Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–19.
ISSN-0037-8712
doi:10.5269/bspm.68889

Periodic Trajectories for an "SEIR" Epidemic Model in a Seasonal Environment with General Incidence Rate

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ABSTRACT: Infectious diseases take different forms. They are caused by bacteria, viruses, parasites or fungi that enter the body to weaken it. These diseases are transmitted from person to person or can be carried by animals. The list of infectious diseases is very long: gastroenteritis, bronchiolitis, tuberculosis, hepatitis, measles, seasonal infection, influenza, malaria, chikungunya, etc. Many epidemic diseases exhibit seasonal peak periods. Studying the population behaviours due to seasonal environment becomes a necessity for predicting the risk of disease transmission and trying to control it. In this work, we considered a four-dimensional system for a fatal disease, in a seasonal environment. We establish the existence, uniqueness, positivity and boundedness of the solution then we prove that it is a periodic orbit. We show that the global dynamics is determined using the basic reproduction number denoted by $\mathcal{R}_0$ and calculated using the spectral radius of an integral operator. We show the global stability of the disease-free periodic solution if $\mathcal{R}_0 < 1$ and we also show the persistence of the disease if $\mathcal{R}_0 > 1$. Finally, we display some numerical investigations supporting the theoretical findings where the trajectories converge to a limit cycle if $\mathcal{R}_0 > 1$.

Key Words: SEIR epidemic model, seasonal environment, Lyapunov stability, periodic solution, uniform persistence, extinction, basic reproduction number.

Contents

1. Introduction

Influenza, gastroenteritis and bronchiolitis are the main infections due to winter viruses. These three pathologies affect millions of people each year and have a strong impact on healthcare structures during the winter. One of the challenges for Public Health in any country is to reduce the risk of contamination. In winter, many viruses are present. Respiratory viruses are responsible for colds, nasopharyngitis, seasonal flu, bronchitis and bronchiolitis in children. For their part, the viruses responsible for gastroenteritis, most often called "rotavirus" and "norovirus". Winter seasonal viruses are transmitted via different vectors [10]:

- Virus-laden droplets emitted during coughing, sneezing (which remain suspended in the air) or by the sputters and saliva of people infected with a respiratory virus

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2010 *Mathematics Subject Classification:* 37J45, 34C25, 92D30, 34D23.
Submitted July 11, 2023. Published December 04, 2025

- Direct contact of the hands of an infected person with another person (for example by shaking hands) or by contact with objects (toys, comforters, pacifiers, elevator buttons, cutlery, etc.) contaminated with an sick person.

The main winter viral diseases are influenza, gastroenteritis and bronchiolitis. Each of these infections has its own symptomatology. Usually, the flu appears suddenly in the form of high fever, body aches, headaches, intense fatigue, general malaise and respiratory symptoms (dry cough, nose that flows). The illness lasts about a week, but fatigue is frequently felt for the next three or four weeks. A dry cough may persist for two weeks. Influenza is often considered a disease of little danger. This is most often the case when it affects young patients in perfect health. But the flu can be serious, even fatal, especially in fragile people, such as the elderly, people with certain chronic diseases, pregnant women, people suffering from obesity or even infants. Complications can then appear, such as [31]:

- a serious lung infection (viral or bacterial pneumonia);
- worsening of an existing chronic disease (diabetes, chronic obstructive pulmonary disease, heart failure, chronic kidney disease, cystic fibrosis, etc).

Bronchiolitis is caused by a virus that affects the lungs and causes swelling of the bronchioles (the smallest bronchi) in children under 2 years old. This virus is very common and very contagious. The bronchioles of infants are smaller than in children and adults. If the infant catches the virus, there is therefore a greater risk that the swelling of his bronchioles will prevent air from passing through. Other winter viruses such as a simple cold in children or adults can also trigger bronchiolitis in infants. The disease usually begins with a cold and a cough, then the child has difficulty breathing and may have difficulty drinking and eating. Coughing fits are very common and may be accompanied by wheezing. Gastroenteritis is an inflammation of the digestive tract, most often caused by viruses called "rotavirus" and "norovirus". It can cause nausea, loss of appetite, vomiting, abdominal cramps, severe diarrhea (loose or watery stools, at least three times a day), dehydration, fever, severe fatigue and aches of head. These viruses are common and highly contagious. Children under 5 years old, and especially infants, are very susceptible to gastroenteritis: the younger the child, the greater the risk of dehydration (loss of water through stools and vomiting). Older people and people with chronic illnesses are also more likely to be dehydrated.

The "Modeling" skill, if we take it in its broadest sense, refers for the mathematician to the fact of using a set of concepts, methods, mathematical theories that will make it possible to describe, understand and predict the evolution of phenomena external to mathematics. Modeling is a way to make the link between reality and mathematics. For several centuries, mathematics has not only been a tool extremely important for acting on and modifying nature, one of the main pillars of technique and technology, but also (and perhaps above all) a major instrument for understand it. In this sense, they are not only a source of utility but also of "truth". In particular, mathematical modeling in epidemiology is a way for studying how a disease is spread, predict the future behaviour, and propose control strategies.

Several works qualitatively proposed and studied some mathematical models describing the dynamical behaviour of infectious disease transmission (see for example [25,9,6,8,15,4,13,3,5,2]). In particular, the SEIR epidemic models with constant coefficients (autonomous systems) have been analysed in several works (see for example [26,28,1,30,21,19,18]). However, seasonality is very repetitive in each of the ecological, biological and human systems [32]. In particular, the climate variation patterns repeated every year by the same way, bird migration is repeated according to the repeated season variation, schools open and close almost periodically each year, etc. Among other things, these seasonal factors affect the pathogens survival in the environment, host behaviour and the abundance of vectors and non-human hosts. Therefore, several diseases show seasonal behaviours and then take into account of seasonally in mathematical modeling become very important. Note that even the simplest mathematical models that take into account seasonality present many difficulties to study [6]. In [7], Bacaër and Gomes discussed the periodic S-I-R model, simple generalization of the classical model of Kermack and McKendrick [22]. In [24], the authors studied a SEIRS epidemic model with periodic fluctuations. They calculated the basic reproduction number, $\mathcal{R}_0$, of the time-averaged system (autonomous). Then, they proved a sufficient

but not necessary condition ($\mathcal{R}_0 < 1$) such that the disease cannot persist in the population in a seasonal environment. In [20], Guerrero-Flores et al. considered a class of SIQRS models with periodic variation in the contact rate. They proved the existence of periodic orbits by using Leray-Schauder degree theory. Zhang and Teng [27] studied an alternative SEIRS epidemic model in a seasonal environment and established some sufficient equivalent conditions for the persistence and the extinction of the disease. These results were improved by Nakata and Kuniya in [25] by giving a threshold value between the uniform persistence and the extinction of the disease. In [8], Bacaër and Guernaoui gave the definition of the basic reproduction number in seasonal environments. In 2008, Wang and Zhao [29] defined $\mathcal{R}_0$ for several compartmental epidemic models in seasonal environments. All these definitions are different, in several cases, from the basic reproduction number defined for time-averaged system. By considering general compartmental epidemic models in seasonal environments, Wang and Zhao [29] showed that $\mathcal{R}_0$ is the threshold value for proving or not the local stability of the disease-free periodic trajectory. In [17], the authors studied the periodic behaviour of an epidemic in a seasonal environment with vaccination. Similarly, in [16,14], the authors analysed the periodic behaviour of several epidemic models in a seasonal environment.

In this paper we proposed an extension of the SEIR model given in [25] taking into account the seasonal environment. We showed that if $\mathcal{R}_0$ is less than 1, thus the disease free periodic solution is globally asymptotically stable and if $\mathcal{R}_0$ is greater than 1, thus the disease persists.
The rest of the paper is organized as follows. In Section 2, we introduce the mathematical model. In Section 3, we studied the case of autonomous system where all parameters are supposed to be constants. In section 4, we consider the non-autonomous system and we give some basic results and we give the definition of $\mathcal{R}_0$. We showed that the value of $\mathcal{R}_0$ around 1 is a threshold value between the disease extinction and the disease uniform persistence. We give numerical examples that supports the theoretical findings in Section 5. Section 6 provide a brief conclusions on our obtained results.

