Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.68875

Fractional Revival on Integral Mixed Circulant Graphs

Rachana Soni*, Neelam Choudhary and Navneet Pratap Singh

ABSTRACT: Fractional revival on graphs can be utilized to transfer quantum information between set of distinct nodes in a quantum spin network. In this work, we prove existence of fractional revival on integral mixed circulant graphs. Spectral characterization and number-theoretic condition for integral mixed circulant graphs is proposed, to possess fractional revival between two distinct nodes, we also present examples in support of the theorem.

Key Words: Quantum state transfer, spectral graph theory, fractional revival, quantum walks.

Contents

1. Introduction

Quantum walk on quantum spin network, is a concept in quantum information science that generalizes classical random walks to the quantum domain. In classical random walks, a particle moves randomly between nodes of a graph, governed by a probability distribution. Contrary to classical random walks, quantum walks utilize principles of quantum mechanics to describe the transformation of a qubit states on a graph. Transfer of information in quantum system with highest fidelity, plays significant role in quantum walk.

Quantum walks come in different versions, each characterized by specific features in sense of time-evolution, quantum interference, and the underlying construction of quantum walk on a network. Two main categories of quantum walks exist. The quantum state's evolution takes discrete time steps in discrete time quantum walk. At every step, the walker undergoes a unitary transformation determined by product of a coin operator and a coin operator. In a continuous time quantum walk, the transformation of quantum state takes place continuously based on the solution Schrödinger equation. Transition of the quantum state over time is controlled by the time evolution operator. Here we are interested in studying continuous time quantum walks on a family of graphs. The evolution of quantum walker in continuous time quantum walk is driven by unitary matrix which is inspired from quantum mechanics part, given by [21],

$$\mathrm{U}(t) = \exp(itA),$$

in which, adjacency matrix A is working as Hamiltonian of the system. Hamiltonian describes the energy of the quantum network and governs the continuous transition of the qubit state. It includes information related to the connectivity of the graph. Laplacian matrix and normalized Laplacian matrix are also considered as the Hamiltonian in many published papers which are milestones in the study of quantum teleportation.

---

* Corresponding author

Submitted July 10, 2023. Published January 23, 2024
2010 *Mathematics Subject Classification:* 05C50, 81P45.

S. Bose [27] initiated the study of quantum state transfer in n qubit quantum network. PST (perfect state transfer) [11]- [13] is said to be happened between set two nodes if $(a,b)^{th}$ entry of the transition matrix is modulus unity or a graph X admits PST [22] between the nodes u and v at the time t if for a unit modulus complex number λ,

$$U(t)e_u = \lambda e_v,$$

where e_u denotes the standard basis vector with u^{th} entry 1 and λ is called phase. Graph X is called *periodic* graph [9] in sense of quantum state transfer, at the node u if $v = u$ in above equation. That is,

$$U(t)e_u = \lambda e_u.$$

Graph is stated to admit pretty good state transfer [9] at some time sequence $\{t_k\}$ of real numbers between the nodes u and v, such that,

$$\lim_{k\to\infty} U(t_k)e_u = \lambda e_v.$$

Fractional revival(FR) [2], [29] is not only analogous to the entanglement but also generalization of two well-studied transport phenomena- periodicity and perfect state transfer. Entanglement is an important transportation phenomenon in quantum information processing theory. Quantum computers and quantum communication protocols often exploit entanglement to perform tasks that are not achievable with classical systems.

If e_a and e_b represent standard basis vectors related to node a and node b, respectively, the graph X exhibits FR from the nodes a to b if [29],

$$\mathrm{U}(t)e_a = \alpha e_a + \beta e_b,$$

in which, α, β are some complex numbers with the condition that $|\alpha|^2 + |\beta|^2 = 1$.

Godsil [9], Cao and Luo [34] discussed perfect state transfer in Cayley graphs. Significant results related to fractional revival are mentioned below:

1. Characterization for undirected paths and cycles to exhibit FR can be seen in [3].
2. FR in infinite families of graphs utilizing association and hamming scheme is studied in [4].
3. Polygamy of FR and effect of graph operations on FR discussed in [5].
4. A systematic study via isospectral deformation of spin chain is given in [29]..
5. Fractional revival in undirected cayley graph characterized in [16], [30].

