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A novel type of spaces satisfying the T_2-separation property and some related results

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ABSTRACT: In this paper, we introduce a new kind of spaces called a $\mathcal{T}$-partial G_b-metric space. In this space, the T_2-separation axiom is verified and many known spaces in the literature are extended. Moreover, we prove a related fixed point theorem and include an example that shows the validity of our results.

Key Words: Fixed point, $\mathcal{T}$-partial G_b-metric space, T_2-separation axiom.

Contents

1. Introduction

In 2009, Mustafa and Sims [7] introduced the concept of G-metric spaces and proved some fixed point results for mappings satisfying various contractive conditions on complete G-metric spaces. Recall that a G-metric $G : X \times X \times X \to \mathbb{R}^+$ is called symmetric if

$$G(x, y, y) = G(y, x, x), \tag{1.1}$$

for all $x, y \in X$.
In 2015, Agarwal et al. in [3] showed that the function $d^G(x, y) = G(x, y, y)$ generates a Hausdorff topology if and only if G is symmetric. So, in order to skip symmetry condition, the authors took the two symmetric equivalent functions $d_m^G(x, y) = \max\{G(x, y, y); G(y, x, x)\}$ and $d_s^G(x, y) = G(x, y, y) + G(y, x, x)$ on X. They proved that G-metric spaces are provided with a Hausdorff topology τ_G generated by d_m^G or by d_s^G. So, a natural question can be posed as follows:
• Can we generate a topology satisfying the T_2-separation axiom without using nor the symmetry condition on G neither any functions equivalent to d_m^G and d_s^G ?
On the other hand, Zand et al. [16] have introduced a new generalized metric space named a G_p-metric space as a generalization of both a G-metric space and a partial metric space introduced by Matthews [6]. It is important to note that in a G_p-metric space the corresponding generated topology loses the T_2-separation axiom. Consequently, a converging sequence may not necessarily possess a unique limit. This lack of uniqueness undermines the fundamental significance of the concept of a limit in calculus and mathematical analysis.
Bakhtin [4] and Czerwik [5] introduced the notion of a b-metric space as a generalization of a metric space. They proved the contraction mapping principle in this novel space. In the same line, Aghajani et al. [2] introduced the class of generalized b-metric spaces (in short: G_b-metric spaces), and then they presented some basic properties of these spaces and proved related remarkable theorems.
The authors in [1] introduced the concept of τ-distance functions in general topological spaces (X, τ). In these spaces, the well-known Banach fixed point theorem is given as follows:

Theorem 1.1 *([1]) Let (X, τ) be a Hausdorff topological space with a τ-distance p. Suppose that X is p-bounded and S-complete. Let $T : X \longrightarrow X$ be a mapping satisfying: there exists $k \in [0, 1)$ such that for all $x, y \in X$, we have $p(Tx, Ty) \leq kp(x, y)$.
Then T has a unique fixed point.*

We recall some facts which will be used in the sequel.

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Definition 1.1 ([1]) Let (X,τ) be a topological space and $p : X \times X \to [0,\infty)$ be a function. For any $\varepsilon > 0$ and any $x \in X$, let $B_p(x,\varepsilon) = \{y \in X : p(x,y) < \varepsilon\}$.

1. The function p is said to be τ-distance if for each $x \in X$ and any neighborhood V of x, there exists $\varepsilon > 0$ such that $B_p(x,\varepsilon) \subset V$.
2. A sequence $\{x_n\}$ in a Hausdorff topological space (X,τ) is a p-Cauchy if $\lim p(x_n, x_m) = 0$.
3. X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim p(x, x_n) = 0$.
4. X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim x_n = x$ with respect to τ.
5. X is said to be p-bounded if $\sup\{p(x,y) : x,y \in X\} < \infty$.

For more details on this topic, we recommend interested readers to consult the latest research articles [9,10,11,12,13,14,15].
The purpose of this paper is to introduce a new class of spaces known as $\mathcal{T}$-partial G_b-spaces by a variation of the definition of G_p-metric spaces. These spaces are presented as an extension of both G-metric spaces and G_b-metric spaces. In a comparison with G_p-metric spaces, our spaces are distinguished by:

- T_2-separation axiom is satisfied,
- self-distance of an arbitrary point need not be equal to zero (as in G_p-metric spaces).

Moreover, we establish a generalization in $\mathcal{T}$-partial G_b-spaces of Theorem 3 stated in [8] with the help of Theorem 1.1 such that:

- the function G can be not symmetric,
- without using compactness of the space X.

At the end of this work, we give a concrete example that illustrates the usability of our results.

2. Main results

At the beginning of this section, we introduce a new definition:

Definition 2.1 Let X be nonempty set and $s \geq 1$ a given real number. A function $G : X \times X \times X \to \mathbb{R}^+$ is a $\mathcal{T}$-partial G_b-metric on X if the following conditions hold:

1. $G(x,y,z) = G(x)$ or $G(x,y,z) = G(y)$ or $G(x,y,z) = G(z)$ then $x = y = z$,
2. $G(x,x,y) \leq G(x,y,z)$ for all $y \neq z$,
3. $sG(x) < G(x,y,z)$ for all $x \neq y$,
4. $G(x,y,z) = G(p(x,y,z))$, where p is a permutation of x,y,z, and
5. $G(x,y,z) \leq s[G(x,a,a) + G(a,y,z) - \min\{G(x), G(y)\}]$ for all $x,y,z,a \in X$, where $G(x) = G(x,x,x)$.

