambda_1}^i \frac{\partial}{\partial x^i} + \frac{\partial {\lambda_1}^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}. \tag{2.2}$$

Let a function f, a vector field λ_1, a 1-form η, (1,1) tensor field ϕ and an affine connection ∇ in M and $f^V, \lambda_1^V, \eta^V, \phi^V, \nabla^V$ and $f^C, \lambda_1^C, \eta^C, \phi^C, \nabla^C$ are vertical and complete lifts of f, λ_1, η, ϕ and ∇, respectively in TM. Then by using mathematical operators [18,11,27]

$$(f\lambda_1)^V = f^V\lambda_1^V, (f\lambda_1)^C = f^C\lambda_1^V + f^V\lambda_1^C, \tag{2.3}$$

$$\lambda_1^V f^V = 0, \lambda_1^V f^C = \lambda_1^C f^V = (\lambda_1 f)^V, \lambda_1^C f^C = (\lambda_1 f)^C, \tag{2.4}$$

$$\eta^V(f^V) = 0, \eta^V(\lambda_1^C) = \eta^C(\lambda_1^V) = \eta(\lambda_1)^V, \eta^C(\lambda_1^C) = \eta(\lambda_1)^C, \tag{2.5}$$

$$\phi^V\lambda_1^C = (\phi\lambda_1)^V, \phi^C\lambda_1^C = (\phi\lambda_1)^C, \tag{2.6}$$

$$[\lambda_1, \lambda_2]^V = [\lambda_1^C, \lambda_2^V] = [\lambda_1^V, \lambda_2^C], [\lambda_1, \lambda_2]^C = [\lambda_1^C, \lambda_2^C], \tag{2.7}$$

$$\nabla^C_{\lambda_1^C}\lambda_2^C = (\nabla_{\lambda_1}\lambda_2)^C, \qquad \nabla^C_{\lambda_1^C}\lambda_2^V = (\nabla_{\lambda_1}\lambda_2)^V. \tag{2.8}$$

$$\nabla^C_{\lambda_1^V}\lambda_2^C = (\nabla_{\lambda_1}\lambda_2)^V, \qquad \nabla^C_{\lambda_1^V}\lambda_2^V = 0. \tag{2.9}$$

Let λ and γ be arbitrary tensor fields in the manifold M. Then by mathematical operators

$$(\lambda \otimes \gamma)^V = \lambda^V \otimes \gamma^V, (\lambda \otimes \gamma)^C = \lambda^C \otimes \gamma^V + \lambda^V \otimes \gamma^C$$

$$(\lambda + \gamma)^V = \lambda^V + \gamma^V, (\lambda + \gamma)^C = \lambda^C + \gamma^C,$$

where λ^C and γ^C are the complete lift of arbitrary tensor fields of λ and γ in the manifold M.

The following notations will be used throughout the paper: let $\Im_0^0(M), \Im_0^1(M), \Im_1^0(M), \Im_1^1(M)$ be the set of functions, vector fields, 1-forms and tensor fields of type (1,1) in M, respectively. Similarly, let $\Im_0^0(TM), \Im_0^1(TM), \Im_1^0(TM), \Im_1^1(TM)$ be the set of functions, vector fields, 1-forms and a tensor fields of type (1,1) in TM, respectively.

2.1. Sasakian manifolds

Let M be n-dimensional differentiable manifold and there is given a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η and a Riemannian metric g satisfying [3]

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0 \quad \eta\circ\phi = 0, \tag{2.10}$$
$$g(\phi\lambda_1, \phi\lambda_2) = g(\lambda_1, \lambda_2) - \eta(\lambda_1)\eta(\lambda_2), \quad \lambda_1, \lambda_2 \in \Im_0^1(M). \tag{2.11}$$

Then the structure (ϕ, ξ, η, g) is said to be an almost contact metric manifold.

From (2.10) and (2.11), we have

$$g(\phi\lambda_1, \lambda_2) = -g(\lambda_1, \phi\lambda_2), \quad g(\lambda_1, \xi) = \eta(\lambda_1), \quad \lambda_1, \lambda_2 \in \Im_0^1(M). \tag{2.12}$$

An almost contact metric manifold is said to be a contact metric manifold if

$$g(\phi\lambda_1, \lambda_2) = d\eta(\lambda_1, \lambda_2), \quad \lambda_1, \lambda_2 \in \Im_0^1(M). \tag{2.13}$$

A contact metric manifold is called K-contact if ξ is a Killing. If (ϕ, ξ, η, g) satisfy the relation

$$(\nabla_{\lambda_1}\phi)\lambda_2 = g(\lambda_1, \lambda_2)\xi - \eta(\lambda_2)\lambda_1 \tag{2.14}$$

where ∇ is the Levi-Civita connection, then M is called a Sasakian manifold. In a Sasakian manifold M, the following relations hold:

$$\nabla_{\lambda_1}\xi = -\phi\lambda_1, \tag{2.15}$$
$$R(\lambda_1, \lambda_2)\xi = \eta(\lambda_2)\lambda_1 - \eta(\lambda_1)\lambda_2, \tag{2.16}$$
$$R(\xi, \lambda_1)\lambda_2 = (\nabla - \lambda_1\phi)\lambda_2, \tag{2.17}$$
$$S(\lambda_1, \xi) = (n-1)\eta(\lambda_1), \tag{2.18}$$
$$S(\phi\lambda_1, \phi\lambda_2) = S(\lambda_1, \lambda_2) - (n-1)\eta(\lambda_1)\eta(\lambda_2), \tag{2.19}$$

$\forall\lambda_1, \lambda_2 \in \Im_0^1(M)$, R and S denote the curvature tensor and Ricci tensor, respectively.

The conharmonic curvature tensor $\tilde{K}$ of an n-dimensional almost contact metric manifold concerning quarter symmetric metric connection $\tilde{\nabla}$ as

$$\begin{aligned}\tilde{K}(\lambda_1, \lambda_2)\lambda_3 &= \tilde{R}(\lambda_1, \lambda_2)\lambda_3 - \frac{1}{n-2}\{\tilde{S}(\lambda_2, \lambda_3)\lambda_1 - \tilde{S}(\lambda_1, \lambda_3)\lambda_2 \\ &+ g(\lambda_2, \lambda_3)\tilde{Q}\lambda_1 - g(\lambda_1, \lambda_3)\tilde{Q}\lambda_2\},\end{aligned} \tag{2.20}$$

$\forall\lambda_1, \lambda_2, \lambda_3 \in \Im_0^1(M)$, $\tilde{R}$ being the curvature tensor of $\tilde{\nabla}$, $\tilde{S}$ is the Ricci tensor of and $\tilde{\nabla}$ and $\tilde{Q}$ is the Ricci operator for defined by $g(\tilde{Q}\lambda_1, \lambda_2) = \tilde{S}(\lambda_1, \lambda_2)$.

Definition 2.1 *An almost contact metric manifold M is said to be quasi-conharmonically flat concerning the quarter symmetric metric connection if*

$$g(\tilde{K}(\lambda_1, \lambda_2)\lambda_3, \phi\lambda_4) = 0, \tag{2.21}$$

ϕ-conharmonically flat concerning the quarter symmetric metric connection if

$$g(\tilde{K}(\phi\lambda_1, \phi\lambda_2)\phi\lambda_3, \phi\lambda_4) = 0, \tag{2.22}$$

and ξ-conharmonically flat concerning the quarter symmetric metric connection if

$$\tilde{K}(\lambda_1, \lambda_2)\xi = 0, \tag{2.23}$$

$\forall\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Im_0^1(M)$.

Let $\{e_i, \xi\}$ be a local orthonormal basis of vector fields in an n-dimensional Sasakian manifold M, then $\{\phi e_i, \xi\}$ is a local orthonormal basis of vector fields. Then the following properties have been established.

$$\sum_{i=1}^{n-1} g(e_i, e_i) = \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n-1, \tag{2.24}$$

$$\begin{aligned}\sum_{i=1}^{n-1} g(e_i, \lambda_3) S(\lambda_2, e_i) &= \sum_{i=1}^{n-1} g(\phi e_i, \lambda_3) S(\lambda_2, \phi e_i)\\ &= S(\lambda_2, \lambda_3) - (n-1)\eta(Z)),\end{aligned} \tag{2.25}$$

$$\begin{aligned}\sum_{i=1}^{n-1} g(e_i, \phi\lambda_3) S(\lambda_2, e_i) &= \sum_{i=1}^{n-1} g(\phi e_i, \phi\lambda_3) S(\lambda_2, \phi e_i)\\ &= S(\lambda_2, \phi\lambda_3) - (n-1)\eta(Z)),\end{aligned} \tag{2.26}$$

$$\sum_{i=1}^{n-1} S(e_i, e_i) = \sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r - (n-1), \tag{2.27}$$

$$\begin{aligned}\sum_{i=1}^{n-1} R(e_i, \lambda_2, \lambda_3, e_i) &= \sum_{i=1}^{n-1} R(\phi e_i, \lambda_2, \lambda_3, \phi e_i)\\ &= S(\lambda_2, \lambda_3) - R(\xi, \lambda_2, \lambda_3, \xi),\end{aligned} \tag{2.28}$$

$$\sum_{i=1}^{n-1} R(\phi e_i, \phi\lambda_2, \phi\lambda_3, \phi e_i) = S(\phi\lambda_2, \phi\lambda_3) - R(\xi, \lambda_2, \lambda_3, \xi), \tag{2.29}$$

$\forall \lambda_2, \lambda_3 \in \Im_0^1(M)$.