Graphs with no multi-edges and loops are considered in this paper. Representation $X = (V(X), E(X))$ depicts a graph with a set of nodes $V(X)$ and set of edges $E(X)$.

In oriented graph X, $(a,b) \in E(X)$ then $(b,a) \notin E(X)$. An edge is undirected with terminal nodes a and b if edges from the both directions (a,b) and (b,a) are in $E(X)$. Graph X is called mixed graph if edge set is union of directed and undirected edges.

Let $X = (Z_n, C)$ be a circulant graph [7] that includes both directed and undirected edges. The node set of X, denoted by $V(X)$, is the cyclic group Z_n, and its edge set, denoted by $E(X)$, follows the rule $(a,b) \in E(X)$ if and only if $b - a \in C$, where the connection set C excludes the identity element of the respective group. The graph X is considered a mixed circulant graph if it contains both types of edges, directed as well as undirected. If the Hermitian adjacency matrix A of the mixed circulant graph has all integer eigenvalues, then X is referred to as an integral mixed circulant graph. Monu Kadyan et al [23], has given conditions for integrality of mixed circulant graphs. Integrality of Cayley graphs is studied well in [1,28].

In the fields of mathematics and computer science, graphs with both directed and undirected edges are referred to as mixed graphs. A greater variety of relationships can be represented by mixed graphs than by either strictly directed or undirected graphs alone because to their hybrid form. Mixed graphs are important because they are versatile and may be used to a wide range of real-world situations where interactions between entities can be either unidirectional or bidirectional.

The use of mixed graphs in the study of graph algorithms is important because it offers a framework for creating more comprehensive solutions that can manage intricate restrictions. Applications for mixed graph algorithms include social network research, network design, routing, and other fields where the kind of links changes [14]. Circulant graph plays significant role in network design and communication [6].

Integral circulant graphs are proved to be a good candidate for quantum state transfer in undirected graphs [19], [20], [32]. Joining all these facts, integral mixed circulant graphs seemed a good candidate to explore possibility of existence of quantum fractional revival in it;

For the undirected abelian Cayley graphs, Cao et. al studied existence of fractional revival and they established spectral characterization and conditions for the graph admitting fractional revival between two nodes in Theorem 2.2 [30].

Generalizing the undirected graphs, in this paper, we present characterization for fractional revival in integral mixed circulant graphs with support of examples. Fractional revival in undirected graphs is studied broadly earlier, so we study fractional revival in mixed graph. Integral mixed circulant graphs seemed good candidate to explore possibility of existence of quantum fractional revival in it. Our problem to determine the time $t > 0$ and two distict nodes, a and b at which fractional revival appears on the graph.

2. Preliminaries

Concept of Cayley graph [25] was brought by mathematician Arther Cayley in 1878. A Cayley graph, denoted by $\mathrm{Cay}(G, C)$, over a finite group G with respect to a connection set C is defined as follows [24]:

- The nodes of the Cayley graph, correspond to the elements of group G.
- Two nodes g and h are connected with an edge if $hg^{-1} \in C$ that is, there is a directed edge from g to h. The connection set C is a subset of G exluding the identity element.

Formally, $\mathrm{Cay}(G, C) = (V, E)$ where:

- $V = G$, the set of nodes is the group itself,
- $E = \{(g, h) \mid hg^{-1} \in C \text{ for } g, h \in G\}$,

A circulant graph [7] is special case of Cayley graph over abelian group when the associated abelian group is cyclic group Z_n of order n. Integral circulant graph has adjacency matrix with all its eigenvalues being integers. The adjacency matrix A of an integral circulant graph is given by:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_0 \end{bmatrix}$$

One of beautiful properties of circulant matrix - it is diagonalizable by the orthogonal matrix V which has kth column in the form of [18],

$$V_k = \frac{1}{\sqrt{n}}[1, w_n^k, \dots, w_n^{(n-1)k}]^T.$$

In mixed integral circulant graphs, Hermitian-adjacency matrix is defined by [15]:

$$A = \begin{cases} 1 & \text{if } (p, q) \in E \ \&\ (q, p) \in E \\ i & \text{if } (p, q) \in E,\ (q, p) \notin E \\ -i & \text{if } (p, q) \notin E,\ (q, p) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Monu Kadyan et al. [23] has given results related to spectrum of mixed circulant graphs as follows.