2. Mathematical model and properties

It would be tempting to mathematically model the spread of dengue realistically while increasing the complexity of the model. We will content ourselves below with an epidemic model with direct and non-vectorial transmission in a periodic seasonal environment, which can be applied to the case of winter viruses. Assume that recovered individuals will not re-infect. Note that even the simplest mathematical models that take into account seasonality present many difficulties to study [6]. The generalized "SEIR" mathematical model for the spread of the epidemic is given by:

$$\begin{cases} \dot{S}(t) &= d(t)S_{in}(t) - \beta(t)\mu(I(t))S(t) - d(t)S(t), \\ \dot{E}(t) &= \beta(t)\mu(I(t))S(t) - \delta(t)E(t) - d(t)E(t), \\ \dot{I}(t) &= \delta(t)E(t) - \kappa(t)I(t) - d(t)I(t), \\ \dot{R}(t) &= \kappa(t)I(t) - d(t)R(t), \end{cases} \tag{2.1}$$

with positive initial condition $(S^0, E^0, I^0, R^0) \in \mathbb{R}^4_+$.

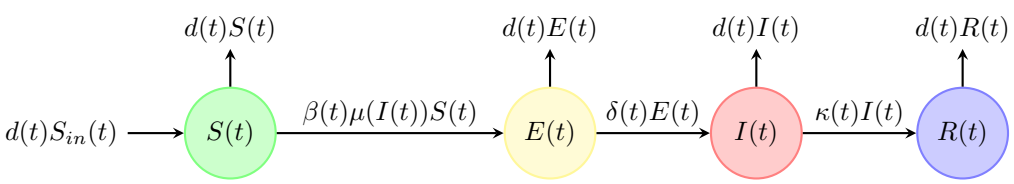

Figure 1: Diagram describing the 'SEIR' dynamical system with the four components S, E, I and R and the transition between them.

$S(t), E(t), I(t)$ and $R(t)$ describe susceptible, exposed (infected but not infectious), infectious and recovered compartments, respectively. $S_{in}(t), d(t), \beta(t), \delta(t)$ and $\kappa(t)$ are continuous functions that are

Notation	Definition
$S(t)$	Instantaneous number of susceptible individuals
$E(t)$	Instantaneous number of exposed individuals
$I(t)$	Instantaneous number of infected individuals
$R(t)$	Instantaneous number of recovered individuals
$d(t)$	Instantaneous mortality rate
$S_{in}(t)$	Instantaneous recruitment rate
$\delta(t)$	Instantaneous infection transmission rate
$1/\delta(t)$	Instantaneous duration time spent in compartment E
$\kappa(t)$	Instantaneous recovery rate
$1/\kappa(t)$	Instantaneous average duration elapsed in compartment I
$\beta(t)$	Instantaneous disease transmission coefficient
$\mu(\cdot)$	Saturated incidence rate

Table 1: Parameters and variables of system (2.1).

positive and T-periodic. $S_{in}(t)$ describes the instantaneous recruitment rate, $\beta(t)$ reflects the disease instantaneous transmission coefficient, $d(t)$ denotes the instantaneous mortality rate, $\delta(t)$ and $\kappa(t)$ are the instantaneous rates of leaving the E and I compartments, respectively. The function $\mu(\cdot)$ describes the saturated incidence rate.

Assumption 1 *μ is a bounded, non-negative $C^1(\mathbb{R}_+)$, concave and increasing function with $\mu(0)=0$.*

Lemma 1 *1. The general non-linear incidence rate μ satisfies $\mu'(x)x \le \mu(x) \le \mu'(0)x$, $\forall x > 0$.*

2. For all $x, x' \in \mathbb{R}_+$, one has $\left(\frac{x}{x'} - \frac{\mu(x)}{\mu(x')}\right) \times \left(\frac{\mu(x')}{\mu(x)} - 1\right) \le 0$.

Proof 1 *1. Let $x, x_1 \in \mathbb{R}_+$, and the function $g_1(x) = \mu(x) - x\mu'(x)$. Since $\mu'(x) \ge 0$ (μ is increasing function) and $\mu''(x) \le 0$ (μ is concave) then $g_1'(x) = -x\mu''(x) \ge 0$ and $g_1(x) \ge g_1(0) = 0$. Therefore, $\mu(x) \ge x\mu'(x)$. Similarly, let $g_2(x) = \mu(x) - x\mu'(0)$ then $g_2'(x) = \mu'(x) - \mu'(0) \le 0$ once μ is a concave function. Thus $g_2(x) \le g_2(0) = 0$ and $\mu(x) \le xf'(0)$.*

2. Let the function $g_3(x) = \frac{\mu(x)}{x}$, $g_3'(x) = \frac{\mu'(x)x - \mu(x)}{x^2} \le 0$ which means that g_3 is decreasing. Since the function μ is increasing then $(g_3(x) - g_3(x')) \times (\mu(x) - \mu(x'))$ is always negative. Then

$$\begin{aligned}(g_3(x) - g_3(x')) \times (\mu(x) - \mu(x')) &= \left(\frac{\mu(x)}{x} - \frac{\mu(x')}{x'}\right) \times (\mu(x) - \mu(x')) \\ &= \frac{\mu(x')\mu(x)}{x}\left(\frac{x}{x'} - \frac{\mu(x)}{\mu(x')}\right) \times \left(\frac{\mu(x')}{\mu(x)} - 1\right) \\ &\le 0.\end{aligned}$$

3. The case of constant parameters

Consider the system(2.1) for the case where all parameters $S_{in}(t) = S_{in}, \beta(t) = \beta, d(t) = d, \delta(t) = \delta$ and $\kappa(t) = \kappa$ are constants.

$$\begin{cases} \dot{S}(t) &= dS_{in} - \beta\mu(I(t))S(t) - dS(t), \\ \dot{E}(t) &= \beta\mu(I(t))S(t) - (\delta + d)E(t), \\ \dot{I}(t) &= \delta E(t) - (\kappa + d)I(t), \\ \dot{R}(t) &= \kappa I(t) - dR(t), \end{cases} \tag{3.1}$$

with positive initial condition $(S^0, E^0, I^0, R^0) \in \mathbb{R}_+^4$.

It is necessary that the state variables $S(t), E(t), I(t)$ and $R(t)$ remain non-negative for all $t \ge 0$.

Proposition 1 *$\Omega = \left\{(S,E,I,R) \in \mathbb{R}^4_+ \ / \ S+E+I+R = S_{in}\right\}$ is a positively invariant compact set for model (3.1).*

Proof 2 *Since $\dot{S} = dS_{in} > 0$ for $S = 0$, $\dot{E} = \beta\mu(I)S \geq 0$ for $E = 0$, $\dot{I} = \delta E \geq 0$ for $I = 0$ and $\dot{R} = \kappa I \geq 0$ for $R = 0$ therefore the model (3.1) admits a non-negative solution.*
By summing the equations of (3.1), we get, for $T = S+E+I+R-S_{in}$, the following equation [19]:

$$\begin{aligned} \dot{T} &= \dot{S}+\dot{E}+\dot{I}+\dot{R} \\ &= dS_{in} - dS - dE - dI - dR \\ &= -dT. \end{aligned}$$

Hence

$$T(t) = T(0)e^{-dt}. \tag{3.2}$$

Hence, Ω is invariant for the model (3.1) due to all variables are non-negative.

The global behavior of our system inevitably depends on the basic reproduction number ($\mathcal{R}_0$), that is, the average number of secondary cases produced by an infectious individual who is introduced into an established population only of susceptible.
To drive $\mathcal{R}_0$ for complex compartmental models, we use the next-generation operator approach proposed by Diekmann et al. [12,11]. A simple calculus permits to obtain the expression of $\mathcal{R}_0$ given by

$$\mathcal{R}_0 = \frac{\delta\beta S_{in}\mu'(0)}{(d+\delta)(d+\kappa)}. \tag{3.3}$$

Proposition 2 *The model (3.1) admits a unique disease-free equilibrium $\mathcal{E}_0 = (S_{in},0,0,0)$ and a unique endemic equilibrium $\mathcal{E}^* = (S^*,E^*,I^*,R^*)$ such that $S^*,E^*,I^*,R^* > 0$.*

Proof 3 *Equilibria of (2.1) satisfies*

$$\begin{cases} 0 &= dS_{in} - \beta\mu(I)S - dS, \\ 0 &= \beta\mu(I)S - (\delta+d)E, \\ 0 &= \delta E - (\kappa+d)I, \\ 0 &= \kappa I - dR, \end{cases} \tag{3.4}$$

which reduces to

$$\begin{cases} S = \dfrac{dS_{in}}{d+\beta\mu(I)}, \\ E = \dfrac{d\beta S_{in}\mu(I)}{(d+\delta)(d+\beta\mu(I))}, \\ I = \dfrac{\delta d\beta S_{in}\mu(I)}{(d+\kappa)(d+\delta)(d+\beta\mu(I))}, \\ R = \dfrac{\kappa}{d}I. \end{cases} \tag{3.5}$$