3. Complete lifts of quarter symmetric metric connection on a Sasakian manifold to its tangent bundle

Let M be a Sasakian manifold and TM its tangent bundle. Taking the complete lift of (1.1), (1.2), (2.10)-(2.19), we infer

$$(\phi^C)^2 = -I + \eta^V \otimes \xi^C - \eta^C \otimes \xi^V, \tag{3.1}$$

$$\eta^C \xi^C = \eta^V \xi^V = 0, \quad \eta^C \xi^V = \eta^V \xi^C = 1 \tag{3.2}$$

$$\phi^C \xi^V = \phi^V \xi^C = \phi^V \xi^V = \phi^C \xi^C = 0 \tag{3.3}$$

$$\eta^V \circ \phi^C = \eta^C \circ \phi^V = \eta^C \circ \phi^C = \eta^V \circ \phi^V = 0, \tag{3.4}$$

$$\begin{aligned}g^C((\phi\lambda_1)^C, (\phi\lambda_2)^C) &= g^C(\lambda_1^C, \lambda_2^C) - \eta^C(\lambda_1^C)\eta^V(\lambda_2^C)\\ &- \eta^V(\lambda_1^C)\eta^C(\lambda_2^C),\end{aligned} \tag{3.5}$$

$\forall \lambda_1^C, \lambda_2^C \in \Im_0^1(TM)$. From (3.3) and (3.5), we have

$$g^C((\phi\lambda_1)^C, \lambda_2^C) = -g(\lambda_1^C, (\phi\lambda_2)^C), \quad g(\lambda_1^C, \xi^C) = \eta^C(\lambda_1^C) \tag{3.6}$$

and

$$g^C((\phi\lambda_1)^C, \lambda_2^C) = d\eta^C(\lambda_1^C, \lambda_2^C), \quad \lambda_1^C, \lambda_2^C \in \Im_0^1(TM). \tag{3.7}$$

Let ∇^C be the complete lift of the Levi-Civita connection ∇ of a Riemannian metric g. Then

$$\begin{aligned}(\nabla^C_{\lambda_1^C}\phi^C)\lambda_2^C &= g^C(\lambda_1^C, \lambda_2^C)\xi^V + g^C(\lambda_1^V, \lambda_2^C)\xi^C\\ &- \eta^C(\lambda_2^C)\lambda_1^V - \eta^V(\lambda_2^C)\lambda_1^C.\end{aligned} \tag{3.8}$$

Furthermore, the following relations are given by

$$\nabla^C_{\lambda_1^C}\xi^C = -(\phi\lambda_1)^C, \tag{3.9}$$

$$R^C(\lambda_1^C,\lambda_2^C)\xi^C = \eta^C(\lambda_2^C)\lambda_1^V + \eta^V(\lambda_2^C)\lambda_1^C - \eta^C(\lambda_1^C)\lambda_2^V - \eta^V(\lambda_1^C)\lambda_2^V, \tag{3.10}$$

$$R^C(\xi^C,\lambda_1^C)\lambda_2^C = (\nabla^C_{\lambda_1^C}\phi^C)\lambda_2^C, \tag{3.11}$$

$$S^C(\lambda_1^C,\xi^C) = (n-1)\eta^C(\lambda_1^C), \tag{3.12}$$

$$S^C((\phi\lambda_1)^C,(\phi\lambda_2)^C) = S^C(\lambda_1^C,\lambda_2^C) - (n-1)\{\eta^C(\lambda_1^C)\eta^V(\lambda_2^C) + \eta^V(\lambda_1^C)\eta^C(\lambda_2^C)\} \tag{3.13}$$

for any vector fields $\lambda_1^C, \lambda_2^C \in \Im_0^1(TM)$, where R^C and S^C denote the complete lift on TM of the curvature tensor R and the Ricci tensor S of M, respectively.

Let $\{e_i^C, \xi^C\}$ be a local orthonormal basis of vector fields in TM, then $\{(\phi e_i)^C, \xi^C\}$ is a local orthonormal basis of vector fields. Taking the complete lift of (2.24)-(2.29), we infer

$$\sum_{i=1}^{n-1} g^C(e_i^C,e_i^C) = \sum_{i=1}^{n-1} g^C((\phi e_i)^C,(\phi e_i)^C) = 2n-1, \tag{3.14}$$

$$\sum_{i=1}^{n-1}(g(e_i,\lambda_3)S(\lambda_2,e_i))^C = (g(\phi e_i,\lambda_3)S(\lambda_2,\phi e_i))^C = S^C(\lambda_2^C,\lambda_3^C) - (2n-1)\eta^C(\lambda_3^C)), \tag{3.15}$$

$$\sum_{i=1}^{n-1}(g(e_i,\phi\lambda_3)S(\lambda_2,e_i))^C = (g(\phi e_i,\phi\lambda_3)S(\lambda_2,\phi e_i))^C = S^C(\lambda_2^C,\phi\lambda_3^C), \tag{3.16}$$

$$\sum_{i=1}^{n-1} S^C(e_i^C,e_i^C) = \sum_{i=1}^{n-1} S^C((\phi e_i)^C,(\phi e_i)^C) = r^C - (2n-1), \tag{3.17}$$

$$\sum_{i=1}^{n-1} R^C(e_i^C,\lambda_2^C,\lambda_3^C,e_i^C) = \sum_{i=1}^{n-1} R^C((\phi e_i)^C,\lambda_2^C,\lambda_3^C,(\phi e_i)^C) = S^C(\lambda_2^C,\lambda_3^C) - R^C(\xi^C,\lambda_2^C,\lambda_3^C,\xi^C), \tag{3.18}$$

$$\sum_{i=1}^{n-1} R^C((\phi e_i)^C,(\phi\lambda_2^C,(\phi\lambda_3^C,(\phi e_i)^C) = S^C((\phi\lambda_2)^C,(\phi\lambda_3)^C) - R^C(\xi^C,\lambda_2^C,\lambda_3^C,\xi^C), \tag{3.19}$$

$\forall \lambda_1^C, \lambda_2^C, \lambda_3^C \in \Im_0^1(TM)$.

4. quarter symmetric metric connection in a Sasakian manifold to its tangent bundle

Let M be an almost contact metric manifold with a Riemannian connection ∇ and TM its tangent bundle. A linear connection $\tilde{\nabla}$ and the tensor H of type (1,1) are related by [4,7]

$$\tilde{\nabla}_{\lambda_1}\lambda_2 = \nabla_{\lambda_1}\lambda_2 + H(\lambda_1,\lambda_2). \tag{4.1}$$

For $\tilde{\nabla}$ to be a quarter symmetric metric connection in M, we have [9]

$$H(\lambda_1,\lambda_2) = \frac{1}{2}[T(\lambda_1,\lambda_2) + T'(\lambda_1,\lambda_2) + T'(\lambda_2,\lambda_1)] \tag{4.2}$$

and

$$g(T'(\lambda_1,\lambda_2),\lambda_3) = g(T(\lambda_3,\lambda_1),\lambda_2). \tag{4.3}$$

Using (1.2) and (4.3), the obtained equation is

$$T'(\lambda_1,\lambda_2) = g(\phi\lambda_2,\lambda_1)\xi - \eta(\lambda_1)\phi\lambda_2. \tag{4.4}$$

Making use of (1.2) and (4.4) in (4.2), the equation (4.2) becomes

$$H(\lambda_1,\lambda_2) = -\eta(\lambda_1)\phi\lambda_2. \tag{4.5}$$

Thus, a quarter symmetric metric connection $\tilde{\nabla}$ on a Sasakian manifold is given by

$$\tilde{\nabla}_{\lambda_1}\lambda_2 = \nabla_{\lambda_1}\lambda_2 - \eta(\lambda_1)\phi\lambda_2. \tag{4.6}$$