Lemma 2.1 *[23] The eigenvalues of the mixed circulant graph $X = (Z_n, C)$ is $\{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$, where,*

$$\lambda_j = \gamma_j + \mu_j$$
$$\lambda_j = \sum_{k \in C \setminus \overline{C}} w_n^{jk} + i \sum_{k \in \overline{C}} \left(w_n^{jk} - w_n^{-jk}\right).$$

Results related to integral graphs can be found in [1], [8], [26], [28], [33].

A graph on n nodes which represents a quantum spin network of n qubits, denoted by X and Hermitian adjacency matrix as A here, in this work. The evolution operator $U(t)$ at the time t written by,

$$U(t) = \exp(itA) \tag{2.1}$$

Hermitian matrix A has the spectral decomposition,

$$A = \sum_{r=0}^{n-1} \lambda_r E_r \tag{2.2}$$

we can write,

$$U(t) = \sum_{r=0}^{n-1} \exp(it\lambda_r) E_r \tag{2.3}$$

Where E_r is the spectral idempotent or orthogonal projection matrix related to eigenvalue λ_r of Hermitian adjacency matrix A.

Definition 2.1 *[29] A graph X is said to admit fractional revival from node a to node b at time τ if*

$$\mathrm{U}(\tau)e_a = \alpha e_a + \beta e_b, \tag{2.4}$$

where $\alpha, \beta \in \mathbb{C}$, the set of complex numbers, with the condition, $|\alpha|^2 + |\beta|^2 = 1$.

Since the characteristic vectors e_a and e_b are orthonormal, we get from equation (2.4),

$$U(t)_{a,c} = \begin{cases} \alpha \text{ if } c = a, \\ \beta \text{ if } c = b, \\ 0, \quad otherwise. \end{cases} \tag{2.5}$$

If given graph exhibits fractional revival between nodes a and b then, from equation (2.5) and the condition $|\alpha|^2 + |\beta|^2 = 1$, we have,

$$|\mathrm{U}(t)_{aa}|^2 + |\mathrm{U}(t)_{ba}|^2 = 1. \tag{2.6}$$

We are going to exploit equation (2.6) to compose our results.

3. Results

Theorem 3.1 *Let $X = (Z_n, C)$ be an integral mixed circulant graph, where $C \subset Z_n$ such that $0 \notin C$ then, fractional revival occurs in the graph X between set of two distinct nodes a and b at some time $t > 0$ if following conditions retain:*

i) $(a - b)$ is even,

ii) There exist pairs of nodes $p, q \in M$ where $M = \{(p, q), p > q \text{ and } 2|(p - q)\}$, *iii)* $t \in 2\pi\mathbb{Z}/N$ *where* $N = \lambda_p - \lambda_q$, $(\lambda_p \neq \lambda_q)$,

iv) n *is multiple of* 4.

Proof: Order of integral mixed circulant graph is multiple of 4 so n is considered even number here [23]. Let A be the Hermitian adjacency matrix of integral mixed circulant graph $G = (Z_n, C)$ of order n. Eigenvalues of the graph is denoted by $SP_G = \{\lambda_0, \lambda_1 \ldots, \lambda_k\}$ and the corresponding orthonormal eigenvectors are $\{v_0, v_1, \ldots, v_k\}$. Then the $(a,b)^{th}$ entry of evolution operator of the quantum walk $U(t)$ of A can be written as

$$U(t)_{ab} = \frac{1}{n}\sum_{k=0}^{n-1} \exp(i\lambda_k t)\ \omega_n^{k(a-b)}, \tag{3.1}$$