We conclude from (3.5) that $I(d+\kappa)(d+\delta)\Big(d+\beta\mu(I)\Big) - \delta d\beta S_{in}\mu(I) = Ig(I) = 0$ where the function g given by $g(I) = (d+\kappa)(d+\delta)(d+\beta\mu(I)) - \delta d\beta S_{in}\dfrac{\mu(I)}{I}$. Let us calculate the derivative of the function g given by

$$g'(I) = \beta(d+\kappa)(d+\delta)\mu'(I) - \delta d\beta S_{in}\frac{\mu'(I)I - \mu(I)}{I^2}.$$

The function μ satisfies $\mu'(I)I \leq \mu(I)$, $\forall I \geq 0$ and all variables are non-negative, hence g is an increasing function and satisfies $g'(I) > 0$. An easy calculation gives

$$\begin{aligned}\lim_{I\to 0} g(I) &= \lim_{I\to 0}(d+\kappa)(d+\delta)(d+\beta\mu(I)) - \delta d\beta S_{in}\frac{\mu(I)}{I}\\ &= d(d+\kappa)(d+\delta) - \delta d\beta S_{in}\mu'(0)\\ &= d(d+\kappa)(d+\delta)(1-\frac{\delta\beta S_{in}\mu'(0)}{(d+\kappa)(d+\delta)}\Big)\\ &= d(d+\kappa)(d+\delta)(1-\mathcal{R}_0),\end{aligned}$$

and

$$\begin{aligned}g(S_{in}) &= (d+\kappa)(d+\delta)(d+\beta\mu(S_{in})) - d\delta\beta\mu(S_{in})\\ &= d(d+\kappa)(d+\delta) + \beta\mu(S_{in})\big((d+\kappa)(d+\delta) - d\delta\big)\\ &> d(d+\kappa)(d+\delta)\\ &> 0.\end{aligned}$$

The equation $Ig(I) = 0$ means that either $I = 0$ or $g(I) = 0$.

- *If $I = 0$ then $S = S_{in}, E = 0$, and $R = 0$. This equilibrium named the disease-free equilibrium will be noted here by $\mathcal{E}_0 = (S_{in}, 0, 0, 0)$.*
- *If $I \neq 0$ and $g(I) = 0$. If $\mathcal{R}_0 > 1$, $g(S_{in}) > 0$ and $\lim_{I\to 0} g(I) < 0$, then $g(I) = 0$ has a unique positive solution I^* inside the interval $(0, S_{in})$ and therefore the equilibrium state $\mathcal{E}^* = (S^*, E^*, I^*, R^*)$ is unique with $S^* = \frac{dS_{in}}{d+\beta\mu(I^*)}$, $E^* = \frac{d\beta S_{in}\mu(I^*)}{(d+\delta)(d+\beta\mu(I^*))}$, and $R^* = \frac{\kappa}{d}I^*$.*

Hereafter, we will discuss the local stability of equilibrium points.

Theorem 1 *$\mathcal{E}_0$ is Locally asymptomatically stable (LAS) if $\mathcal{R}_0 < 1$, however, it is unstable once $\mathcal{R}_0 > 1$.*

Proof 4 *The Jacobian matrix at the equilibrium point $\mathcal{E}_0$ is*

$$J_0 = \begin{pmatrix} -d & 0 & -\beta\mu'(0)S_{in} & 0\\ 0 & -(d+\delta) & \beta\mu'(0)S_{in} & 0\\ 0 & \delta & -(d+\kappa) & 0\\ 0 & 0 & \kappa & -d\end{pmatrix}$$

J_0 admits two eigenvalues given by $\lambda_1 = -d < 0$ and $\lambda_2 = -d < 0$. The other two eigenvalues are eigenvalues of the sub-matrix

$$S_{J_0} = \begin{pmatrix} -(d+\delta) & \beta\mu'(0)S_{in}\\ \delta & -(d+\kappa)\end{pmatrix}.$$

The trace is given by

$$Trace\ (S_{J_0}) = -(2d+\delta+\kappa) < 0$$

and the determinant is given by

$$Det\ (S_{J_0}) = (d+\delta)(d+\kappa) - \delta\beta\mu'(0)S_{in} = (d+\delta)(d+\kappa)(1-\mathcal{R}_0).$$

If $\mathcal{R}_0 < 1$, then we have negative eigenvalues. Therefore, $\mathcal{E}_0$ is LAS. Whereas, if $\mathcal{R}_0 > 1$, we have at least one non-negative eigenvalue, and then $\mathcal{E}_0$ is unstable.

Theorem 2 *$\mathcal{E}^*$ is LAS once $\mathcal{R}_0 > 1$.*

Proof 5 *For the equilibrium point $\mathcal{E}^*$, the Jacobian is given by*

$$J^* = \begin{pmatrix} -(d+\beta\mu(I^*)) & 0 & -\beta\mu'(I^*)S^* & 0 \\ \beta\mu(I^*) & -(d+\delta) & \beta\mu'(I^*)S^* & 0 \\ 0 & \delta & -(d+\kappa) & 0 \\ 0 & 0 & \kappa & -d \end{pmatrix}$$

and its characteristic polynomial is then

$$\begin{aligned}
P(\lambda) &= \begin{vmatrix} -(\lambda+d+\beta\mu(I^*)) & 0 & -\beta\mu'(I^*)S^* & 0 \\ \beta\mu(I^*) & -(\lambda+d+\delta) & \beta\mu'(I^*)S^* & 0 \\ 0 & \delta & -(\lambda+d+\kappa) & 0 \\ 0 & 0 & \kappa & -(\lambda+d) \end{vmatrix} \\
&= \begin{vmatrix} -(\lambda+d) & -(\lambda+d+\delta) & 0 & 0 \\ \beta\mu(I^*) & -(\lambda+d+\delta) & \beta\mu'(I^*)S^* & 0 \\ 0 & \delta & -(\lambda+d+\kappa) & 0 \\ 0 & 0 & \kappa & -(\lambda+d) \end{vmatrix} \\
&= -(\lambda+d)\begin{vmatrix} -(\lambda+d) & -(\lambda+d+\delta) & 0 \\ \beta\mu(I^*) & -(\lambda+d+\delta) & \beta\mu'(I^*)S^* \\ 0 & \delta & -(\lambda+d+\kappa) \end{vmatrix} \\
&= (\lambda+d)(\lambda+d)\begin{vmatrix} -(\lambda+d+\delta) & \beta\mu'(I^*)S^* \\ \delta & -(\lambda+d+\kappa) \end{vmatrix} \\
&\quad +\beta\mu(I^*)(\lambda+d)\begin{vmatrix} -(\lambda+d+\delta) & 0 \\ \delta & -(\lambda+d+\kappa) \end{vmatrix} \\
&= (\lambda+d)(\lambda+d)((\lambda+d+\delta)(\lambda+d+\kappa) - \delta\beta\mu'(I^*)S^*) + \beta\mu(I^*)(\lambda+d)(\lambda+d+\delta)(\lambda+d+\kappa).
\end{aligned}$$

Then $\lambda_1 = -d < 0$ is an eigenvalue. The other eigenvalues are roots of the polynomial

$$\begin{aligned}
Q(\lambda) &= (\lambda+d)\Big((\lambda+d+\delta)(\lambda+d+\kappa) - \delta\beta\mu'(I^*)S^*\Big) + \beta\mu(I^*)(\lambda+d+\delta)(\lambda+d+\kappa) \\
&= \lambda^3 + \lambda^2(3d+\delta+\kappa+\beta\mu(I^*)) + \lambda\Big(d(d+\delta) + d(d+\kappa) + (d+\delta)(d+\kappa) \\
&\quad -\delta\beta\mu'(I^*)S^* + \beta\mu(I^*)(2d+\delta+\kappa)\Big) + d(d+\delta)(d+\kappa) - d\delta\beta\mu'(I^*)S^* \\
&\quad +\beta\mu(I^*)(d+\delta)(d+\kappa) \\
&= \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0
\end{aligned}$$

where

$$\begin{aligned}
a_2 &= 3d+\delta+\kappa+\beta\mu(I^*), \\
a_1 &= d(d+\delta) + d(d+\kappa) + (d+\delta)(d+\kappa) - \delta\beta\mu'(I^*)S^* + \beta\mu(I^*)(2d+\delta+\kappa), \\
a_0 &= d(d+\delta)(d+\kappa) - d\delta\beta\mu'(I^*)S^* + \beta\mu(I^*)(d+\delta)(d+\kappa).
\end{aligned}$$