The pair (X,G) is called a $\mathcal{T}$-partial G_b-metric space with coefficient s.

It is clear that every G_b-metric space is a $\mathcal{T}$-partial G_b-metric space with $G(x) = 0$ for all $x \in X$. However, the converse of this fact need not hold, as we will present in the following examples:

Example 2.1 *Let (X, G_b) be a G_b-metric space of coefficient s. Then (X,G) is a $\mathcal{T}$-partial G_b-metric space of coefficient s for $G(x,y,z) = G_b(x,y,z) + \epsilon$, for all $x,y,z \in X$ with $\epsilon > 0$. Indeed:*
Let $x,y,z,a \in X$, we have

$$\begin{aligned} G(x,y,z) &= G_b(x,y,z) + \epsilon \\ &\leq s[G_b(x,a,a) + G_b(a,y,z)] + s\epsilon \\ &< s[(G_b(x,a,a) + \epsilon) + (G_b(a,y,z) + \epsilon) - \epsilon] \\ &= s[G(x,a,a) + G(a,y,z) - \min\{G(x), G(y)\}]. \end{aligned} \tag{2.1}$$

Hence, (X,G) is a $\mathcal{T}$-partial G_b-metric space. Moreover, X is not a G_b-metric, since $G(x) = \epsilon \neq 0$ for every $x \in X$.

The following are related topological notions of a $\mathcal{T}$-partial G_b-metric space:

Definition 2.2 Let (X, G) be a $\mathcal{T}$-partial G_b-metric, $x \in X$ and $\varepsilon > 0$.

1. $B_G(x, \varepsilon) = \{y \in X : G(x, y, y) < G(x) + \varepsilon\}$ is called the open ball with center x and radius ε.
2. A sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n,m\to\infty} G(x, x_n, x_m) = G(x)$.
3. A sequence $\{x_n\} \subset X$ is a Cauchy sequence if $\lim_{m,n\to\infty} G(x_n, x_m, x_m)$ exists and is finite.
4. X is complete if every Cauchy sequence $\{x_n\} \subset X$ converges to a point $x \in X$.
5. X is said to be bounded if $\sup\{G(x, y, z) : x, y, z \in X\} < \infty$.

Lemma 2.1 *Let (X, G) be a $\mathcal{T}$-partial G_b-metric space and $p : X \times X \to \mathbb{R}^+$ be a function defined by*

$$p(x, y) = e^{G(x,y,y)} - 1. \tag{2.2}$$

Then p is a τ_G-distance on X, where τ_G is the topology induced by G.

Proof: Let (X, τ_G) be the topological space with the topology τ_G and V an arbitrary neighborhood of an arbitrary $x \in X$, then there exists $\varepsilon > 0$ such that $B_G(x, \varepsilon) \subset V$, where $B_G(x, \varepsilon) = \{y \in X, G(x, y, y) < G(x) + \varepsilon\}$ is the open ball in (X, G).
It is easy to see that $B_p(x, e^\varepsilon - 1) \subset B_G(x, \varepsilon)$, indeed:
Let $y \in B_p(x, e^\varepsilon - 1)$, then $p(x, y) < e^\varepsilon - 1$, which implies that $e^{G(x,y,y)} < e^{G(x)+\varepsilon}$. Therefore, $G(x, y, y) < G(x) + \varepsilon$. □

Lemma 2.2 *Let (X, G) be a bounded $\mathcal{T}$-partial G_b-metric space, then (X, p) is a bounded topological space with the τ-distance p defined in Lemma 2.1.*

Lemma 2.3 *Let (X, G) be a complete $\mathcal{T}$-partial G_b-metric space, then (X, τ_G) is a S-complete topological space.*

Proof: Let $\{x_n\}$ be a p-Cauchy sequence, which implies that $\lim_{n,m} p(x_n, x_m) = 0$, and hence $\lim_{n,m} G(x_n, x_m, x_m) = 0$. Therefore, $\{x_n\} \subset (X, G)$ is a Cauchy sequence. Now, since (X, G) is complete, there exists $u \in X$ such that $\lim p(u, x_n) = 0$. □

Proposition 2.1 *A $\mathcal{T}$-partial G_b-metric on a nonempty X generates a Hausdorff topology τ_G on X with a base of the family of open balls $\{B_G(x, \varepsilon) : x \in X, \varepsilon > 0\}$.*

Proof: Let $x \neq y \in X$, denote $B_{x,\varepsilon} = B_G(x, \varepsilon)$ and $B_{y,\varepsilon} = B_G(y, \varepsilon)$ with $\varepsilon > 0$. We put $d_z := G(x, y, z) - s \max\{G(x), G(y)\} > 0$, where $z \in X$.
There exists an element $z_0 \in X$ such that:

$$B_{x,\frac{d_{z_0}}{2s}} \cap B_{y,\frac{d_{z_0}}{2s}} = \emptyset. \tag{2.3}$$