from the left hand side of (2.6), we get,

$$|\ \mathrm{U}(t)_{aa}|^2 + |\mathrm{U}(t)_{ba}|^2 = \left|\frac{1}{n}\sum_{k=0}^{n-1}\exp(i\lambda_k t)\right|^2 + \left|\frac{1}{n}\sum_{k=0}^{n-1}\exp(i\lambda_k t)\ \omega_n^{k(b-a)}\right|^2$$

$$= \frac{1}{n^2}\left(\left(\sum_{k=0}^{n-1} exp(i\lambda_k t)\right)\left(\overline{\sum_{k=0}^{n-1} exp(i\lambda_k t)}\right) + \left(\sum_{k=0}^{n-1}\exp(i\lambda_k t)\ \omega_n^{k(a-b)}\right)\left(\overline{\sum_{k=0}^{n-1}\exp(i\lambda_k t)\ \omega_n^{k(b-a)}}\right)\right)$$

$$= \frac{1}{n^2}\left(\left(\sum_{k=0}^{n-1} exp(i\lambda_k t)\right)\left(\sum_{k=0}^{n-1} \overline{exp(i\lambda_k t)}\right) + \left(\sum_{k=0}^{n-1}\exp(i\lambda_k t)\ \omega_n^{k(a-b)}\right)\left(\sum_{k=0}^{n-1}\overline{\exp(i\lambda_k t)}\ \overline{\omega_n^{k(b-a)}}\right)\right)$$

$$= \frac{1}{n^2}\left(\sum_{\forall p,q\in G}\exp(i(\lambda_p-\lambda_q)t + \sum_{\forall p,q\in G}\exp(i(\lambda_p-\lambda_q)t)\exp\left((p-q)\frac{2\pi i}{n}(a-b)\right)\right)$$

$$= \frac{1}{n^2}\sum_{\forall p,q\in G}\left(\exp(i(\lambda_p-\lambda_q)t\left(1+\exp\left((p-q)\frac{2\pi i}{n}(b-a)\right)\right)\right)$$

$$= \frac{1}{n^2}\sum_{\forall p,q\in G}\exp\left(it\left(\sum_{k\in C\setminus\overline{C}} w_n^{pk} - w_n^{qk} + i\sum_{k\in\overline{C}}\left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right)\right)\right)\left(1 + w_n^{(p-q)(b-a)}\right)$$

Eigenvalues of mixed circulant graph G is given by $\lambda_p = \sum_{k\in C\setminus\overline{C}} w_n^{pk} + i\ \sum_{k\in\overline{C}}\left(w_n^{pk} - w_n^{-pk}\right)$ (accordingly λ_q) [23], where $w_n = \exp\dfrac{2\pi i}{n}$.

$$= \frac{1}{n^2}\sum_{\forall p,q\in G}\Bigg[\exp\left(it\left(\sum_{k\in C\setminus\overline{C}} w_n^{pk} - w_n^{qk} + i\sum_{k\in\overline{C}}\left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right)\right)\right)$$
$$+ \exp\left(it\left(\sum_{k\in C\setminus\overline{C}} w_n^{pk} - w_n^{qk} + i\sum_{k\in\overline{C}}\left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right)\right)\right) w_n^{(p-q)(b-a)}\Bigg]$$

$$= \frac{1}{n^2}\sum_{\forall p,q\in G}\Bigg[\exp\left(it\left(\sum_{k\in C\setminus\overline{C}} w_n^{pk} - w_n^{qk} + i\sum_{k\in\overline{C}}\left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right)\right)\right)$$
$$+ \exp\left(it\left(\sum_{k\in C\setminus\overline{C}} w_n^{pk} - w_n^{qk} + i\sum_{k\in\overline{C}}\left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right)\right) + \frac{2\pi i}{n}(p-q)(b-a)\right)\Bigg]$$