Taking the complete lift of (1.1), (1.2), (1.3), (4.1), (4.2) and (4.3), the obtained equations are

$$\begin{aligned} T^C(\lambda_1^C,\lambda_2^C) &= \tilde{\nabla}^C_{\lambda_1^C}\lambda_2^C - \tilde{\nabla}^C_{\lambda_2^C}\lambda_1^C - [\lambda_1^C,\lambda_2^C] && (4.7)\\ &= \eta^C(\lambda_2^C)(\phi\lambda_1)^V - \eta^V(\lambda_1^C)(\phi\lambda_2)^C \\ &- \eta^C(\lambda_2^C)(\phi\lambda_1)^V - \eta^V(\lambda_1^C)(\phi\lambda_2)^C, && (4.8) \end{aligned}$$

$$\tilde{\nabla}^C_{\lambda_1^C}\lambda_2^C = \nabla^C_{\lambda_1^C}\lambda_2^C + H^C(\lambda_1^C,\lambda_2^C), \tag{4.9}$$

where

$$H^C(\lambda_1^C,\lambda_2^C) = \frac{1}{2}[T^C(\lambda_1^C,\lambda_2^C) + T'^C(\lambda_1^C,\lambda_2^C) + T'^C(\lambda_2^C,\lambda_1^C)] \tag{4.10}$$

and

$$g^C(T'^C(\lambda_1^C,\lambda_2^C),\lambda_3^C) = g^C(T^C(\lambda_3^C,\lambda_1^C),\lambda_2^C), \forall\lambda_1,\lambda_2 \in \Im_0^1(M). \tag{4.11}$$

From (4.7) and (4.11), we provide

$$\begin{aligned} T'^C(\lambda_1^C,\lambda_2^C) &= g^C((\phi\lambda_2)^C,\lambda_1^C)\xi^C + g^C((\phi\lambda_2)^V,\lambda_1^C)\xi^C \\ &- \eta^C(\lambda_1^C)(\phi\lambda_2)^V - \eta^V(\lambda_1^C)(\phi\lambda_2)^C. && (4.12) \end{aligned}$$

Using (4.7) and (4.12) in (4.13), we get

$$H^C(\lambda_1^C,\lambda_2^C) = -\eta^C(\lambda_1^C)(\phi\lambda_2)^V - \eta^V(\lambda_1^C)(\phi\lambda_2)^C, \tag{4.13}$$

where H^C is the complete lift of H.

Thus a quarter symmetric metric connection $\tilde{\nabla}^C$ in a Sasakian manifold concerning ∇^C on TM is given by

$$\tilde{\nabla}^C_{\lambda_1^C}\lambda_2^C = \nabla^C_{\lambda_1^C}\lambda_2^C - \eta^C(\lambda_1^C)(\phi\lambda_2)^V - \eta^V(\lambda_1^C)(\phi\lambda_2)^C. \tag{4.14}$$

Therefore the equation (4.14) is the relation between the Levi-Civita connection ∇^C and the quarter symmetric metric connection $\tilde{\nabla}^C$ on TM. Thus

Theorem 4.1 *Let $\tilde{\nabla}$ be the quarter symmetric metric connection on a Sasakian manifold M and $\tilde{\nabla}^C$ be its complete lift on the tangent bundle TM. Then the relation between ∇^C and $\tilde{\nabla}^C$ on TM is given by the equation (4.14).*

Let $\tilde{\nabla}$ be the quarter symmetric metric connection on M and $\tilde{R}$ the curvature tensor of $\tilde{\nabla}$. Then the curvature tensor $\tilde{R}^C$ concerning $\tilde{\nabla}^C$ on the tangent bundle TM is given by

$$\tilde{R}^C(\lambda_1^C,\lambda_2^C)\lambda_3^C = \tilde{\nabla}^C_{\lambda_1^C}\tilde{\nabla}^C_{\lambda_2^C}\lambda_3^C - \tilde{\nabla}^C_{\lambda_2^C}\tilde{\nabla}^C_{\lambda_1^C}\lambda_3^C - \tilde{\nabla}^C_{[\lambda_1^C,\lambda_2^C]}\lambda_3^C, \tag{4.15}$$

where

$$\tilde{R}(\lambda_1,\lambda_2)\lambda_3 = \tilde{\nabla}_{\lambda_1}\tilde{\nabla}_{\lambda_2}\lambda_3 - \tilde{\nabla}_{\lambda_2}\tilde{\nabla}_{\lambda_1}\lambda_3 - \tilde{\nabla}_{[\lambda_1,\lambda_2]}\lambda_3. \tag{4.16}$$

From (4.14) it follows that

$$\tilde{\nabla}^C_{\lambda_1^C}\tilde{\nabla}^C_{\lambda_2^C}\lambda_3^C = \tilde{\nabla}^C_{\lambda_1^C}\nabla^C_{\lambda_2^C}\lambda_3^C - \tilde{\nabla}^C_{\lambda_1^C}\eta^C(\lambda_2^C)(\phi\lambda_3)^V - \tilde{\nabla}^C_{\lambda_1^C}\eta^V(\lambda_2^C)(\phi\lambda_3)^C. \tag{4.17}$$

In view of (4.14), (4.15) and (4.17), we get

$$\begin{aligned}\tilde{R}^C(\lambda_1^C,\lambda_2^C)\lambda_3^C &= R^C(\lambda_1^C,\lambda_2^C)\lambda_3^C - 2d\eta^C(\lambda_1^C,\lambda_2^C)(\phi\lambda_3)^V \\ &- 2d\eta^V(\lambda_1^V,\lambda_2^C)(\phi\lambda_3)^C + \eta^V(\lambda_1^C)(\nabla^C_{\lambda_2^C}\phi^C)\lambda_3^C \\ &- \eta^V(\lambda_2^C)(\nabla^C_{\lambda_1^C}\phi^C)\lambda_3^C,\end{aligned} \tag{4.18}$$

$\forall\lambda_1^C,\lambda_2^C,\lambda_3^C \in \Im_0^1(TM), \phi^C \in \Im_1^1(TM), \eta^C \in \Im_1^0(TM)$, as $(\nabla_{\lambda_1}\phi)^V = 0$ ([27], p. 45).

Using the equation (3.8), the obtained equation is

$$\begin{aligned}\tilde{R}^C(\lambda_1^C,\lambda_2^C)\lambda_3^C &= R^C(\lambda_1^C,\lambda_2^C)\lambda_3^C - 2d\eta^C(\lambda_1^C,\lambda_2^C)(\phi\lambda_3)^V \\ &- 2d\eta^V(\lambda_1^C,\lambda_2^C)(\phi\lambda_3)^C + \eta^V(\lambda_1^C)g^C(\lambda_2^C,\lambda_3^C)\xi^C \\ &+ \eta^C(\lambda_1^C)g^C(\lambda_2^V,\lambda_3^C)\xi^C + \eta^C\lambda_1^C)g^C(\lambda_2^C,\lambda_3^C)\xi^V \\ &- \eta^V(\lambda_2^C)g^C(\lambda_1^C,\lambda_3^C)\xi^C - \eta^C(\lambda_2^C)g^C(\lambda_1^V,\lambda_3^C)\xi^C \\ &- \eta^C\lambda_2^C)g^C(\lambda_1^C,\lambda_3^C)\xi^V - \{\eta^V(\lambda_1^C)\eta^C(\lambda_3^C)\lambda_2^C \\ &+ \eta^C(\lambda_1^C)\eta^V(\lambda_3^C)\lambda_2^C + \eta^C(\lambda_1^C)\eta^C(\lambda_3^C)\lambda_2^V \\ &- \eta^V(\lambda_2^C)\eta^C(\lambda_3^C)\lambda_1^C + \eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\lambda_1^C \\ &+ \eta^C(\lambda_2^C)\eta^C(\lambda_3^C)\lambda_1^V\},\end{aligned} \tag{4.19}$$

where $R^C(\lambda_1^C,\lambda_2^C)\lambda_3^C$ is the curvature tensor of the connection ∇^C. Thus

Theorem 4.2 *Let $\tilde{R}$ be the curvature tensor of a Sasakian manifold M concernig quarter symmetric metric connection $\tilde{\nabla}$. Then the curvature tensor $\tilde{R}^C$ concernig the quarter symmetric metric connection $\tilde{\nabla}^C$ on TM is given by the equation (4.19).*

Let M be a Sasakian manifold and TM its tangent bundle. If $\tilde{R}^C$ be the curvature tensor concernig the quarter symmetric metric connection $\tilde{\nabla}^C$ on the tangent bundle TM, then from (4.19) and (3.10), we get

$$\begin{aligned}\tilde{R}^C(\lambda_1^C,\lambda_2^C)\xi^C &= 2\{\eta^V(\lambda_2^C)\lambda_1^C + \eta^C(\lambda_2^C)\lambda_1^V - \eta^V(\lambda_1^C)\lambda_2^C \\ &- \eta^C(\lambda_1^C)\lambda_2^V\},\end{aligned} \tag{4.20}$$

$$\begin{aligned}\tilde{R}^C(\lambda_1^C,\xi^C)\lambda_2^C &= 2\{\eta^V(\lambda_2^C)\lambda_1^C + \eta^C(\lambda_2^C)\lambda_1^V - g^C(\lambda_1^V,\lambda_2^C)\xi^C \\ &- g^C(\lambda_1^C,\lambda_2^C)\xi^V)\}\end{aligned} \tag{4.21}$$

and

$$\tilde{R}^C(\xi^C,\lambda_1^C)\xi^C = 2\{\eta^V(\lambda_2^C)\xi^C + \eta^C(\lambda_2^C)\xi^V - \lambda_1^C\}. \tag{4.22}$$