Since $\mu'(I^*) \le \dfrac{\mu(I^*)}{I^*}$, $\delta\beta\dfrac{\mu(I^*)S^*}{I^*} = (d+\delta)(d+\kappa)$, and therefore $(d+\delta)(d+\kappa) - \delta\beta\mu'(I^*)S^* \ge 0$, we obtain

$$\begin{aligned}
a_2 &= d+d+\delta+d+\kappa+\beta\mu(I^*) > 0, \\
a_1 &= d(d+\delta) + d(d+\kappa) + (d+\delta)(d+\kappa) - \delta\beta\mu'(I^*)S^* + \beta\mu(I^*)(d+\delta+d+\kappa) \\
&> d(d+\delta) + d(d+\kappa) + \beta\mu(I^*)(d+\delta+d+\kappa) > 0, \\
a_0 &= d((d+\delta)(d+\kappa) - \delta\beta\mu'(I^*)S^*) + \beta\mu(I^*)(d+\delta)(d+\kappa) \\
&> \beta\mu(I^*)(d+\delta)(d+\kappa) > 0, \\
a_2a_1 - a_0 &= \Big(3d+\delta+\kappa+\beta\mu(I^*)\Big)\Big(d(d+\delta) + d(d+\kappa) + (d+\delta)(d+\kappa) - \delta\beta\mu'(I^*)S^* \\
&\quad +\beta\mu(I^*)(2d+\delta+\kappa)\Big) - d(d+\delta)(d+\kappa) + d\delta\beta\mu'(I^*)S^* - \beta\mu(I^*)(d+\delta)(d+\kappa) \\
&> \Big(3d+\delta+\kappa+\beta\mu(I^*)\Big)\Big(d(d+\delta) + d(d+\kappa) + \beta\mu(I^*)(2d+\delta+\kappa)\Big) \\
&\quad -d(d+\delta)(d+\kappa) + d\delta\beta\mu'(I^*)S^* - \beta\mu(I^*)(d+\delta)(d+\kappa) \\
&> \Big(2d+\delta+\beta\mu(I^*)\Big)\Big(d(d+\delta) + d(d+\kappa) + \beta\mu(I^*)(2d+\delta+\kappa)\Big) + d\delta\beta\mu'(I^*)S^* > 0.
\end{aligned}$$

By the Routh-Hurwitz criterion, we have negative eigenvalues. Thus, the equilibrium $\mathcal{E}^*$ is LAS once it exists ($\mathcal{R}_0 > 1$).

Here, we discuss the global behaviour of the equilibrium points.

Theorem 3 *$\mathcal{E}_0$ is globally asymptotically stable (GAS) if $\mathcal{R}_0 \leq 1$.*

Proof 6 *Define the Lyapunov function given by: $\mathcal{F}_0 = \delta E + (d+\delta)I$.*
Since we have from lemma 1 that $\mu(I) \leq \mu'(0)I$, then the derivative of $\mathcal{F}_0$

$$\begin{aligned}
\dot{\mathcal{F}}_0 &= \delta\dot{E} + (d+\delta)\dot{I} \\
&= \delta(\beta\mu(I)S - (d+\delta)E) + (d+\delta)(\delta E - (d+\kappa)I) \\
&= \delta\beta\mu(I)S - (d+\delta)(d+\kappa)I \\
&\leq \delta\beta\mu'(0)IS - (d+\delta)(d+\kappa)I \\
&= (d+\delta)(d+\kappa)\Big(\frac{\delta\beta\mu'(0)S}{(d+\delta)(d+\kappa)} - 1\Big)I \\
&\leq (d+\delta)(d+\kappa)\Big(\frac{\delta\beta\mu'(0)S_{in}}{(d+\delta)(d+\kappa)} - 1\Big)I\,, \ \textit{since } S \leq S_{in} \\
&= (d+\delta)(d+\kappa)(\mathcal{R}_0 - 1)I, \ \textit{for all } (S,E,I,R) \in \Omega.
\end{aligned}$$

$\dot{\mathcal{F}}_0 \leq 0$ due to $\mathcal{R}_0 \leq 1$, $\forall\, S,E,I,R > 0$. Let $W_0 = \{(S,E,I,R) : \dot{\mathcal{F}}_0 = 0\}$. We can show that $W_0 = \mathcal{E}_0$, thus, using LaSalle's invariance principle [23] we conclude that $\mathcal{E}_0$ is GAS once $\mathcal{R}_0 \leq 1$. Hence, as $t \to +\infty$, the solution of (3.1) converges to $\mathcal{E}_0$.

Theorem 4 *The endemic equilibrium point, $\mathcal{E}^*$, is GAS once $\mathcal{R}_0 > 1$.*

Proof 7 *Again we use a Lyapunov function given by*

$$\mathcal{F}^* = \left(S - S^*\ln\left(\frac{S}{S^*}\right)\right) + \left(E - E^*\ln\left(\frac{E}{E^*}\right)\right) + \frac{d+\delta}{\delta}\left(I - I^*\ln\left(\frac{I}{I^*}\right)\right).$$

The function $\mathcal{F}^*$ admits its minimum value $\mathcal{F}^*_{min} = S^* + E^* + \frac{d+\delta}{\delta}I^*$ when $S = S^*, E = E^*, I = I^*$.
The derivative of $\mathcal{F}^*$ with respect to time, along solutions of model (3.1) is expressed as

$$\begin{aligned}
\dot{\mathcal{F}}^* &= \left(1 - \frac{S^*}{S}\right)\dot{S} + \left(1 - \frac{E^*}{E}\right)\dot{E} + \frac{d+\delta}{\delta}\left(1 - \frac{I^*}{I}\right)\dot{I} \\
&= \left(1 - \frac{S^*}{S}\right)(dS_{in} - dS - \beta\mu(I)S) + \left(1 - \frac{E^*}{E}\right)(\beta\mu(I)S - (d+\delta)E) \\
&\quad + \frac{d+\delta}{\delta}\left(1 - \frac{I^*}{I}\right)(\delta E - (d+\kappa)I)\,.
\end{aligned}$$

Applying the steady state conditions for $\mathcal{E}^*$ where $dS_{in} = dS^* + \beta\mu(I^*)S^*$, $\beta\mu(I^*)S^* = (d+\delta)E^*$, $\delta E^* = (d+\kappa)I^*$, we get

$$\begin{aligned}
\dot{\mathcal{F}}^* &= \left(1 - \frac{S^*}{S}\right)\Big(dS^* - dS + \beta\mu(I^*)S^* - \beta\mu(I)S\Big) + \beta\mu(I)S - (d+\delta)E - \beta\mu(I)S\frac{E^*}{E} \\
&\quad + (d+\delta)E^* + \frac{d+\delta}{\delta}\Big(\delta E - (d+\kappa)I - \delta E\frac{I^*}{I} + (d+\kappa)I^*\Big) \\
&= \left(1 - \frac{S^*}{S}\right)(dS^* - dS) + \beta\mu(I^*)S^* - \beta\mu(I^*)S^*\frac{S^*}{S} + \beta\mu(I)S^* - \beta\mu(I)S\frac{E^*}{E} \\
&\quad + (d+\delta)E^* - \frac{d+\delta}{\delta}(d+\kappa)I - (d+\delta)E\frac{I^*}{I} + \frac{d+\delta}{\delta}(d+\kappa)I^* \\
&= d\left(1 - \frac{S^*}{S}\right)(S^* - S) + \beta\mu(I^*)S^* - \beta\mu(I^*)S^*\frac{S^*}{S} + \beta\mu(I)S^* - \beta\mu(I)S\frac{E^*}{E} + \beta\mu(I^*)S^* \\
&\quad - \beta\mu(I^*)S^*\frac{I}{I^*} - \beta\mu(I^*)S^*\frac{E}{E^*}\frac{I^*}{I} + \beta\mu(I^*)S^* \\
&= d\left(1 - \frac{S^*}{S}\right)(S^* - S) + \beta\mu(I^*)S^*\Big(4 - \frac{S^*}{S} - \frac{E}{E^*}\frac{I^*}{I} - \frac{I}{I^*}\frac{\mu(I^*)}{\mu(I)} - \frac{S}{S^*}\frac{E^*}{E}\frac{\mu(I)}{\mu(I^*)}\Big) \\
&\quad + \beta\mu(I^*)S^*\left(\frac{I}{I^*} - \frac{\mu(I)}{\mu(I^*)}\right)\left(\frac{\mu(I^*)}{\mu(I)} - 1\right).
\end{aligned}$$

