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Power centralizing semiderivations of Lie ideals in prime rings *

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ABSTRACT: If a semiderivation $\mathcal{F}$ with associated automorphism ξ is induced on a non-central Lie ideal $\mathcal{L}$ of $\mathfrak{A}$ such that $[\mathcal{F}(\eta),\eta]^n \in \mathcal{Z}(\mathcal{R})$, where n is a fixed positive integer, and $\eta \in \mathcal{L}$, then it has been proven that either $Char(\mathfrak{A}) = 0$ or $Char(\mathfrak{A}) > n+1$ then $\mathfrak{A}$ satisfies a standard identity in 4 variables usually denoted by s_4.

Key Words: Prime ring, automorphism, Lie ideal, semiderivation.

Contents

1. Introduction

Throughout this paper, $\mathfrak{A}$ will represent a prime ring with centre $\mathcal{Z}(\mathfrak{A})$, and its Martindale quotient ring will be denoted by $\mathcal{Q}$ with its extended centroid $\mathcal{C}$(for further details see [2]). Also, It is pertinent to mention that for any $\eta,\omega \in \mathfrak{A}$, the commutator of η and ω will be $[\eta,\omega] = \eta\omega - \omega\eta$. It is also worthy to mention that a ring $\mathfrak{A}$ is called a prime ring if $\mathtt{a}\mathfrak{A}\mathtt{b} = 0$ always implies that either $\mathtt{a} = 0$ or $\mathtt{b} = 0$. Again, let $\mathbb{A}$ and $\mathbb{B}$ be any two proper subsets of $\mathfrak{A}$, then the symbol $[\mathbb{A},\mathbb{B}]$ will stand for the additive subgroup generated by $[\mathtt{x},\mathtt{y}]$ with $\mathtt{x} \in \mathbb{A}$ and $\mathtt{y} \in \mathbb{B}$. Also by a Lie Ideal we mean an additive subgroup $\mathcal{L}$ of $\mathfrak{A}$ such that if $[\eta,r] \in \mathcal{L}$ for all $\eta \in \mathcal{L}$ and $r \in \mathfrak{A}$. A mapping $g : \mathfrak{A} \to \mathfrak{A}$ is commuting (resp. centralizing) on a subset $\mathcal{S}$ of $\mathfrak{A}$ if $[g(\eta),\eta] = 0$ (resp. $[g(\eta),\eta] \in \mathcal{Z}(\mathcal{R})$) for all $\eta \in \mathcal{S}$. Analogously, a mapping $g : \mathfrak{A} \to \mathfrak{A}$ is power-commuting (resp. power-centralizing) on a subset $\mathcal{S}$ of $\mathfrak{A}$ if $[g(\eta),\eta]^n = 0$ (resp. $[g(\eta),\eta]^n \in \mathcal{Z}(\mathcal{R})$) for all $\eta \in \mathcal{S}$ and $n \geq 1$. An additive mapping $\mathcal{D} : \mathfrak{A} \to \mathfrak{A}$ is a derivation on $\mathfrak{A}$, if $\mathcal{D}(\eta\omega) = \mathcal{D}(\eta)\omega + \eta\mathcal{D}(\omega)$ holds for all $\eta,\omega \in \mathfrak{A}$.

In [3], semiderivation was introduced by Bergen. An additive mapping $\mathcal{F} : \mathfrak{A} \to \mathfrak{A}$ is a semiderivation associated with a mapping $\mathcal{G} : \mathfrak{A} \to \mathfrak{A}$, whenever

$$\mathcal{F}(\eta\omega) = \mathcal{F}(\eta)\mathcal{G}(\omega) + \eta\mathcal{F}(\omega) = \mathcal{F}(\eta)\omega + \mathcal{G}(\eta)\mathcal{F}(\omega)$$

and $\mathcal{F}(\mathcal{G}(\eta)) = \mathcal{G}(\mathcal{F}(\eta))$ holds for all $\eta,\omega \in \mathfrak{A}$. For $\mathcal{G} = 1_{\mathfrak{A}}$, the identity map on $\mathfrak{A}$, $\mathcal{F}$ is clearly a derivation. In prime rings, the only semiderivation according to Brešer are ordinary derivations and mappings of the form $\mathcal{F}(\eta) = \gamma(\eta - \mathcal{G}(\eta))$, where $\zeta \in \mathcal{C}$ and $\mathcal{G}$ is an endomorphism.