Taking the inner product of (4.19) with λ_4^C, we have

$$\begin{aligned} g^C(\tilde{R}^C(\lambda_1^C,\lambda_2^C)\lambda_3^C,\lambda_4^C) &= g^C(R^C(\lambda_1^C,\lambda_2^C)\lambda_3^C,\lambda_4^C) \\ &- 2d\eta^C(\lambda_1^C,\lambda_2^C)g^C((\phi\lambda_3)^V,\lambda_4^C) \\ &- 2d\eta^C(\lambda_1^V,\lambda_2^C)g^C((\phi\lambda_3)^C,\lambda_4^C) \\ &+ \eta^V(\lambda_1^C)\eta^C(\lambda_4^C)g^C(\lambda_2^C,\lambda_3^C) \\ &+ \eta^C(\lambda_1^C)\eta^V(\lambda_4^C)g^C(\lambda_2^C,\lambda_3^C) \\ &+ \eta^C(\lambda_1^C)\eta^C(\lambda_4^C)g^C(\lambda_2^V,\lambda_3^C) \\ &- \eta^V(\lambda_2^C)\eta^C(\lambda_4^C)g^C(\lambda_1^C,\lambda_3^C) \\ &- \eta^C(\lambda_2^C)\eta^V(\lambda_4^C)g^C(\lambda_1^C,\lambda_3^C) \\ &- \eta^C(\lambda_2^C)\eta^C(\lambda_4^C)g^C(\lambda_1^V,\lambda_3^C) \\ &- \{\eta^V(\lambda_1^C)\eta^C(\lambda_3^C)g^C(\lambda_2^C,\lambda_4^C) \\ &+ \eta^C(\lambda_1^C)\eta^V(\lambda_3^C)g^C(\lambda_2^C,\lambda_4^C) \\ &+ \eta^C(\lambda_1^C)\eta^C(\lambda_3^C)g^C(\lambda_2^V,\lambda_4^C) \\ &- \eta^V(\lambda_2^C)\eta^C(\lambda_3^C)g^C(\lambda_1^C,\lambda_4^C) \\ &- \eta^C(\lambda_2^C)\eta^V(\lambda_3^C)g^C(\lambda_1^C,\lambda_4^C) \\ &- \eta^C(\lambda_2^C)\eta^C(\lambda_3^C)g^C(\lambda_1^V,\lambda_4^C)\}, \end{aligned} \tag{4.23}$$

$\forall\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C\in\Im_0^1(TM)$. The following theorem is obtained:

Theorem 4.3 *Let M be a sasakian manifold and TM its tangent bundle. If $\tilde{R}^C(\lambda_1^C,\lambda_2^C)\lambda_3^C$ be the curvature tensor concernig the quarter symmetric metric connection $\tilde{\nabla}^C$ on the tangent bundle TM, then*

i. $\tilde{R}^C(\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C)+\tilde{R}^C(\lambda_2^C,\lambda_1^C,\lambda_3^C,\lambda_4^C)=0$,

ii. $\tilde{R}^C(\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C)+\tilde{R}^C(\lambda_2^C,\lambda_1^C,\lambda_4^C,\lambda_3^C)=0$,

iii. $\tilde{R}^C(\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C)-\tilde{R}^C(\lambda_3^C,\lambda_4^C,\lambda_1^C,\lambda_2^C)=0$,

$\forall\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C\in TM$.

Let M be a Sasakian manifold and TM its tangent bundle. Let $\tilde{S}^C$ and S^C be the Ricci tensors of connections $\tilde{\nabla}^C$ and ∇^C, respectively. Contracting (4.23) over λ_1^C and λ_4^C, the obtained equation is

$$\begin{aligned} \tilde{S}^C(\lambda_2^C,\lambda_3^C) &= S^C(\lambda_2^C,\lambda_3^C)-2d\eta^C((\phi\lambda_3)^C,\lambda_2^C)+g^C(\lambda_2^C,\lambda_3^C) \\ &+ (n-2)\{\eta^V(\lambda_2^C)\eta^C(\lambda_3^C)+\eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\}, \end{aligned} \tag{4.24}$$

$\forall\lambda_2^C,\lambda_3^C\in TM$.

Thus, the Ricci tensor of the quarter symmetric metric connection on TM is symmetric. In consequence of (3.12) and (4.24), we infer

$$\begin{aligned} \tilde{S}^C((\phi\lambda_2)^C,(\phi\lambda_3)^C) &= \tilde{S}^C(\lambda_2^C,\lambda_3^C)-2(n-1)\{\eta^V(\lambda_2^C)\eta^C(\lambda_3^C) \\ &+ \eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\}, \end{aligned} \tag{4.25}$$

$$\begin{aligned} \tilde{S}^C(\lambda_2^C,\xi^C) &= 2(n-1)\{\eta^V(\lambda_2^C)\eta^C(\lambda_3^C) \\ &+ \eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\}. \end{aligned} \tag{4.26}$$

Again, contracting (4.24) over λ_2^C and λ_3^C, then

$$\tilde{r}^C=r^C+2(n-1). \tag{4.27}$$

The following theorem is obtained:

Theorem 4.4 *Let $\tilde{\nabla}$ be the quarter symmetric metric connection on a Sasakian manifold and $\tilde{\nabla}^C$ be the complete lift of $\tilde{\nabla}$ on the tangent bundle TM of M. Then*

i. *The Ricci tensor $\tilde{S}^C$ is given by the equation (4.24),*

ii. $\tilde{S}^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)=\tilde{S}^C(\lambda_2,\lambda_3)-2(n-1)\{\eta^V(\lambda_2^C)\eta^C(\lambda_3^C)+\eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\}$

iii. The Ricci tensor $\tilde{S}^C$ is symmetric,
iv. The scalar curvature $\tilde{r}$ is given by (4.27), where $\tilde{S}^C$ and S^C are the Ricci tensors of connections $\tilde{\nabla}^C$ and ∇^C, respectively on TM.

The following properties hold in a Sasakian manifold concerning the connection $\tilde{\nabla}$

$$\begin{aligned}\sum_{i=1}^{n-1} g(e_i,\lambda_3)\tilde{S}(\lambda_2,e_i) &= \sum_{i=1}^{n-1} g(\phi e_i,\lambda_3)\tilde{S}(\lambda_2,\phi e_i)\\ &= \tilde{S}(\lambda_2,\lambda_3)-2(n-1)\eta(\lambda_2)\eta(\lambda_3),\end{aligned} \tag{4.28}$$

$\forall \lambda_2,\lambda_3 \in M$.

$$\tilde{R}(\xi,\lambda_2,\lambda_3,\xi) = -2d\eta(\phi\lambda_3,\lambda_2), \tag{4.29}$$
$$\tilde{S}(\xi,\xi) = 2(n-1) \tag{4.30}$$

and

$$\tilde{Q}\xi = 2(n-1)\xi. \tag{4.31}$$

From (4.29), (2.28) and (2.29) it follows that

$$\begin{aligned}\sum_{i=1}^{n-1} \tilde{R}(e_i,\lambda_2,\lambda_3,e_i) &= \sum_{i=1}^{n-1} \tilde{R}(\phi e_i,\lambda_2,\lambda_3,\phi e_i) = \tilde{S}(\lambda_2,\lambda_3)\\ &+ 2d\eta(\phi\lambda_3,\lambda_2),\end{aligned} \tag{4.32}$$
$$\begin{aligned}\sum_{i=1}^{n-1} \tilde{R}(e_i,\phi\lambda_2,\phi\lambda_3,e_i) &= \sum_{i=1}^{n-1} \tilde{R}(\phi e_i,\phi\lambda_2,\phi\lambda_3,\phi e_i)\\ &= \tilde{S}(\lambda_2,\lambda_3)-2g(\lambda_2,\lambda_3)\\ &- 2(n-2)\eta(\lambda_2)\eta(\lambda_3),\end{aligned} \tag{4.33}$$

Also from (4.30) and (2.27), we get

$$\sum_{i=1}^{n-1} \tilde{S}(e_i,e_i) = \sum_{i=1}^{n-1} \tilde{S}(\phi e_i,\phi e_i) = \tilde{r}-2(n-1), \tag{4.34}$$
$$\sum_{i=1}^{n-1} \tilde{S}(\phi e_i,\phi\lambda_3)g(\phi\lambda_2,\phi e_i) = g(\phi\lambda_2,\phi\lambda_3),, \tag{4.35}$$

$\forall \lambda_2^C,\lambda_3^{\in}\Im_0^1(M)$.