Using the rule $x_1+x_2+x_3+x_4 \geq 4\sqrt[4]{x_1\cdot x_2\cdot x_3\cdot x_4}, x_1,x_2,x_3,x_4\geq 0$, *we obtain*

$$\left(4-\frac{S^*}{S}-\frac{E}{E^*}\frac{I^*}{I}-\frac{I}{I^*}\frac{\mu(I^*)}{\mu(I)}-\frac{S}{S^*}\frac{E^*}{E}\frac{\mu(I)}{\mu(I^*)}\right)\leq 0.$$

Now, using Lemma 1 one has $\left(\frac{I}{I^*}-\frac{\mu(I)}{\mu(I^*)}\right)\left(\frac{\mu(I^*)}{\mu(I)}-1\right)\leq 0$. *Therefore* $\dot{\mathcal{F}}^*\leq 0$. *Hence,* $\mathcal{E}^*$ *is stable. It remains to show that* $\mathcal{E}^*$ *is asymptotically stable using the LaSalle's invariance principle.*
Then $\dot{\mathcal{F}}^*(S,E,I,R)=0$ *if and only if* $S=S^*$ *and* $\frac{E}{E^*}=\frac{I}{I^*}$. *Let* $a=\frac{E}{E^*}=\frac{I}{I^*}$, *then* $E=aE^*$ *and* $I=aI^*$.
The equilibrium point $\mathcal{E}^*$ *satisfies* $dS_{in}=dS^*+\beta\mu(I^*)S^*, \beta\mu(I^*)S^*=(d+\delta)E^*, \delta E^*=(d+\kappa)I^*$, *and* $\kappa I^*=dR^*$.
Then $a=1$. *Thus* $I=I^*$ *and* $E=E^*$. *Therefore* $\dot{\mathcal{F}}^*(S,E,I,R)=0$ *if and only if* $S=S^*, E=E^*, I=I^*$ *and* $R=R^*$.
Therefore, the largest invariant set inside the set $\{(S,E,I,R)\,|\dot{\mathcal{F}}^*=0\}$ *is reduced to the singleton set* $\{\mathcal{E}^*\}$. *Using LaSalle's invariance principle [23] we conclude that* $\mathcal{E}^*$ *is GAS when* $\mathcal{R}_0>1$.

4. Seasonal environment and periodic solution

Return now to the main model (3.1) where all parameters are continuous and T-periodic positive functions:

$$\left\{\begin{array}{lcl}\dot{S}(t) &=& d(t)S_{in}(t)-d(t)S(t)-\beta(t)\mu(I(t))S(t),\\ \dot{E}(t) &=& \beta(t)\mu(I(t))S(t)-(d(t)+\delta(t))E(t),\\ \dot{I}(t) &=& \delta(t)E(t)-(d(t)+\kappa(t))I(t),\\ \dot{R}(t) &=& \kappa(t)I(t)-d(t)R(t),\end{array}\right. \tag{4.1}$$

with positive initial condition $(S^0,E^0,I^0,R^0)\in\mathbb{R}^4_+$.
Let $(\mathbb{R}^m,\mathbb{R}^m_+)$ to be the ordered m-dimensional Euclidean space associated with norm $\|\cdot\|$. For $X_1,X_2\in\mathbb{R}^m$, we denote by $X_1\geq X_2$ if $X_1-X_2\in\mathbb{R}^m_+$. We denote by $X_1>X_2$ if $X_1-X_2\in\mathbb{R}^m_+\backslash\{0\}$. We denote by $X_1\gg X_2$ if $X_1-X_2\in \mathrm{Int}(\mathbb{R}^m_+)$. Consider a T-periodic $m\times m$ matrix function denoted by $C(t)$ which is continuous, irreducible and cooperative. Let us denote by $\phi_C(t)$ the fundamental matrix, solution of the following system

$$\dot{x}(t)=C(t)x(t). \tag{4.2}$$

Let denote the spectral radius of the matrix $\phi_C(T)$ by $r(\phi_C(T))$. Therefore, all entries of $\phi_C(t)$ are positive for each $t>0$. Let apply the theorem of Perron-Frobenius to deduce that $r(\phi_C(T))$ is the principal eigenvalue of $\phi_C(T)$ (simple and admits an eigenvector $y^*\gg 0$). For the rest of the paper, the following lemma will be useful.

Lemma 2 *[33]. There exists a positive* T*-periodic function* $y(t)$ *such that* $x(t)=y(t)e^{kt}$ *will be a solution of system (4.2) where* $k=\frac{1}{T}\ln(r(\phi_C(T)))$.

Let start by proving the existence (and uniqueness) of the disease free periodic trajectory of model (4.1). Let consider the following equation

$$\dot{S}(t) \quad = \quad d(t)S_{in}(t)-d(t)S(t), \tag{4.3}$$

with initial condition $S^0\in\mathbb{R}_+$. (4.3) admits a unique T-periodic solution $S^*(t)$ with $S^*(t)>0$ which is globally attractive in $\mathbb{R}_+$ and hence, system (4.1) has a unique disease free periodic solution $(S^*(t),0,0,0)$. For a continuous, positive T-periodic function $g(t)$, we set $g^u=\max\limits_{t\in[0,T)}g(t)$ and $g^l=\min\limits_{t\in[0,T)}g(t)$.

Let $N(t)=S(t)+E(t)+I(t)+R(t)$ be the population size at time t and $\bar{N}=\frac{d^u}{d^l}S^u_{in}$. Then, we obtain

Lemma 3 *$\Omega^u = \{(S,E,I,R) \in \mathbb{R}^4_+; 0 \leq S+E+I+R \leq \bar{N}\}$ is a positively invariant attractor set for system (4.1). Furthermore, we have*

$$\lim_{t\to\infty}\Big(S(t)+E(t)+I(t)+R(t)-S^*(t)\Big) = \lim_{t\to\infty}\Big(N(t)-S^*(t)\Big) = 0. \tag{4.4}$$

Proof 8 *From system (4.1), we have*

$$\dot{N}(t) = d(t)S_{in}(t) - d(t)N(t) \leq d^u S^u_{in} - d^l N(t) \leq 0 \text{ once } \frac{d^u}{d^l}S^u_{in} \leq N(t). \tag{4.5}$$

This means that Ω^u is a forward invariant compact absorbing set of all solutions of system (4.1).

Next, in subsection 4.1, we define $\mathcal{R}_0$, the basic reproduction number and we will prove that the disease free periodic trajectory $(0,0,S^*(t),0)$ is globally asymptotically stable (and therefore, the disease dies out) once $\mathcal{R}_0 < 1$. Then, in subsection 4.2, we will prove that $I(t)$ is uniform persistence (and then the disease persists) once $\mathcal{R}_0 > 1$. Therefore, we deduce that $\mathcal{R}_0$ is the threshold parameter between the uniform persistence and the extinction of the disease.

4.1. Disease Free Periodic Solution

We start by giving the definition of the basic reproduction number of model (4.1), by using the theory given in [29] where $\mathcal{F}(t,X) = \begin{pmatrix} \beta(t)\mu(I(t))S(t) \\ \delta(t)E(t) \\ 0 \\ 0 \end{pmatrix}$, $\mathcal{V}^-(t,X) = \begin{pmatrix} (d(t)+\delta(t))E(t) \\ (d(t)+\kappa(t))I(t) \\ \beta(t)\mu(I(t))S(t)+(d(t)+p(t))S(t) \\ d(t)R(t) \end{pmatrix}$ and $\mathcal{V}^+(t,X) = \begin{pmatrix} 0 \\ 0 \\ d(t)S_{in}(t) \\ \kappa(t)I(t) \end{pmatrix}$ with $X = \begin{pmatrix} E \\ I \\ S \\ R \end{pmatrix}$.
Our aim is to check the conditions (A1)-(A7) in [29, Section 1]. Note that system(4.1) can have the following form

$$\dot{X} = \mathcal{F}(t,X) - \mathcal{V}(t,X) = \mathcal{F}(t,X) - \mathcal{V}^-(t,X) + \mathcal{V}^+(t,X). \tag{4.6}$$

The first five conditions (A1)-(A5) are fulfilled.