E.C. Posner studied prime ring centralizing derivations and established that if $\mathfrak{A}$ is a prime ring and $\mathcal{D}$ is a non-zero derivation of $\mathfrak{A}$ such that $[\mathcal{D}(\eta),\eta] \in \mathcal{Z}(\mathcal{R})$, then $\mathfrak{A}$ is commutative for all $\eta \in \mathfrak{A}$. Then, Lanski extended Posner's result to Lie ideals. [13]. In [5] Carini and De Filippis studied the power centralizing derivations on Lie ideals of prime rings. More precisely, they proved that if $\mathfrak{A}$ is a prime ring with $char(\mathfrak{A})$ $\neq 2$, $\mathcal{L}$ a non-central Lie ideal of $\mathfrak{A}$ and $\mathcal{D}$ a non-zero derivation of $\mathfrak{A}$ such that $[\mathcal{D}(\eta),\eta]^n \in \mathcal{Z}(\mathcal{R})$ for all $\eta \in \mathcal{L}$, then $\mathfrak{A}$ satisfies s_4. Huang [10] investigated the semiderivations of Lie ideals with power values in prime rings. To be more precise, he obtained the following result.

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Theorem 1.1 *Let $\mathfrak{A}$ be a prime ring, $\mathcal{L}$ a non-central Lie ideal, $\mathcal{F}$ a non-zero semiderivation associated with an automorphism ξ such that $\mathcal{F}(\eta)^n = 0$ for all $\eta \in \mathcal{L}$, where n is a fixed integer. If either $char(\mathfrak{A}) > n+1$ or $char(\mathfrak{A}) = 0$, then $\mathfrak{A}$ satisfies s_4, the standard identity in 4 variables.*

Motivated by these results, we continue this line of investigation by studying the power centralizing semiderivations on Lie ideals. In fact, we arrived at the following conclusion:

Theorem 1.2 *Let $\mathfrak{A}$ be a prime ring, $\mathcal{L}$ a non-central Lie ideal of $\mathfrak{A}$ and $\mathcal{F}$ be a semiderivation with associated automorphism ξ such that*

$$[\mathcal{F}(\eta), \eta]^n \in \mathcal{Z}(\mathcal{R}), \quad where\ n\ is\ a\ fixed\ positive\ integer\ .$$

If either $Char(\mathfrak{A}) = 0$ or $Char(\mathfrak{A}) > n+1$, then $\mathfrak{A}$ satisfies s_4.

2. Preliminaries

We will go through a few introductory concepts that are essential for the proof of the main theorem. If there is an invertible element $q \in \mathcal{Q}$ such that $\xi(\eta) = q\eta q{-}1\ \forall\ \eta \in \mathfrak{A}$, the automorphism ξ is called $\mathcal{Q}$-inner. Also, the standard identity s_4 in four variables is defined as:

$$s_4 = \sum (-1)^{\mu} \mathtt{X}_{\mu(1)} \mathtt{X}_{\mu(2)} \mathtt{X}_{\mu(3)} \mathtt{X}_{\mu(4)}$$

where $(-1)^{\mu}$ is a sign of permutation μ of the symmetric group of degree 4.
The following results are important in the development of the proof of our main theorem:

Fact 2.1 *(See [8, Theorem 2]) Let $\mathfrak{A}$ be a prime ring of characteristic different from 2, $\mathcal{I}$ a non-zero ideal of $\mathfrak{A}$ and $n \geq 1$ a fixed positive integer, such that*

$$\Big[[r_1, r_2], [r_3, r_4]\Big]^n \in \mathcal{Z}(\mathcal{R})$$

for any $r_1, r_2, r_3, r_4 \in \mathcal{I}$, then $\mathfrak{A}$ satisfies s_4.