Taking the complete lift of (4.28)-(4.35), we infer

$$\begin{aligned}\sum_{i=1}^{n-1}(g(e_i,\lambda_3)\tilde{S}(\lambda_2,e_i))^C &= \sum_{i=1}^{n-1}(g(\phi e_i,\lambda_3)\tilde{S}(\lambda_2,\phi e_i))^C\\ &= \tilde{S}^C(\lambda_2^C,\lambda_3^C)-2(n-1)\{\eta^V(\lambda_2^C)\eta^C(\lambda_3^C)\\ &+ \eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\},\end{aligned} \tag{4.36}$$

$$\tilde{R}^C(\xi^C,\lambda_2^C,\lambda_3^C,\xi^C) = -2d\eta^C((\phi\lambda_3)^C,\lambda_2^C), \tag{4.37}$$
$$\tilde{S}^C(\xi^C,\xi^C) = 2(n-1) \tag{4.38}$$

and

$$(\tilde{Q}\xi)^C = 2(n-1)\xi^C. \tag{4.39}$$

From (4.37), (3.18) and (3.19) it follows that

$$\begin{aligned}\sum_{i=1}^{n-1}(\tilde{R}(e_i,\lambda_2,\lambda_3,e_i))^C &= \sum_{i=1}^{n-1}(\tilde{R}(\phi e_i,\lambda_2,\lambda_3,\phi e_i))^C = \tilde{S}^C(\lambda_2^C,\lambda_3^C)\\ &+ 2d\eta^C((\phi\lambda_3)^C,\lambda_2^C), \qquad (4.40)\\ \sum_{i=1}^{n-1}(\tilde{R}(e_i,\phi\lambda_2,\phi\lambda_3,e_i))^C &= \sum_{i=1}^{n-1}(\tilde{R}(\phi e_i,\phi\lambda_2,\phi\lambda_3,\phi e_i))^C\\ &= \tilde{S}^C(\lambda_2^C,\lambda_3^C) - 2g^C(\lambda_2^C,\lambda_3^C) - 2(n-2)\{\eta^V(\lambda_2^C)\eta^C(\lambda_3^C)\\ &+ \eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\}. \qquad (4.41)\end{aligned}$$

Also from (4.38) and (3.17) we get

$$\begin{aligned}\sum_{i=1}^{n-1}(\tilde{S}(e_i,e_i))^C &= \sum_{i=1}^{n-1}(\tilde{S}(\phi e_i,\phi e_i))^C = \tilde{r}^C - 2(n-1), \qquad (4.42)\\ \sum_{i=1}^{n-1}(\tilde{S}(\phi e_i,\phi\lambda_3)g(\phi\lambda_2,\phi e_i))^C &= g^C((\phi\lambda_2)^C,(\phi\lambda_3)^C), \qquad (4.43)\end{aligned}$$

$\forall \lambda_2^C, \lambda_3^C \in \Im_0^1(M)$.

5. Conharmonic curvature tensor concerning the quarter symmetric metric connection $\tilde{\nabla}^C$ on TM

Let $\tilde{\nabla}$ be the quarter symmetric metric connection and $\tilde{K}$ be the conharmonic curvature tensor concerning $\tilde{\nabla}$. Then the conharmonic curvature tensor $\tilde{K}^C$ concerning $\tilde{\nabla}^C$ on TM is given by

$$\begin{aligned}\tilde{K}^C(\lambda_1^C,\lambda_2^C)\lambda_3^C &= \tilde{R}^C(\lambda_1^C,\lambda_2^C)\lambda_3^C\\ &- \frac{1}{2n-2}\{\tilde{S}^C(\lambda_2^C,\lambda_3^C)\lambda_1^V + \tilde{S}^V(\lambda_2^C,\lambda_3^C)\lambda_1^C\\ &- \tilde{S}^C(\lambda_1^C,\lambda_3^C)\lambda_2^V - \tilde{S}^V(\lambda_1^C,\lambda_3^C)\lambda_2^C\\ &+ g^C(\lambda_2^C,\lambda_3^C)(\tilde{Q}\lambda_1)^V + g^V(\lambda_2^C,\lambda_3^C)(\tilde{Q}\lambda_1)^C\\ &- g^C(\lambda_1^C,\lambda_3^C)(\tilde{Q}\lambda_2)^V - g^V(\lambda_1^C,\lambda_3^C)(\tilde{Q}\lambda_2)^C\}, \qquad (5.1)\end{aligned}$$

$\forall \lambda_2^C, \lambda_3^C \in \Im_0^1(M)$.

Theorem 5.1 *Let M be an n-dimensional $(n>3)$ Sasakian manifold concerning quarter symmetric metric connection $\tilde{\nabla}$ and $\tilde{\nabla}^C$ be the complete lift of $\tilde{\nabla}$ on TM of M. If TM of M satisfies*

$$g^C(\tilde{K}^C(\lambda_1^C,\lambda_2^C)\lambda_3^C,(\phi\lambda_4)^C) = 0, \qquad (5.2)$$

then the scalar curvature corresponding to the connection $\tilde{\nabla}^C$ is zero and TM is an η^C-Einstein manifold with respect to the connection $\tilde{\nabla}^C$.

Proof. Let M be an n-dimensional ($n > 3$) Sasakian manifold and TM its tangent bundle. From (5.1) we get

$$\begin{aligned}
g^C(\tilde{K}^C((\phi\lambda_1)^C,\lambda_2^C)\lambda_3^C,(\phi\lambda_4)^C &= g^C(\tilde{R}^C((\phi\lambda_1)^C,\lambda_2^C)\lambda_3^C,(\phi\lambda_4)^C \\
&- \frac{1}{2n-2}\{\tilde{S}^C(\lambda_2^C,\lambda_3^C)g^V((\phi\lambda_1)^C,(\phi\lambda_4)^C) \\
&+ \tilde{S}^V(\lambda_2^C,\lambda_3^C)g^C((\phi\lambda_1)^C,(\phi\lambda_4)^C) \\
&- \tilde{S}^C((\phi\lambda_1)^C,\lambda_3^C)g^V(\lambda_2^C,(\phi\lambda_4)^C \\
&- \tilde{S}^V((\phi\lambda_1)^C,\lambda_3^C)g^C(\lambda_2,\phi\lambda_4)^C \\
&+ g^C(\lambda_2^C,\lambda_3^C)\tilde{S}^V((\phi\lambda_1)^C,(\phi\lambda_4)^C) \\
&+ g^V(\lambda_2^C,\lambda_3^C)\tilde{S}^C((\phi\lambda_1)^C,(\phi\lambda_4)^C) \\
&- g^C((\phi\lambda_1)^C,\lambda_3^C)\tilde{S}^V(\lambda_2^C,(\phi\lambda_4)^C \\
&- g^V((\phi\lambda_1)^C,\lambda_3^C)\tilde{S}^C(\lambda_2,\phi\lambda_4)^C\}, \qquad (5.3)
\end{aligned}$$

$\forall\lambda_1^C,\lambda_2^C\lambda_3^C,\lambda_4^C \in \Im_0^1(TM)$.

Let $\{e_i^C,\xi^C\}$ be a local orthonormal basis on TM where $\{e_i,\xi\}$ is a local orthonormal basis on M, then (5.3) gives

$$\begin{aligned}
\sum_{i=1}^{n-1} g^C(\tilde{K}^C((\phi e_i)^C,\lambda_2^C)\lambda_3^C,(\phi e_i)^C &= \sum_{i=1}^{n-1} g^C(\tilde{R}^C((\phi e_i)^C,\lambda_2^C)\lambda_3^C,(\phi e_i)^C \\
&- \frac{1}{n-2}\sum_{i=1}^{n-1}\{\tilde{S}^C(\lambda_2^C,\lambda_3^C)g^V((\phi e_i)^C,(\phi e_i)^C) \\
&+ \tilde{S}^V(\lambda_2^C,\lambda_3^C)g^C((\phi e_i)^C,(\phi e_i)^C) \\
&- \tilde{S}^C((\phi e_i)^C,\lambda_3^C)g^V(\lambda_2^C,(\phi e_i)^C \\
&- \tilde{S}^V((\phi e_i)^C,\lambda_3^C)g^C(\lambda_2,\phi e_i)^C \\
&+ g^C(\lambda_2^C,\lambda_3^C)\tilde{S}^V((\phi e_i)^C,(\phi e_i)^C) \\
&+ g^V(\lambda_2^C,\lambda_3^C)\tilde{S}^C((\phi e_i)^C,(\phi e_i)^C) \\
&- g^C((\phi e_i)^C,\lambda_3^C)\tilde{S}^V(\lambda_2^C,(\phi e_i)^C \\
&- g^V((\phi e_i)^C,\lambda_3^C)\tilde{S}^C(\lambda_2,\phi e_i)^C\}, \qquad (5.4)
\end{aligned}$$

$\forall\lambda_2^C,\lambda_3^C \in \Im_0^1(TM)$.