Now for all p and $q \in G$ such that $p \geq q$, and $p-q$ is even and we call this set M, then we have,

$$= \frac{1}{n^2} \sum_{\forall p,q \in M} \Bigg[2Re\left(\exp\left(it\left(\sum_{k \in C \setminus \overline{C}} w_n^{pk} - w_n^{qk} + i \sum_{k \in \overline{C}} \left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right) \right)\right)\right)$$

$$+ 2Re\left(\exp\left(it\left(\sum_{k \in C \setminus \overline{C}} w_n^{pk} - w_n^{qk} + i \sum_{k \in \overline{C}} \left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right) \right) + \frac{2\pi i}{n}(p-q)(b-a) \right)\right) \Bigg]$$

$$= \frac{1}{n^2}\Bigg(2 \sum_{\forall p,q \in M,} \Bigg[\cos t \left(\sum_{k \in C \setminus \overline{C}} w_n^{pk} - w_n^{qk} + i \sum_{k \in \overline{C}} \left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right) \right)$$

$$+ \cos t \left[\sum_{k \in C \setminus \overline{C}} w_n^{pk} - w_n^{qk} + i \sum_{k \in \overline{C}} \left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right) + \frac{2\pi}{n}(p-q)(a-b) \right) \Bigg] \Bigg)$$

Since (p, q) is even,

$$= \frac{1}{n^2}\left(2 \sum_{\forall p,q \in M} \left[2\cos t \left(\sum_{k \in C \setminus \overline{C}} w_n^{pk} - w_n^{qk} + i \sum_{k \in \overline{C}} \left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right) \right) \right] \right)$$

$$= \frac{4}{n^2} \sum_{\forall p,q \in M} \left[\cos t \left(\sum_{k \in C \setminus \overline{C}} w_n^{pk} - w_n^{qk} + i \sum_{k \in \overline{C}} \left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right) \right) \right]$$

From equation (2.6),

$$|\, U(t)_{aa}|^2 + |U(t)_{ba}|^2 = 1$$

That is,

$$\sum_{\forall p,q \in M} \cos t \left(\sum_{k \in C \setminus \overline{C}} w_n^{pk} - w_n^{qk} + i \sum_{k \in \overline{C}} \left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right) \right) = \frac{n^2}{4} \tag{3.2}$$

In case of integral mixed circulant graphs,

$$\left(\sum_{k \in C \setminus \overline{C}} w_n^{pk} - w_n^{qk} + i \sum_{k \in \overline{C}} \left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right) \right) \in \mathbb{Z}$$

$$\text{As, } \left(\sum_{k \in C \setminus \overline{C}} w_n^{pk} - w_n^{qk} + i \sum_{k \in \overline{C}} \left(w_n^{pk} - w_n^{-pk} - w_n^{qk} + w_n^{-qk}\right) \right) = \lambda_p - \lambda_q \ (= N(say))$$

Equation (3.2) turns out to be,

$$\sum_{\forall p,q \in M} \cos Nt = \frac{n^2}{4} \tag{3.3}$$

Here, $|M| = n^2/4$ as $|p-q| = n^2/2$ and number of set (p, q) such that $p > q$ and $p-q$ is even is $\frac{n^2}{2}/2$, that is, $n^2/4$. Thus, for $N \in \mathbb{Z}$

$$\cos Nt = 1 \ or \ t \in 2\pi\mathbb{Z}/N.$$

□

4. Examples

Example 4.1 Fractional revival in Integral mixed circulant graph $G = (Z_4, \{1, 2\})$ occurs between antipodal nodes 0 and 2 at the time $t = \pi/2$. With simple calculation we can see that spectrum of the graph $G = (Z_4, \{1, 2\})$ related to Hermitian adjacency matrix is $Sp_A = \{-3, 1, 1, 1\}$. For $N = \lambda_p - \lambda_q = 4, p, q \in M, t = 2\pi/4 = \pi/2$.