Fact 2.2 *Let $\mathfrak{A}$ be a prime ring and $\mathcal{I}$ a two sided ideal of $\mathfrak{A}$. Then $\mathcal{I}$, $\mathfrak{A}$, $\mathcal{Q}$ satisfy the same generalized polynomial identities with coefficients in $\mathcal{Q}$ (see [7]). Furthermore, $\mathcal{I}$, $\mathfrak{A}$ and $\mathcal{Q}$ satisfy the same generalized polynomial identities with automorphisms(see [6, Theorem 1]).*

3. Main Results

Proof of the Theorem 1.2 First, we must remember that if ξ is an identity map on $\mathfrak{A}$, then $\mathcal{F}$ is nothing more than a derivation and thus by [5], our conclusion follows. As a result, we proceed with the assumption that ξ is not an identity map on $\mathfrak{A}$. In this case, there exists a non-zero two-sided ideal $\mathcal{I}$ of $\mathfrak{A}$ such that $0 \neq [\mathcal{I}, \mathfrak{A}] \subseteq \mathcal{L}$. In particular, $[\mathcal{I}, \mathcal{I}] \subseteq \mathcal{L}$. Hence, without generality, we can assume that $\mathcal{L} = [\mathcal{I}, \mathcal{I}] \subseteq \mathcal{L}$. Hence in view of Bršar [4], $\mathcal{F}(\eta) = \gamma(\eta - \xi(\eta))$ for all $\eta \in \mathfrak{A}$, where $0 \neq \gamma \in \mathcal{C}$. As a result of our hypothesis, we have

$$\Big[\gamma[\eta, \omega] - \gamma\xi[\eta, \omega], [\eta, \omega]\Big]^n \in \mathcal{Z}(\mathcal{R}), \text{ for all } \eta, \omega \in \mathcal{Q}$$

Since $0 \neq \gamma \in \mathcal{C}$, the above identity reduces to

$$[\xi[\eta, \omega], [\eta, \omega]]^n \in \mathcal{Z}(\mathcal{R}), \text{ for all } \eta, \omega \in \mathcal{Q} \tag{3.1}$$

In the light of Kharchenko's theorem [12], we divide the proof into the following two cases.
Case 1. Let ξ be $\mathcal{Q}$-outer automorphism. Since $char(\mathfrak{A}) > n+1$ or $char(\mathfrak{A}) = 0$, by Chuang [6], it follows that

$$[[r, s], [\eta, \omega]]^n \in \mathcal{Z}(\mathcal{R}), \text{ for all } \eta, \omega, r, s \in \mathcal{I}$$

Hence by using the fact 2.1, we get the desired conclusion.
Case 2. Suppose that ξ is $\mathcal{Q}$-inner automorphism. Thus there exits an element $q \in \mathcal{Q} - \mathcal{C}$ such that $\xi(\eta) = q\eta q^{-1}$ for all $\eta \in \mathfrak{A}$, for all $\eta \in \mathfrak{A}$. Thus

$$\left[\left[[q\eta q^{-1}, q\omega q^{-1}], [\eta, \omega]\right]^n, r\right] = 0 \tag{3.2}$$