Using (4.40), (3.14), (4.36) and (4.42) in (5.4), we obtain

$$\begin{aligned}
\sum_{i=1}^{n-1} g^C(\tilde{K}^C((\phi e_i)^C,\lambda_2^C)\lambda_3^C,(\phi e_i)^C &= \tilde{S}^C(\lambda_2^C,\lambda_3^C) + 2d\eta^C((\phi\lambda_3)^C),\lambda_2^C) \\
&- \frac{1}{2n-2}\{(22n-3)\tilde{S}^C(\lambda_2^C,\lambda_3^C) \\
&- d\eta^C((\phi\lambda_3)^C,\lambda_2^C) \\
&- (\tilde{r}^C - 2(2n-1))g^C(\lambda_2^C,\lambda_3^C) \\
&+ 8n\eta^C(\lambda_2)\eta^C(\lambda_3^C) \\
&+ 8n\eta^V(\lambda_2)\eta^C(\lambda_3^C)\}, \qquad (5.5)
\end{aligned}$$

$\forall\lambda_2^C,\lambda_3^C \in \Im_0^1(TM)$.

If TM of M satisfies (5.2), then from (5.5) we obtain

$$\begin{aligned} \tilde{S}^C(\lambda_2^C, \lambda_3^C) + 2d\eta^C((\phi\lambda_3)^C),)\lambda_2^C) &= \frac{1}{2n-2}\{(2n-3)\tilde{S}^C(\lambda_2^C, \lambda_3^C) \\ &- d\eta^C((\phi\lambda_3)^C), \lambda_2^C) \\ &- (\tilde{r} - 2(2n-1))g^C(\lambda_2^C, \lambda_3^C) \\ &+ 8n\eta^C(\lambda_2^C)\eta^C(\lambda_3^C) \\ &+ 8n\eta^V(\lambda_2^C)\eta^C(\lambda_3^C)\}, \end{aligned} \tag{5.6}$$

$\forall \lambda_2^C, \lambda_3^C \in \Im_0^1(TM)$.

Using (2.13) in (5.6), we get

$$\begin{aligned} \tilde{S}^C(\lambda_2^C, \lambda_3^C) &= (\tilde{r} - 2)g^C(\lambda_2^C, \lambda_3^C) + 8n\eta^C(\lambda_2^C)\eta^C(\lambda_3^C) \\ &+ 8n\eta^V(\lambda_2^C)\eta^C(\lambda_3^C), \end{aligned} \tag{5.7}$$

$\forall \lambda_2^C, \lambda_3^C \in \Im_0^1(TM)$.

Putting $\lambda_3^C = \xi^C$ in (5.7) and using (4.26) and $\eta^C(\lambda_3^C) = 1$, we obtain $\tilde{r} = 0$. Then (5.7) becomes

$$\begin{aligned} \tilde{S}^C(\lambda_2^C, \lambda_3^C) &= -2g^C(\lambda_2^C, \lambda_3^C) + 4n\eta^C(\lambda_2^C)\eta^C(\lambda_3^C) \\ &+ 4n\eta^V(\lambda_2^C)\eta^C(\lambda_3^C). \end{aligned} \tag{5.8}$$

This means that TM is an η^C-Einstein with respect to the connection $\tilde{\nabla}^C$.

Theorem 5.2 *Let M be an n-dimensional $(n > 3)$ Sasakian manifold and TM its tangent bundle. Then TM is quasi-conharmonically flat concerning the quarter symmetric metric connection $\tilde{\nabla}^C$ if and only if*

$$\begin{aligned} \tilde{R}^C(\lambda_1^C, \lambda_2^C)\lambda_3^C, (\phi\lambda_4)^C &= -\frac{4}{2n-2}\{g^C(\lambda_2^C, \lambda_3^C)g^V(\lambda_1^C, (\phi\lambda_4)^C) \\ &+ g^V(\lambda_2^C, \lambda_3^C)g^C(\lambda_1^C, (\phi\lambda_4)^C) \\ &- g^C(\lambda_1^C, \lambda_3^C)g^V(\lambda_2^C, (\phi\lambda_4)^C \\ &- g^V(\lambda_1^C, \lambda_3^C)g^C(\lambda_2^C, \phi\lambda_4)^C)\} \\ &+ \frac{4n}{2n-2}\{\eta^C(\lambda_2^C)\eta^C(\lambda_3^C)g^V(\lambda_1^C, (\phi\lambda_4)^C) \\ &+ \eta^C(\lambda_2^C)\eta^V(\lambda_3^C)g^C(\lambda_1^C, (\phi\lambda_4)^C) \\ &+ \eta^V(\lambda_2^C)\eta^C(\lambda_3^C)g^C(\lambda_1^C, (\phi\lambda_4)^C) \\ &- \eta^C(\lambda_1^C)\eta^C(\lambda_3^C)g^V(\lambda_2^C, (\phi\lambda_4)^C) \\ &- \eta^C(\lambda_1^C)\eta^V(\lambda_3^C)g^C(\lambda_2^C, (\phi\lambda_4)^C) \\ &- \eta^V(\lambda_1^C)\eta^C(\lambda_3^C)g^C(\lambda_2^C, (\phi\lambda_4)^C)\}, \end{aligned} \tag{5.9}$$

$\forall \lambda_1^C, \lambda_2^C, \lambda_3^C, \lambda_4^C \in \Im_0^1(TM)$.

Proof. Let M be quasi-conharmonically flat and TM it tangent bundle. Using (5.8) in

$$\begin{aligned} g^C(\tilde{K}^C(\lambda_1^C, \lambda_2^C)\lambda_3^C, (\phi\lambda_4)^C &= \tilde{R}^C(\lambda_1^C, \lambda_2^C)\lambda_3^C, (\phi\lambda_4)^C \\ &- \frac{1}{2n-2}\{\tilde{S}^C(\lambda_2^C, \lambda_3^C)g^V(\lambda_1^C, (\phi\lambda_4)^C) \\ &+ \tilde{S}^V(\lambda_2^C, \lambda_3^C)g^C(\lambda_1^C, (\phi\lambda_4)^C) \\ &- \tilde{S}^C(\lambda_1^C, \lambda_3^C)g^V(\lambda_2^C, (\phi\lambda_4)^C \\ &- \tilde{S}^V(\lambda_1^C, \lambda_3^C)g^C(\lambda_2, \phi\lambda_4)^C \\ &+ g^C(\lambda_2^C, \lambda_3^C)\tilde{S}^V(\lambda_1^C, (\phi\lambda_4)^C) \\ &+ g^V(\lambda_2^C, \lambda_3^C)\tilde{S}^C(\lambda_1^C, (\phi\lambda_4)^C) \\ &- g^C(\lambda_1^C, \lambda_3^C)\tilde{S}^V(\lambda_2^C, (\phi\lambda_4)^C \\ &- g^V(\lambda_1^C, \lambda_3^C)\tilde{S}^C(\lambda_2, \phi\lambda_4)^C\}, \end{aligned} \tag{5.10}$$

we get (5.9). The converse is obvious.

Theorem 5.3 *Let M be an n-dimensional $(n>3)$ Sasakian manifold and TM its tangent bundle. Then TM is ϕ^C-conharmonically flat concerning the quarter symmetric metric connection $\tilde{\nabla}^C$ if and only if TM satisfies*

$$\begin{aligned}\tilde{R}^C((\phi\lambda_1)^C,(\phi\lambda_2)^C)(\phi\lambda_3)^C,(\phi\lambda_4)^C &= -\frac{4}{2n-2}\{g^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)g^V((\phi\lambda_1)^C,(\phi\lambda_4)^C)\\ &+ g^V((\phi\lambda_2)^C,(\phi\lambda_3)^C)g^C((\phi\lambda_1)^C,(\phi\lambda_4)^C)\\ &- g^C((\phi\lambda_1)^C,(\phi\lambda_3)^C)g^V((\phi\lambda_2)^C,(\phi\lambda_4)^C\\ &- g^V((\phi\lambda_1)^C,(\phi\lambda_3)^C)g^C((\phi\lambda_2)^C,(\phi\lambda_4)^C)\},\end{aligned}\tag{5.11}$$

$\forall\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C\in\Im_0^1(TM)$.