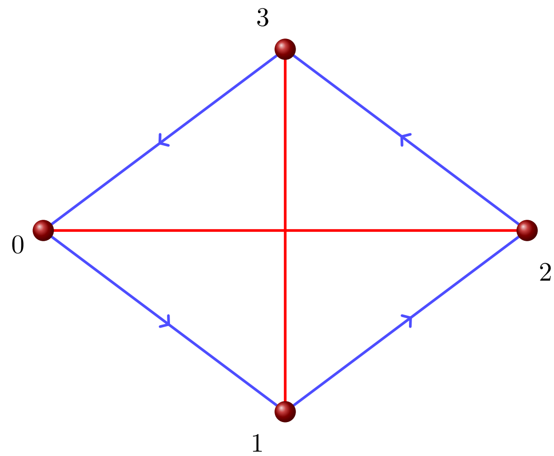

Figure 1: $(Z_4, \{1, 2\})$

Example 4.2 Integral mixed circulant graph of order 8, that is, $(Z_8, \{1, 4, 5, 6\}$ whose spectrum of Hermitian-adjacency matrix A is $Sp_A = \{-3, -3, -3, 1, 1, 1, 1, 5\}$ has fractional revival between its antipodal nodes 0 and 4 at the time $t = \pi/2$. Here $N = \lambda_p - \lambda_q = 4$ for $p, q \in M$ so $t = 2\pi/4 = \pi/2$.

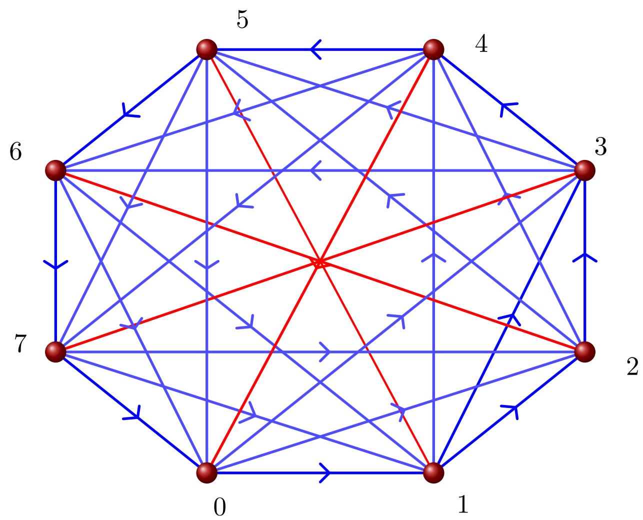

Figure 2: $(Z_8, \{1, 4, 5, 6\})$

Remark 4.1 [3] Let $X = (Z_n, C)$ be the integral mixed circulant graph on n nodes and let C be a subset of Z_n with adjacency matrix A then, X has (α, β) quantum fractional revival from nodes a to b if and only if

$$U(t) = \alpha I_n + \beta T$$

where I_n is identity matrix of dimension n. And T is the permutation matrix with no fixed point, of order 2. Morever, $\alpha\overline{\beta} + \overline{\alpha}\beta = 0$.

Proof: Suppose for some t we have, $U(t) = \alpha I_n + \beta T$, where T is a permutation matrix. For some a and every b in T,

$$T_{0,a} = 1,\ T_{0,b} = 0,\ T_{a,0} = 1,\ T_{a,b} = 0,\ \forall\ b \in n$$

So evolution operator results into,

$$U_{0,0} = \alpha,\ U_{0,a} = \beta,\ U_{0,b} = 0,\ \forall\ b \in n \tag{4.1}$$

From equation (4.1) and (2.5) we conclude graph X admits $(\alpha, \beta)-$ FR.

Conversely, Suppose graph X admits FR between a and b, evolution operator of integral mixed circulant graph at some t is

$$U(t) = \frac{1}{n}\sum_{j=0}^{(n-1)} \exp(i\lambda_j t) u_j u_j^*$$

where vector $u_j = [1, \omega_n^j, \ldots, \omega_n^{(n-1)j}]^t$ and $(a, b)^{th}$ entry of the evolution matrix is

$$U(t)_{a,b} = \frac{1}{n}\sum_{j=0}^{(n-1)} \exp(i\lambda_j t)\omega_n^{(a-b)}$$

if graph G has $(\alpha, \beta)-$ FR between a, b then

$$U(t)_{a,a} = \frac{1}{n}\sum_{j=0}^{(n-1)} \exp(i\lambda_j t) = \alpha$$

$$U(t)_{a,b} = \frac{1}{n}\sum_{j=0}^{(n-1)} \exp(i\lambda_j t)\omega_n^{(a-b)} = \beta$$

and

$$U(t)_{a,c} = 0,\ \forall\ c \neq b$$

Clearly, $U(t)$ can be expressed as,

$$U(t) = \alpha I_n + \beta T$$

for some permutation matrix T. Now we prove T is of order 2.