The system (4.6) admits a disease free periodic trajectory $X^*(t) = \begin{pmatrix} 0 \\ 0 \\ S^*(t) \\ 0 \end{pmatrix}$. Let $f(t,X(t)) = \mathcal{F}(t,X) - \mathcal{V}^-(t,X) + \mathcal{V}^+(t,X)$ and $M(t) = \left(\frac{\partial f_i(t,X^*(t))}{\partial X_j}\right)_{3\leq i,j\leq 4}$ where $f_i(t,X(t))$ and X_i are the i-th component of $f(t,X(t))$ and X, respectively. By an easy calculus, we get $M(t) = \begin{pmatrix} -(d(t)+p(t)) & 0 \\ 0 & -d(t) \end{pmatrix}$ and then $r(\phi_M(T)) < 1$. Therefore $X^*(t)$ is linearly asymptotically stable in the subspace $\Gamma_s = \{(0,0,S,R) \in R^4_+\}$. Thus, the condition (A6) in [29, Section 1] is satisfied.

Now, let us define $\mathbf{F}(t)$ and $\mathbf{V}(t)$ to be two by two matrices given by $\mathbf{F}(t) = \left(\frac{\partial \mathcal{F}_i(t,X^*(t))}{\partial X_j}\right)_{1\leq i,j\leq 2}$ and $\mathbf{V}(t) = \left(\frac{\partial \mathcal{V}_i(t,X^*(t))}{\partial X_j}\right)_{1\leq i,j\leq 2}$ where $\mathcal{F}_i(t,X)$ and $\mathcal{V}_i(t,X)$ are the i-th component of $\mathcal{F}(t,X)$ and $\mathcal{V}(t,X)$, respectively. By an easy calculus, we obtain from system (4.6)

$$\mathbf{F}(t) = \begin{pmatrix} 0 & \beta(t)\mu'(0)S^*(t) \\ \delta(t) & 0 \end{pmatrix}, \mathbf{V}(t) = \begin{pmatrix} d(t)+\delta(t) & 0 \\ 0 & d(t)+\kappa(t) \end{pmatrix}.$$

Consider $Z(t_1,t_2)$ to be the two by two matrix solution of the system $\frac{d}{dt}Z(t_1,t_2) = -\mathbf{V}(t_1)Z(t_1,t_2)$ for any $t_1 \geq t_2$, with $Z(t_1,t_1) = I$, the two by two identity matrix. Thus, condition (A7) was satisfied.

Let define C_T to be the ordered Banach space of T-periodic functions defined on $\mathbb{R} \mapsto \mathbb{R}^2$, associated to the maximum norm $\|.\|_\infty$ and the positive cone $C_T^+ = \{\psi \in C_T : \psi(s) \geq 0, \text{ for any } s \in \mathbb{R}\}$. Define the linear operator $K : C_T \to C_T$ by

$$(K\psi)(s) = \int_0^\infty Z(s, s-w)\mathbf{F}(s-w)\psi(s-w)dw, \quad \forall s \in \mathbb{R}, \psi \in C_T \tag{4.7}$$

Let now define the basic reproduction number, $\mathcal{R}_0$, of model (4.1) by $\mathcal{R}_0 = r(K)$.

Therefore, we conclude the local asymptotic stability of the disease free periodic solution $\mathcal{E}_0(t) = (S^*(t), 0, 0, 0)$ for (4.1) as follows.

Theorem 5 *[29, Theorem 2.2].*

- $\mathcal{R}_0 < 1$ *if and only if* $r(\phi_{F-V}(T)) < 1$.
- $\mathcal{R}_0 = 1$ *if and only if* $r(\phi_{F-V}(T)) = 1$.
- $\mathcal{R}_0 > 1$ *if and only if* $r(\phi_{F-V}(T)) > 1$.

Therefore, $\mathcal{E}_0(t)$ is unstable if $\mathcal{R}_0 > 1$ and it is asymptotically stable if $\mathcal{R}_0 < 1$.

Theorem 6 *$\mathcal{E}_0(t)$ is globally asymptotically stable if $\mathcal{R}_0 < 1$. It is unstable if $\mathcal{R}_0 > 1$.*

Proof 9 *Using the Theorem 5, we have $\mathcal{E}_0(t)$ is locally stable once $\mathcal{R}_0 < 1$ and it is unstable once $\mathcal{R}_0 > 1$. Therefore, it remains to prove the global attractivity of $\mathcal{E}_0(t)$ when $\mathcal{R}_0 < 1$. Consider the case where $\mathcal{R}_0 < 1$. Using the limit (4.4) in Lemma 3, for any $\delta_1 > 0$, there exists $T_1 > 0$ satisfying $S(t) + E(t) + I(t) + R(t) \leq S^*(t) + \delta_1$ for $t > T_1$. Then $S(t) \leq S^*(t) + \delta_1$ and we deduce that*

$$\begin{cases} \dot{E}(t) & \leq & \beta(t)\mu(I(t))S^*(t) + \delta_1) - (d(t) + \delta(t))E(t), \\ \dot{I}(t) & = & \delta(t)E(t) - (\kappa(t) + d(t))I(t) \end{cases} \tag{4.8}$$

for $t > T_1$. Let $M_2(t)$ to be the following 2×2 matrix function

$$M_2(t) = \begin{pmatrix} 0 & \beta(t)\mu'(0) \\ 0 & 0 \end{pmatrix}. \tag{4.9}$$

By Theorem 5, we have $r(\varphi_{F-V}(T)) < 1$. Let chose $\delta_1 > 0$ such that $r(\varphi_{F-V+\delta_1 M_2}(T)) < 1$. Consider the system hereafter system

$$\begin{cases} \dot{\bar{E}}(t) & = & \beta(t)\mu(\bar{I}(t))S^*(t) + \delta_1) - (d(t) + \delta(t))\bar{E}(t), \\ \dot{\bar{I}}(t) & = & \delta(t)\bar{E}(t) - (\kappa(t) + d(t))\bar{I}(t). \end{cases} \tag{4.10}$$

Applying Lemma 2 and using the standard comparison principle, we deduce that there exists a positive T-periodic function $y_1(t)$ satisfying $x(t) \leq y_1(t)e^{k_1 t}$ where $x(t) = \begin{pmatrix} E(t) \\ I(t) \end{pmatrix}$ and $k_1 = \frac{1}{T}\ln(r(\varphi_{F-V+\delta_1 M_2}(T)) < 0$. Thus, $\lim_{t\to\infty} E(t) = 0$ and $\lim_{t\to\infty} I(t) = 0$. Therefore, we deduce that $\lim_{t\to\infty} R(t) = 0$. Furthermore, we have $\lim_{t\to\infty} S(t) - S^*(t) = \lim_{t\to\infty} N(t) - E(t) - I(t) - R(t) - S^*(t) = 0$. Then, we deduce that the disease free periodic solution $\mathcal{E}_0(t)$ is globally attractive which complete the proof.

For the following subsection, we consider only the case where $\mathcal{R}_0 > 1$.

4.2. Endemic Periodic Solution

From Lemma 3, system (4.1) admits a positively invariant compact set Ω^u. Now, since the recovery variable do not affect the other equations of system (4.1), then the model (4.1) will be reduced as follows.

$$\left\{\begin{array}{lcl} \dot{S}(t) &=& d(t)S_{in}(t) - d(t)S(t) - \beta(t)\mu(I(t))S(t), \\ \dot{E}(t) &=& \beta(t)\mu(I(t))S(t) - (d(t)+\delta(t))E(t), \\ \dot{I}(t) &=& \delta(t)E(t) - (d(t)+\kappa(t))I(t), \end{array}\right. \tag{4.11}$$

with initial condition $(S^0, E^0, I^0) \in \mathbb{R}^3_+$ such that $S^0 > 0, E^0 > 0$ and $I^0 > 0$.