Proof: Let M be an n-dimensional Sasakian manifold and TM its tangent bundle. From (5.1), we get

$$\begin{aligned}g^C(\tilde{K}^C((\phi\lambda_1)^C,(\phi\lambda_2)^C)(\phi\lambda_3)^C,(\phi\lambda_4)^C) &= g^C(\tilde{R}^C((\phi\lambda_1)^C,(\phi\lambda_2)^C)(\phi\lambda_3)^C,(\phi\lambda_4)^C)\\ &- \frac{1}{2n-2}\{\tilde{S}^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)g^V((\phi\lambda_1)^C,(\phi\lambda_4)^C)\\ &+ \tilde{S}^V((\phi\lambda_2)^C,(\phi\lambda_3)^C)g^C((\phi\lambda_1)^C,(\phi\lambda_4)^C)\\ &- \tilde{S}^C((\phi\lambda_1)^C,(\phi\lambda_3)^C)g^V((\phi\lambda_2)^C,(\phi\lambda_4)^C\\ &- \tilde{S}^V((\phi\lambda_1)^C,(\phi\lambda_3)^C)g^C((\phi\lambda_2)^C,(\phi\lambda_4)^C)\\ &+ \tilde{S}^C((\phi\lambda_1)^C,(\phi\lambda_4)^C)g^V((\phi\lambda_2)^C,(\phi\lambda_3)^C)\\ &+ \tilde{S}^V((\phi\lambda_1)^C,(\phi\lambda_4)^C)g^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)\\ &- \tilde{S}^C((\phi\lambda_2)^C,(\phi\lambda_4)^Cg^V((\phi\lambda_1)^C,(\phi\lambda_3)^C)\\ &- \tilde{S}^V((\phi\lambda_2)^C,(\phi\lambda_4)^C)g^C((\phi\lambda_1)^C,(\phi\lambda_3)^C)\},\end{aligned}\tag{5.12}$$

$\forall\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C\in\Im_0^1(TM)$.

Let $\{e_i^C,\xi^C\}$ be a local orthonormal basis on TM where $\{e_i,\xi\}$ is a local orthonormal basis on M, from (5.12) it follows that

$$\begin{aligned}g^C(\tilde{K}^C((\phi\lambda_1)^C,(\phi\lambda_2)^C)(\phi\lambda_3)^C,(\phi\lambda_4)^C) &= g^C(\tilde{R}^C((\phi\lambda_1)^C,(\phi\lambda_2)^C)(\phi\lambda_3)^C,(\phi\lambda_4)^C)\\ &- \frac{1}{2n-2}\{\tilde{S}^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)g^V((\phi\lambda_1)^C,(\phi\lambda_4)^C)\\ &+ \tilde{S}^V((\phi\lambda_2)^C,(\phi\lambda_3)^C)g^C((\phi\lambda_1)^C,(\phi\lambda_4)^C)\\ &- \tilde{S}^C((\phi\lambda_1)^C,(\phi\lambda_3)^C)g^V((\phi\lambda_2)^C,(\phi\lambda_4)^C\\ &- \tilde{S}^V((\phi\lambda_1)^C,(\phi\lambda_3)^C)g^C((\phi\lambda_2)^C,(\phi\lambda_4)^C)\\ &+ \tilde{S}^C((\phi\lambda_1)^C,(\phi\lambda_4)^C)g^V((\phi\lambda_2)^C,(\phi\lambda_3)^C)\\ &+ \tilde{S}^V((\phi\lambda_1)^C,(\phi\lambda_4)^C)g^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)\\ &- \tilde{S}^C((\phi\lambda_2)^C,(\phi\lambda_4)^Cg^V((\phi\lambda_1)^C,(\phi\lambda_3)^C)\\ &- \tilde{S}^V((\phi\lambda_2)^C,(\phi\lambda_4)^C)g^C((\phi\lambda_1)^C,(\phi\lambda_3)^C)\}\end{aligned}\tag{5.13}$$

Making use of (3.14), (4.41), (4.42) and (4.43), the equation (5.13) becomes

$$\begin{aligned}g^C(\tilde{K}^C((\phi\lambda_1)^C,(\phi\lambda_2)^C)(\phi\lambda_3)^C,(\phi\lambda_4)^C) &= \tilde{S}^C(\lambda_2^C,\lambda_3^C)-2g^C(\lambda_2^C,\lambda_3^C)\\ &- 2(2n-2)\{\eta^C(\lambda_2^C)\eta^V(\lambda_3^C)+\eta^V(\lambda_2^C)\eta^C(\lambda_3^C)\}\\ &- \frac{1}{2n-2}\{(2n-3)\tilde{S}^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)\\ &+ (\tilde{r}^C-2(2n-1))g^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)\}\end{aligned}\tag{5.14}$$

$\forall\lambda_2^C,\lambda_3^C\in\Im_0^1(M)$.

If TM is ϕ^C-conharmonically flat concerning the connection $\tilde{\nabla}^C$, using (2.23), (4.26) and (3.5) in (5.14) we get

$$\begin{aligned}\tilde{S}^C(\lambda_2^C,\lambda_3^C) &= (\tilde{r}^C-2)g^C(\lambda_2^C,\lambda_3^C)-(4n-\tilde{r})\{\eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\\ &+ \eta^V(\lambda_2^C)\eta^C(\lambda_3^C)\},\end{aligned} \tag{5.15}$$

$\forall\lambda_2^C,\lambda_3^C\in\Im_0^1(TM)$.

Putting $\lambda_4=\xi$ in (5.15) and using (4.26) and $\eta^C(\xi^C)=1$, we get $\tilde{r}=0$ and consequently (5.15) reduces to (5.8). By replacing λ_1 by $\phi\lambda_1$ and λ_2 by $\phi\lambda_2$ in (5.8) one can get $\tilde{S}^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)=-2g^C((\phi\lambda_2)^C,(\phi\lambda_3)^C)$ for $\forall\lambda_2^C,\lambda_3^C\in\Im_0^1(TM)$. Now using this value in (5.12) with (2.22), we obtain (5.11). The converse is obvious.

Theorem 5.4 *Let M be an n-dimensional $(n>3)$ Sasakian manifold and TM its tangent bundle. Then the following statements are equivalent:*
i. TM is conharmonically flat concerning the connection $\tilde{\nabla}^C$.
ii. TM is ϕ^C-conharmonically flat concerning the connection $\tilde{\nabla}^C$.
iii. The curvature tensor concerning the connection $\tilde{\nabla}^C$ of TM is given by

$$\begin{aligned}\tilde{R}^C(\lambda_1^C,\lambda_2^C)\lambda_3^C &= -\frac{4}{2n-2}\{g^C(\lambda_2^C,\lambda_3^C)\lambda_1^C+g^V(\lambda_2^C,\lambda_3^C)\lambda_1^C\\ &- g^C(\lambda_1^C,\lambda_3^C)\lambda_2^C-g^V(\lambda_1^C,\lambda_3^C)\lambda_2^C\}\\ &+ \frac{4n}{2n-2}\{\eta^C(\lambda_2^C)\xi^C g^V(\lambda_1^C,\lambda_3^C)+\eta^C(\lambda_2^C)\xi^V g^C(\lambda_1^C,\lambda_3^C)\\ &+ \eta^V(\lambda_1^C)\xi^C g^V(\lambda_2^C,\lambda_3^C)-\eta^C(\lambda_1^C)\xi^C g^V(\lambda_2^C,\lambda_3^C)\\ &+ \eta^C(\lambda_1^C)\xi^V g^C(\lambda_2^C,\lambda_3^C)+\eta^V(\lambda_1^C)\xi^C g^V(\lambda_2^C,\lambda_3^C)\\ &- \eta^V(\lambda_2^C)\eta^C(\lambda_3^C)\lambda_1^C-\eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\lambda_1^C\\ &- \eta^C(\lambda_2^C)\eta^C(\lambda_3^C)\lambda_1^V+\eta^V(\lambda_1^C)\eta^C(\lambda_3^C)\lambda_2^C\\ &- \eta^C(\lambda_1^C)\eta^V(\lambda_3^C)\lambda_2^C\},\end{aligned} \tag{5.16}$$

$\forall\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C\in\Im_0^1(TM)$.