Indeed: If $a \in B_{x,\frac{d_z}{2s}} \cap B_{y,\frac{d_z}{2s}}$ for all $z \in X$, we have

$$\begin{aligned} G(x, y, a) &\leq s[G(x, a, a) + G(a, y, a) - \min\{G(x), G(y)\}] \\ &< s\left[G(x) + \frac{d_a}{2s} + G(y) + \frac{d_a}{2s} - \min\{G(x), G(y)\}\right] \\ &= s\left[\frac{G(x, y, a)}{s} + G(x) + G(y) - \min\{G(x), G(y)\} - \max\{G(x), G(y)\}\right] \\ &= G(x, y, a) + s[G(x) + G(y) - \min\{G(x), G(y)\} - \max\{G(x), G(y)\}]. \end{aligned} \tag{2.4}$$

It is readily apparent that:

$$G(x)+G(y)-\min\{G(x),G(y)\}-\max\{G(x),G(y)\}=0. \tag{2.5}$$

Therefore, we obtain $G(x,y,a)<G(x,y,a)$, which this is a contradiction. Furthermore, we have

$$x\in B_{x,\frac{d_{z_0}}{2s}} \text{ and } y\in B_{y,\frac{d_{z_0}}{2s}}. \tag{2.6}$$

The proof is finished. □

The main result of this work is the following:

Theorem 2.1 *Let $T:X\longrightarrow X$ be a mapping of a bounded complete $\mathcal{T}$-partial G_b-metric space (X,G) such that*

$$\inf_{x\neq y\in X}\{G(x,y,y)-G(Tx,Ty,Ty)\}>0. \tag{2.7}$$

Then T has a unique fixed point.

Proof: We set $\alpha=\inf\limits_{x\neq y\in X}\{G(x,y,y)-G(Tx,Ty,Ty)\}$. Hence, for all $x\neq y\in X$, we get

$$G(Tx,Ty,Ty)\leq G(x,y,y)-\alpha, \tag{2.8}$$

which implies that

$$e^{G(Tx,Ty,Ty)}\leq ke^{G(x,y,y)}, \tag{2.9}$$

for all $x\neq y\in X$ where $k=e^{-\alpha}<1$. Also,

$$p(Tx,Ty)\leq kp(x,y), \tag{2.10}$$

for all $x\neq y\in X$ where $k<1$ and p is the function defined in Lemma 2.1.
Finally, using Lemmas 2.1, 2.1, 2.2, 2.3 and Theorem 1.1, we conclude that T has a fixed point in X. □

Example 2.2 *Let $X=[0,1]\times[0,1]$, consider a mapping $T:X\to X$ defined by*

$$T(x_1,x_2)=(0,1-x_1). \tag{2.11}$$

Let us denote $\mathrm{X}=(x_1,x_2)$, $\mathrm{Y}=(y_1,y_2)$ and $\mathrm{Z}=(z_1,z_2)$ and define a function $G:X\times X\times X\to\mathbb{R}^+$ by

$$G(\mathrm{X},\mathrm{Y},\mathrm{Z})=\begin{cases}2+\frac{1}{9}\left(|x_1-y_1|+|x_1-z_1|+|y_1-z_1|\right)^2; & \mathrm{X}\neq\mathrm{Y},\\ 1+\frac{4}{9}|y_2-z_2|^2; & \mathrm{X}=\mathrm{Y}.\end{cases} \tag{2.12}$$

It is easy to see that G is a T-partial G-metric on X of coefficient $s=2$.
We have

$$G(\mathrm{X},\mathrm{Y},\mathrm{Y})-G(\mathrm{TX},\mathrm{TY},\mathrm{TY})\geq 1, \tag{2.13}$$

for all $\mathrm{X}\neq\mathrm{Y}$.
In other words, we obtain $\inf_{\mathrm{X}\neq\mathrm{Y}\in X}\{G(\mathrm{X},\mathrm{Y},\mathrm{Y})-G(\mathrm{TX},\mathrm{TY},\mathrm{TY})\}>0$. Hence, all assumptions of Theorem 2.1 are satisfied and T has the unique fixed point $\left(0,\frac{1}{2}\right)=T\left(0,\frac{1}{2}\right)$.

If we take $G(x)=0$, we obtain:

Corollary 2.1 *Let $T:X\longrightarrow X$ be a mapping of a bounded complete G_b-metric space (X,G_b) such that $\inf\limits_{x\neq y\in X}\{G_b(x,y,y)-G_b(Tx,Ty,Ty)\}>0$. Then T has a unique fixed point.*

For $G(x)=0$ and $s=1$, we get:

Corollary 2.2 *Let $T:X\longrightarrow X$ be a mapping of a bounded complete G-metric space (X,G) such that $\inf\limits_{x\neq y\in X}\{G(x,y,y)-G(Tx,Ty,Ty)\}>0$. Then T has a unique fixed point.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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