is a generalized polynomial identity for $\mathfrak{A}$. Since $\mathfrak{A}$ and $\mathcal{Q}$ satisfy the same generalized polynomial identities with coefficients in $\mathcal{Q}$ (by the Fact 2.2), it follows that (3.2) is a generalized polynomial identities for $\mathcal{Q}$. In case $\mathcal{C}$ is infinite, we have $\mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ satisfies (3.2), where $\overline{\mathcal{C}}$ is algebraic closure of $\mathcal{C}$. Since both $\mathcal{Q}$ and $\mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ are centrally closed (Theorems 2.5 and 3.5 in [9]), we may replace $\mathfrak{A}$ by $\mathcal{Q}$ or $\mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ according as $\mathcal{C}$ is finite or infinite. Thus, we can consider the case when $\mathfrak{A}$ is centrally closed over $\mathcal{C}$, which is either finite or algebraically closed, without losing generality. By Martindale's theorem [14], $\mathfrak{A}$ is a primitive ring having a non-zero socle with $\mathcal{C}$ as the associated division ring. In the light of Jacobson theorem (page 75 in [11]) $\mathfrak{A}$ is isomorphic to a dense ring of linear transformations on some vector space $\mathcal{V}$ over $\mathcal{C}$. Let ${}_{\mathfrak{A}}\mathcal{V}$ be faithful irreducible left $\mathfrak{A}$ module with division ring $\mathcal{D} = End({}_{\mathfrak{A}}\mathcal{V})$. Since $\mathcal{C}$ is finite and algebraically closed, $\mathcal{D}$ must coincide with $\mathcal{C}$. By the density theorem $\mathfrak{A}$ acts dense on ${}_{\mathcal{D}}\mathcal{V}$. Now in order to prove our result, we have to show that $dim_{\mathcal{D}}\mathcal{V} \leq 2$
Suppose on the contrary we assume that $dim_{\mathcal{D}}\mathcal{V} \geq 3$. Thus there exists some $w \in \mathcal{V}$ such that $w, qv, q^{-1}v$ are linearly $\mathcal{D}$-independent.
By the density of $\mathfrak{A}$, there exists $\eta, \omega, r \in \mathcal{I}$ such that

$$\eta v = v,\ \omega v = v,\ \eta q^{-1}v = 0,\ \omega q^{-1}v = v, rv = 0$$

$$\eta qv = 0,\ \omega qv = w,\ \eta w = w,\ rw = v$$

Thus by (3.2), we get a contradiction:

$$0 = \left[\left[[q\eta q^{-1}, q\omega q^{-1}], [\eta, \omega]\right]^n, r\right]v = v \neq 0.$$

Hence $dim_{\mathcal{D}}\mathcal{V} \leq 2$ and we are done. The proof of the theorem is now complete.

As an application of our result, we can prove the following theorem in prime case. We omit the proof for the sake of brevity.

Theorem 3.1 *Let $\mathfrak{A}$ be a prime ring, $\mathcal{I}$ a non-zero ideal, $\mathcal{F}$ a non-zero semiderivation with associated automorphism ξ such that $\left[\mathcal{F}([\eta, \omega]), [\eta, \omega]\right]^n \in \mathcal{Z}(\mathcal{R})$ for all $\eta, \omega \in \mathcal{I}$, where n is a fixed positive integer, If either $char(\mathfrak{A}) > n + 1$ or $char(\mathfrak{A}) = 0$, then $\mathfrak{A}$ is commutative.*

Finally, we close our discussion, with the following example which shows that the assumption $\mathfrak{A}$ to be prime is not superfluous in the Theorem 3.1

Example 3.1 Let $\mathbb{Z}$ be the ring of integers. Take
$\mathfrak{A} = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \middle|\ u, v, w \in \mathbb{Z} \right\}$ and $\mathcal{I} = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle|\ u, v \in \mathbb{Z} \right\}$
Next, we define a mapping $\mathcal{F} : \mathfrak{A} \to \mathfrak{A}$ by

$$\mathcal{F}\begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2u & v \\ 0 & 0 & 2w \\ 0 & 0 & 0 \end{pmatrix}$$

Thus it is easy to see that $\mathcal{I}$ is a non-zero ideal of $\mathfrak{A}$ and $\mathcal{F}$ is a non-zero semiderivation of $\mathfrak{A}$ such that $\left[\mathcal{F}([\eta, \omega]), [\eta, \omega]\right]^n \in \mathcal{Z}(\mathcal{R})$ for all $\eta, \omega \in \mathcal{I}$. However, $\mathfrak{A}$ is not commutative.

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