Proof. Let M be an n-dimensional $(n>3)$ Sasakian manifold and TM its tangent bundle. From (2.21) and (2.23), obviously (i) $\Rightarrow$ (ii). Now, Let (ii) be true. In view of (4.20) and (4.21) we can verify

$$\begin{aligned}\tilde{R}^C(\phi^2\lambda_1^C,\phi^2\lambda_2^C,\phi^2\lambda_3^C,\phi^2\lambda_4^C) &= \tilde{R}^C(\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C)+2\{g^V(\lambda_1^C,\lambda_3^C)\eta^C(\lambda_2^C)\eta^C(\lambda_4^C)\\ &+ g^C(\lambda_1^C,\lambda_3^C)\eta^V(\lambda_2^C)\eta^C(\lambda_4^C)+g^C(\lambda_1^C,\lambda_3^C)\eta^C(\lambda_2^C)\eta^V(\lambda_4^C)\\ &- g^V(\lambda_2^C,\lambda_3^C)\eta^C(\lambda_1^C)\eta^C(\lambda_4^C)-g^C(\lambda_2^C,\lambda_3^C)\eta^V(\lambda_1^C)\eta^C(\lambda_4^C)\\ &- g^C(\lambda_2^C,\lambda_3^C)\eta^C(\lambda_1^C)\eta^V(\lambda_4^C)+g^V(\lambda_2^C,\lambda_4^C)\eta^C(\lambda_1^C)\eta^C(\lambda_3^C)\\ &+ g^C(\lambda_2^C,\lambda_4^C)\eta^V(\lambda_1^C)\eta^C(\lambda_3^C)+g^C(\lambda_2^C,\lambda_4^C)\eta^C(\lambda_1^C)\eta^V(\lambda_3^C)\\ &- g^V(\lambda_1^C,\lambda_4^C)\eta^C(\lambda_2^C)\eta^C(\lambda_3^C)-g^C(\lambda_1^C,\lambda_4^C)\eta^V(\lambda_2^C)\eta^C(\lambda_3^C)\\ &- g^C(\lambda_1^C,\lambda_4^C)\eta^C(\lambda_2^C)\eta^V(\lambda_3^C)\},\end{aligned} \tag{5.17}$$

$\forall\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C\in\Im_0^1(TM)$.

By replacing $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ to $\phi\lambda_1,\phi\lambda_2,\phi\lambda_3,\phi\lambda_4$, respectively in (5.11) and using (5.17) we obtain (5.16). Hence, (ii) $\Rightarrow$ (iii). Also, let (iii) be true. On contracting (5.16) it follows (5.8). Using (5.8) and (5.16) in (2.20) we get (i).

This completes the proof.

Theorem 5.5 *Let M be an n-dimensional $(n>3)$ η-Einstein manifold and TM its tangent bundle. Then TM is ϕ^C-conharmonically flat with repsect to the quarter-symmetric metric connection $\tilde{\nabla}^C$.*

Proof. Suppose M be an n-dimensional η-Einstein manifold and TM its tangent bundle. Then there exists functions α and β such that

$$\tilde{S}^C(\lambda_1^C,\lambda_2^C)=g^C((\tilde{Q}\lambda_1)^C,\lambda_2^C)=g^C(\lambda_1^C,\lambda_2^C)+\beta\eta^C(\lambda_1^C)\eta^V(\lambda_2^C)+\beta\eta^V(\lambda_1^C)\eta^V(\lambda_2^C). \tag{5.18}$$

From (4.38), we also get

$$\alpha+\beta=2(n-1). \tag{5.19}$$

On the other hand, the scalar curvature concerning the connection $\tilde{\nabla}^C$ satisfies:

$$\tilde{r}^C=\sum_{i=1}^{n}\tilde{S}^C(\lambda_1^C,\lambda_2^C)=n\alpha+\beta. \tag{5.20}$$

By virtue of (5.19) and (5.20), we have from (5.18) that

$$\begin{aligned}\tilde{S}^C(\lambda_1^C,\lambda_2^C) &= \left(\frac{\tilde{r}^C}{2n-1}-2\right)g^C(\lambda_1^C,\lambda_2^C)\\ &+ \left(4n-\frac{\tilde{r}^C}{2n-1}\right)\{\eta^C(\lambda_1^C)\eta^V(\lambda_2^C)+\eta^V(\lambda_1^C)\eta^V(\lambda_2^C)\}.\end{aligned} \tag{5.21}$$

Let $\{e_i^C,\xi^C\}$ be a local orthonormal basis on TM where $\{e_i,\xi\}$ is a local orthonormal basis on M. Then from (5.21) we get $\tilde{r}=0$ and consequently (5.21) reduces to (5.8). By taking account of (4.20) and (5.21) in formula (2.20) we get the required result.

Theorem 5.6 *Let M be a Sasakian manifold and TM its tangent bundle. TM is ϕ^C-conharmonically flat concerning the quarter-symmetric metric connection $\tilde{\nabla}^C$ if and only if TM is an η^C-Einstein manifold concerning the connection $\tilde{\nabla}^C$.*

Proof. We just need to prove that a ϕ^C-conharmonically flat Sasakian manifold concerning the connection $\tilde{\nabla}^C$ is an η^C-Einstein manifold concerning the connection $\tilde{\nabla}^C$. The converse follows from Theorem (3.7).

For a ϕ^C-conharmonically flat Sasakian manifold concerning the connection $\tilde{\nabla}^C$ and by (2.20) and $g^C(\lambda_1^C,\xi^C)=\eta^C(\lambda_1^C)$, we have

$$\begin{aligned}g^C(\tilde{K}^C(\lambda_1^C,\lambda_2^C)\xi^C,\lambda_4^C) &= g^C(\tilde{R}^C(\lambda_1^C,\lambda_2^C)\xi^C,\lambda_4^C)\\ &- \frac{1}{2n-2}\{\tilde{S}^V(\lambda_2^C,\xi^C)g^C(\lambda_1^C,\lambda_4^C)+\tilde{S}^C(\lambda_2^C,\xi^C)g^V(\lambda_1^C,\lambda_4^C)\\ &- \tilde{S}^V(\lambda_1^C,\xi^C)g^C(\lambda_2^C,\lambda_4^C)-\tilde{S}^C(\lambda_1^C,\xi^C)g^V(\lambda_2^C,\lambda_4^C)\\ &+ \eta^C(\lambda_2^C)\tilde{S}^V(\lambda_1^C,\lambda_4^C)+\eta^V(\lambda_2^C)\tilde{S}^C(\lambda_1^C,\lambda_4^C)\\ &+ \eta^C(\lambda_1^C)\tilde{S}^V(\lambda_2^C,\lambda_4^C)+\eta^V(\lambda_1^C)\tilde{S}^C(\lambda_2^C,\lambda_4^C)\},\end{aligned} \tag{5.22}$$

$\forall\lambda_1^C,\lambda_2^C,\lambda_3^C,\lambda_4^C\in\Im_0^1(TM)$.

Let $\{e_i^C,\xi^C\}$ be a local orthonormal basis on TM where $\{e_i,\xi\}$ is a local orthonormal basis on M From (5.22) we get

$$\begin{aligned}g^C(\tilde{K}^C(e_i^C,\lambda_2^C)\xi^C,e_i^C) &= g^C(\tilde{R}^C(e_i^C,\lambda_2^C)\xi^C,e_i^C)\\ &- \frac{1}{2n-2}\{\tilde{S}^V(\lambda_2^C,\xi^C)g^C(e_i^C,e_i^C)+\tilde{S}^C(\lambda_2^C,\xi^C)g^V(e_i^C,e_i^C)\\ &- \tilde{S}^V(e_i^C,\xi^C)g^C(\lambda_2^C,e_i^C)-\tilde{S}^C(e_i^C,\xi^C)g^V(\lambda_2^C,e_i^C)\\ &+ \eta^C(\lambda_2^C)\tilde{S}^V(e_i^C,e_i^C)+\eta^V(\lambda_2^C)\tilde{S}^C(e_i^C,e_i^C)\\ &+ \eta^C(e_i^C)\tilde{S}^V(\lambda_2^C,e_i^C)+\eta^V(e_i^C)\tilde{S}^C(\lambda_2^C,e_i^C)\},\end{aligned} \tag{5.23}$$

$\forall\lambda_2^C\in\Im_0^1(TM)$.

If TM is ϕ^C-conharmonically flat concerning the connection $\tilde{\nabla}^C$ and using (2.23), (3.14), (4.36) and (4.42) in the above equation, we get $\tilde{r} = 0$. Now putting $\lambda_2 = \xi$ in (5.23) and using (3.1)-(3.3) and (4.26) we have

$$\begin{aligned} g^C(\tilde{K}^C(\lambda_1^C, \xi^C)\xi^C, \lambda_4^C) &= g^C(\tilde{R}^C(\lambda_1^C, \xi^C)\xi^C, \lambda_4^C) \\ &- \frac{1}{2n-2}\{\tilde{S}^C(\lambda_1^C, \lambda_4^C) - 4(n-1)(\eta^C(\lambda_1^C)\eta^V(\lambda_4^C) \\ &+ \eta^V(\lambda_1^C)\eta^C(\lambda_4^C)\}, \end{aligned} \tag{5.24}$$

$\forall \lambda_2, \lambda_4 \in \Im_0^1(M)$. Using (2.22) and (4.22) in the above equation, we get (5.8).

Conflict of interest

The authors declare no conflicts of interest in this paper.

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