Since $U(t)$ is unitary matrix,

$$U(t)U^*(t) = I_n$$

or

$$\alpha\overline{\alpha}I_n + \beta\overline{\beta}T^2 + (\alpha\overline{\beta} + \overline{\alpha}\beta)T = I_n$$

implies,

$$T^2 = I_n \text{ and } \alpha\overline{\beta} + \overline{\alpha}\beta = 0.$$

□

Corollary 4.1 *If integral mixed circulant graph $X = (Z_n, C)$ has fractional revival from the node a to b at $t \in 2\pi\mathbb{Z}/N$, where $N = \lambda_p - \lambda_q,\ \lambda_p \neq \lambda_q$ then it will have fractional revival at the same time t from the node b to node a.*

5. Conclusion

In this work, we present characterization for fractional revival in integral mixed circulant graphs with examples. Fractional revival which is analogue to quantum entanglement and one of quantum state transfer phenomena, has been explored in undirected graphs largely earlier, here we worked for integral mixed circulant graph where Hermitian-adjacency matrix of the graph was taken as Hamiltonian of system.

There are a few research questions for future work. Study of fractional revival in mixed Cayley graphs over other finite groups can be tried. What effect will edges have on the event of fractional revival between nodes in integral mixed circulant graphs when they are added or removed? Important insights can be gained from results pertaining to FR following graph operations in mixed graphs.

Acknowledgments

Authors are grateful to Bennett University for providing financial support.

References

1. Abdollahi, A., & Vatandoost, E., *Which Cayley graphs are integral?.*, The Electronic Journal of Combinatorics, 16(1), (2009).
2. Bernard, P. A., Chan, A., Loranger, É., Tamon, C., & Vinet, L., *A graph with fractional revival*, Physics Letters A, 382(5), 259-264, (2018).
3. Chan, A., Coutinho, G., Tamon, C., Vinet, L., & Zhan, H., *Quantum fractional revival on graphs*, Discrete Applied Mathematics, 269, 86–98, (2019).
4. Chan, A., Coutinho, G., Tamon, C., Vinet, L., & Zhan, H. , *Fractional revival and association schemes*, Discrete Mathematics, 343 (11), 112018, (2020).
5. Chan, A., Coutinho, G., Drazen, W., Eisenberg, O., Godsil, C., Kempton, M., Lippner, G., Tamon, C. & Zhan, H., *Fundamentals of fractional revival in graphs.*, Linear Algebra and its Applications, 655, 129-158, (2022).
6. Monakhova, E. A., Romanov, A. Y., & Lezhnev, E. V., *Shortest path search algorithm in optimal two-dimensional circulant networks: Implementation for networks-on-chip*, IEEE Access, 8, 215010-215019, (2020).
7. Elspas, B., & Turner, J.. *Graphs with circulant adjacency matrices*, Journal of Combinatorial Theory, 9(3), 297-307, (1970).
8. Steinberg, B., *Representation theory of finite groups: an introductory approach*, New York: Springer, (2012).
9. Godsil, C., *State transfer on graphs*, Discrete Math., 312, 123–147, (2012).
10. Meyer, C. D.,*Matrix analysis and applied linear algebra*, (Vol. 188). Siam, (2023).
11. Coutinho, G., *Quantum state transfer in graphs*, PhD thesis, University of Waterloo, 2014.
12. Coutinho, G., & Liu, H., *No Laplacian perfect state transfer in trees*, SIAM Journal on Discrete Mathematics, 29(4), 2179-2188, (2015).
13. Thongsomnuk, I., & Meemark, Y., *Perfect state transfer in unitary Cayley graphs and gcd-graphs*, Linear and Multilinear Algebra, 67(1), 39-50, (2019).
14. Bang-Jensen, J., & Gutin, G. Z., *Digraphs: Theory, Algorithms and Applications*, Springer Science & Business Media, (2008).
15. Liu, J., & Li, X., *Hermitian-adjacency matrices and Hermitian energies of mixed graphs*, Linear Algebra and its Applications, 466, 182-207, (2015).
16. Wang, J., Wang, L., & Liu, X., *Fractional revival on Cayley graphs over abelian groups*, arXiv preprint arXiv:2206.12584, (2022).
17. Zhou, J., Bu, C., & Shen, J., *Some results for the periodicity and perfect state transfer*, The Electronic Journal of Combinatorics, P184-P184, (2011).
18. Babai, L., *Spectra of Cayley graphs.* Journal of Combinatorial Theory, Series B, 27(2), 180-189, (1979).
19. Bašić, M., Petković, M. D., & Stevanović, D., *Perfect state transfer in integral circulant graphs*, Applied Mathematics Letters, 22(7), 1117-1121, (2009).
20. Bašić, M., *Characterization of quantum circulant networks having perfect state transfer*, Quantum Information Processing, 12(1), 345-364, (2013).
21. Christandl, M., Datta, N., Dorlas, T. C., Ekert, A., Kay, A., & Landahl, A. J.,*Perfect transfer of arbitrary states in quantum spin networks*, Physical Review A, 71(3), 032312, (2005).