Let us define the function $P : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ to be the Poincaré map associated to system (4.11) such that $X_0 \mapsto u(T, X^0)$, where $u(t, X^0)$ is the unique solution of the system (4.11) with the initial condition $u(0, X^0) = X^0 \in \mathbb{R}^3_+$.
Let us define

$$\Gamma = \left\{(S, E, I) \in \mathbb{R}^3_+\right\},\ \Gamma_0 = Int(\mathbb{R}^3_+) \text{ and } \partial\Gamma_0 = \Gamma \setminus \Gamma_0.$$

Note that from Lemma 3, both Γ and Γ_0 are positively invariant. P is point dissipative. Define

$$M_\partial = \left\{(S^0, E^0, I^0) \in \partial\Gamma_0 : P^n(S^0, E^0, I^0) \in \partial\Gamma_0, \text{ for any } n \geq 0\right\}.$$

In order to apply the theory of uniform persistence detailed in Zhao [34] (also in [33, Theorem 2.3]), we prove that

$$M_\partial = \{(S, 0, 0),\ S \geq 0\}. \tag{4.12}$$

Note that $M_\partial \supseteq \{(S, 0, 0),\ S \geq 0\}$. To show that $M_\partial \setminus \{(S, 0, 0),\ S \geq 0\} = \emptyset$. Let consider $(S^0, E^0, I^0) \in M_\partial \setminus \{(S, 0, 0),\ S \geq 0\}$.

If $I^0 = 0$ and $0 < E^0$, thus $E(t) > 0$ for any $t > 0$. Then, it holds that $\dot{I}(t)_{|t=0} = \delta(0)E^0 > 0$. If $I^0 > 0$ and $E^0 = 0$, then $I(t) > 0$ and $S(t) > 0$ for any $t > 0$. Therefore, for any $t > 0$, we have

$$E(t) = \left[E^0 + \int_0^t \beta(\omega)\mu(I(\omega))S(\omega)e^{\int_0^\omega (\delta(u)+m(u))du} d\omega\right] e^{-\int_0^t (\delta(u)+m(u))du} > 0,$$

for all $t > 0$. This means that $(S(t), E(t), I(t)) \notin \partial\Gamma_0$ for $0 < t \ll 1$. Therefore, Γ_0 is positively invariant from which we deduce (4.12). Using the previous discussion, we deduce that there exists one fixed point $(S^*(0), 0, 0)$ of P in M_∂. We deduce, therefore, the uniform persistence of the disease as follows.

Theorem 7 *Consider the case $\mathcal{R}_0 > 1$. (4.11) admits at least one positive periodic trajectory and $\exists\ \gamma > 0$ satisfying $\forall\ (S^0, E^0, I^0) \in \mathbb{R}_+ \times Int(\mathbb{R}^2_+)$,*

$$\liminf_{t\to\infty} I(t) \geq \gamma > 0.$$

Proof 10 *Let start by proving that P is uniformly persistent respecting to $(\Gamma_0, \partial\Gamma_0)$, which will prove that the trajectory of the reduced system (4.11) is uniformly persistent respecting to $(\Gamma_0, \partial\Gamma_0)$ using [34, Theorem 3.1.1]. Recall that using Theorem 5, we obtain $r(\varphi_{F-V}(T)) > 1$. Therefore, $\exists\ \eta > 0$ small enough and satisfying $r(\varphi_{F-V-\eta M_2}(T)) > 1$. Let us consider the following perturbed equation*

$$\dot{S}_\alpha(t) = d(t)S_{in}(t) - d(t)S_\alpha(t) - \beta(t)\mu(\alpha)S_\alpha(t). \tag{4.13}$$

The function P associated to the perturbed system (4.13) has a unique positive fixed point $\bar{S}^0_\alpha$ that it is globally attractive in $\mathbb{R}_+$. Applying the implicit function theorem to deduce that $\bar{S}^0_\alpha$ is continuous respecting to α. Therefore, we can chose $\alpha > 0$ small enough and satisfying $\bar{S}_\alpha(t) > \bar{S}(t) - \eta,\ \forall\ t > 0$. Let $M_1 = (\bar{S}^0, 0, 0)$. Since the trajectory is continuous respecting to the initial condition, $\exists\ \alpha^*$ satisfying $(S^0, E^0, I^0) \in \Gamma_0$ with $\|(S^0, E^0, I^0) - u(t, M_1)\| \leq \alpha^*$, it holds that,

$$\|u(t, (S^0, E^0, I^0)) - u(t, M_1)\| < \alpha \text{ for } 0 \leq t \leq T.$$

We prove by contradiction that

$$\limsup_{n\to\infty} d(P^n(S^0,E^0,I^0),M_1) \geq \alpha^* \text{ for any } (S^0,E^0,I^0)\in\Gamma_0. \tag{4.14}$$

Suppose that $\limsup_{n\to\infty} d(P^n(S^0,E^0,I^0),M_1) < \alpha^*$ *for some* $(S^0,E^0,I^0)\in\Gamma_0$. *Without loss of generality, we can assume that* $d(P^n(S^0,E^0,I^0),M_1) < \alpha^*$ *for any* $n>0$. *Then, from the above discussion, we have that*

$$\|u(t,P^n(S^0,E^0,I^0)) - u(t,M_1)\| < \alpha \text{ for any } n>0 \text{ and } 0\leq t\leq T.$$

For all $t\geq 0$, *let* $t = nT+t_1$, *with* $t_1\in[0,T)$ *and* $n=[\frac{t}{T}]$ *(greatest integer* $\leq \frac{t}{T}$*). Then, we get*

$$\|u(t,(S^0,E^0,I^0)) - u(t,M_1)\| = \|u(t_1,P^n(S^0,E^0,I^0)) - u(t_1,M_1)\| < \alpha \text{ for all } t\geq 0.$$

Set $(S(t),E(t),I(t)) = u(t,(S^0,E^0,I^0))$. *Therefore* $0\leq I(t)\leq \alpha, t\geq 0$ *and*

$$\dot{S}(t) \quad \geq \quad d(t)S_{in}(t) - d(t)S(t) - \beta(t)\mu(\alpha)S(t). \tag{4.15}$$

The fixed point $\bar{S}^0_\alpha$ *of the function* P *associated to the perturbed system (4.13) is globally attractive such that* $\bar{S}_\alpha(t) > \bar{S}(t) - \eta$, *then* $\exists\, T_2>0$ *large enough and satisfying* $S(t) > \bar{S}(t)-\eta$ *for* $t>T_2$. *Therefore, for* $t>T_2$

$$\begin{cases} \dot{E}(t) & \geq & \beta(t)\mu(I(t))(\bar{S}(t)-\eta) - \delta(t)E(t) - d(t)E(t), \\ \dot{I}(t) & = & \delta(t)E(t) - \kappa(t)I(t) - d(t)I. \end{cases} \tag{4.16}$$

Note that we have $r(\varphi_{F-V-\eta M_2}(T)) > 1$. *Applying Lemma 2 and the comparison principle, there exists a positive* T*-periodic trajectory* $y_2(t)$ *satisfying* $J(t)\geq e^{k_2 t}y_2(t)$ *with* $k_2 = \frac{1}{T}\ln\, r\,(\varphi_{F-V-\eta M_2}(T)) > 0$, *which implies that* $\lim_{t\to\infty} I(t) = \infty$ *which is impossible since the trajectories are bounded. Therefore, the inequality (4.14) is satisfied and* P *is weakly uniformly persistent respecting to* $(\Gamma_0,\partial\Gamma_0)$. *By applying Lemma 3,* P *has a global attractor. We deduce that* $M_1 = (\bar{S}^0,0,0)$ *is an isolated invariant set inside* X *and* $W^s(M_1)\cap\Gamma_0 = \emptyset$. *All trajectory inside* M_∂ *converges to* M_1 *which is acyclic in* M_∂. *Applying [34, Theorem 1.3.1 and Remark 1.3.1], we deduce that* P *is uniformly persistent respecting to* $(\Gamma_0,\partial\Gamma_0)$. *Furthermore, using [34, Theorem 1.3.6],* P *admits a fixed point* $(\tilde{S}^0,\tilde{E}^0,\tilde{I}^0)\in\Gamma_0$. *Note that* $(\tilde{S}^0,\tilde{E}^0,\tilde{I}^0)\in R_+\times Int(R^2_+)$.