22. Christandl, M., Datta, N., Ekert, A., & Landahl, A. J., *Perfect state transfer in quantum spin networks*, Physical review letters, 92(18), 187902, (2004).

23. Kadyan, M., & Bhattacharjya, B., *Integral mixed circulant graphs*, Discrete Math., 346 (1), 113142, (2023).

24. Biggs, N., *Algebraic Graph Theory* (No. 67). Cambridge university press, (1993).

25. Cayley, P., *Desiderata and suggestions: No. 2. The Theory of groups: graphical representation*, American journal of mathematics, 1(2), 174-176, (1878).

26. Angeles-Canul, R. J., Norton, R. M., Opperman, M. C., Paribello, C. C., Russell, M. C., & Tamon, C., *Perfect state transfer, integral circulants, and join of graphs*, Quantum Information and Computation, 10(3-4), 325-342, (2010).

27. Bose, S.,*Quantum Communication through an Unmodulated Spin Chain*, Phys. Rev. Lett., 91 (20), 207901, (2003).

28. So, W., *Integral circulant graphs*, Discrete Math, 306 (1), 153-158, (2006).

29. Genest, V. X., Vinet, L., & Zhedanov, A., *Quantum spin chains with fractional revival*, Annals of Physics, 371, 348-367, (2016).

30. Cao, X., & Luo, G., *Fractional revival on abelian Cayley graphs*, arXiv preprint arXiv:2208.05107, (2022).

31. Song, X., *Quantum state transfer on integral oriented circulant graphs*, Applied Mathematics and Computation, 464, 128391, (2024).

32. Song, X., & Lin, H., *State transfer on integral mixed circulant graphs*, Discrete Mathematics, 347(1), 113727, (2024).

33. Liu, X., & Zhou, S., *Eigenvalues of Cayley Graphs*, The Electronic Journal of Combinatorics, P2-9, (2022).

34. Tan, Y. Y., Feng, K., & Cao, X., *Perfect state transfer on abelian Cayley graphs*, Linear Algebra and its Applications, 563, 331-352, (2019).

Rachana Soni,
Department of Mathematics,
School of Computer Science Engineering and Technology
Bennett University, Greater Noida, U.P.
India.
E-mail address: `e20soe813@bennett.edu.in`

and

Neelam Choudhary,
School of Artificial Intelligence
Bennett University, Greater Noida, U.P.
India.
E-mail address: `neelam.choudhary@bennett.edu.in`

and

Navneet Pratap Singh,
School of Artificial Intelligence
Bennett University, Greater Noida, U.P.
India.
E-mail address: `navneet.singh@bennett.edu.in`