We prove also by contradiction that $\tilde{S}^0>0$. *Assume that* $\tilde{S}^0=0$. *Using the first equation of the reduced system (4.11),* $\tilde{S}(t)$ *verifies*

$$\dot{\tilde{S}}(t) \geq d(t)S_{in}(t) - \beta(t)\mu(\tilde{I}(t))\tilde{S}(t) - d(t)\tilde{S}(t), \tag{4.17}$$

with $\tilde{S}^0 = \tilde{S}(mT) = 0, m=1,2,3,\cdots$. *Applying Lemma 3,* $\forall\ \delta_3>0$, *there exists* $T_3>0$ *large enough and satisfying* $\tilde{I}(t)\leq \bar{N}+\delta_3, t>T_3$. *Then, we obtain*

$$\dot{\tilde{S}}(t) \geq d(t)S_{in}(t) - \beta(t)\mu(\bar{N}+\delta_3)\tilde{S}(t) - d(t)\tilde{S}(t), \text{ for } t\geq T_3 \tag{4.18}$$

There exists $\bar{m}$ *large enough and satisfying* $mT>T_3$ *for all* $m>\bar{m}$. *Applying the comparison principle, we deduce*

$$\tilde{S}(mT) = \Big[\tilde{S}^0 + \int_0^{mT} d(\omega)S_{in}(\omega)e^{\int_0^\omega(\beta(u)\mu(\bar{N}+\delta_3)+d(u))du}d\omega\Big]e^{-\int_0^{mT}(\beta(u)\mu(\bar{N}+\delta_3)+d(u))du} > 0$$

for any $m>\bar{m}$ *which is impossible. Therefore,* $\tilde{S}^0>0$ *and* $(\tilde{S}^0,\tilde{E}^0,\tilde{I}^0)$ *is a positive* T*-periodic trajectory of the reduced system (4.11).*

5. Examples and numerical results

We performed numerical simulations on the system (2.1) using classical Monod function to express the transmission rate $\mu(I) = \dfrac{\mu_{max} I}{k+I}$ where k and μ_{max} are constants. One can readily check that the transmission rate μ satisfies the Assumption 1. The parameters of the model are T-periodic functions having the following forms:

$$\begin{cases} S_{in}(t) &= S_{in0}(1 + S_{in1}\cos(2\pi(t+\phi))), \\ d(t) &= d_0(1 + d_1\cos(2\pi(t+\phi))), \\ \beta(t) &= \beta_0(1 + \beta_1\cos(2\pi(t+\phi))), \\ \delta(t) &= \delta_0(1 + \delta_1\cos(2\pi(t+\phi))), \\ \kappa(t) &= \kappa_0(1 + \kappa_1\cos(2\pi(t+\phi))). \end{cases} \tag{5.1}$$

S_{in1}, d_1, β_1, δ_1 and κ_1 measure the amplitude of the seasonal variation in each of the parameters with $|S_{in1}| < 1$, $|d_1| < 1$, $|\beta_1| < 1$, $|\delta_1| < 1$, and $|\kappa_1| < 1$. ϕ is the phase shift. Some fixed constants used for the numerical simulations are given in Table 2.

Parameter	S_{in0}	d_0	β_0	δ_0	κ_0	ϕ
Value	10	0.8	2	1	0.8	4

Table 2: Some fixed parameters for numerical simulations.

We will consider three cases. The first case is dedicated for the case of constant parameters (autonomous system) to validate the obtained theoretical results concerning the local and global stability of the equilibrium points $\mathcal{E}_0$ and $\mathcal{E}^*$. The second case deals only with a seasonal contact ($\beta(t)$ is a periodic function) where the other parameters are constants (partially non-autonomous system). The third case considers all parameters as periodic functions (non-autonomous system).

5.1. The case of the autonomous system

In a first step, we consider that all parameters of the system (2.1) are constants ($S_{in1} = d_1 = \beta_1 = \delta_1 = \kappa_1 = 0$). Thus, the model is given by

$$\begin{cases} \dot{S}(t) &= d_0 S_{in0} - d_0 S(t) - \beta_0 \mu(I(t)) S(t), \\ \dot{E}(t) &= \beta_0 \mu(I(t)) S(t) - (d_0 + \delta_0) E(t), \\ \dot{I}(t) &= \delta_0 E(t) - (d_0 + \kappa_0) I(t), \\ \dot{R}(t) &= \kappa_0 I(t) - d_0 R(t), \end{cases} \tag{5.2}$$

with positive initial condition $(S^0, E^0, I^0, R^0) \in \mathbb{R}^4_+$. We give some numerical results that confirm the stability of the equilibrium points of (5.2). In Fig. 2, we give the results for the case where $\mathcal{R}_0 > 1$. The approximated solution of the given model (5.2) approaches asymptotically to $\mathcal{E}^*$, which confirms that $\mathcal{E}^*$ is globally asymptotically stable once $\mathcal{R}_0 > 1$. In Fig. 3, we give the results for the case where $\mathcal{R}_0 < 1$. The approximated solution of the given model (5.2) approaches the equilibrium $\mathcal{E}_0$, which confirms that $\mathcal{E}_0$ is globally asymptotically stable once $\mathcal{R}_0 \leq 1$.

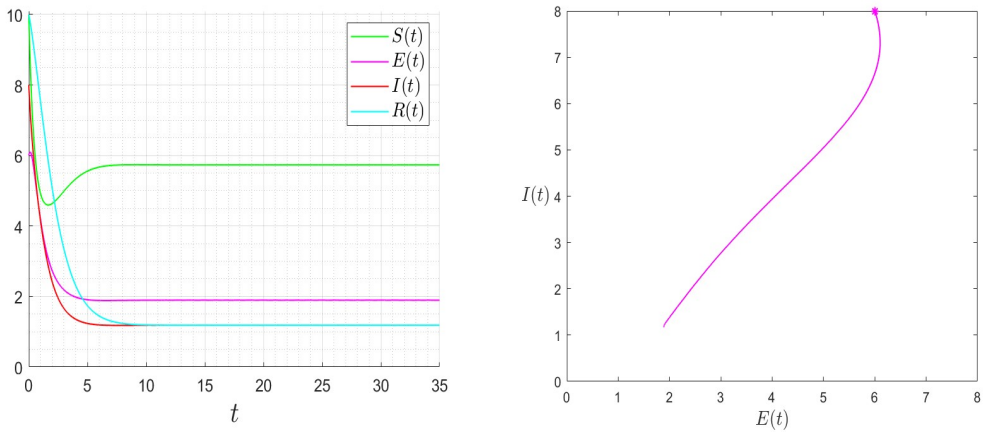

Figure 2: Behaviour of the solution of system (2.1) for $k = 2$ and $\mu_{max} = 0.8$ then $\mathcal{R}_0 \approx 2.778 > 1$.

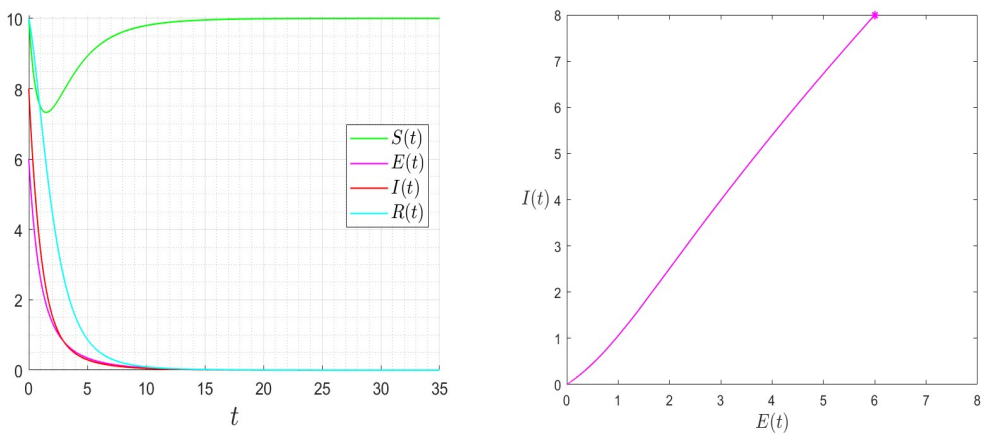

Figure 3: Behaviour of the solution of system (2.1) for $k = 4$ and $\mu_{max} = 0.4$ then $\mathcal{R}_0 \approx 0.694 < 1$.

5.2. The case of the partially non-autonomous system

In a second step, we performed numerical simulations on the system (2.1) using linear function to express the transmission rate where only the seasonally forced T-periodic function β is depending on time, t. The other parameters are constant ($d_1 = S_{in1} = \delta_1 = \kappa_1 = 0$).
Thus the model is given by

$$\begin{cases} \dot{S}(t) &= d_0 S_{in0} - d_0 S(t) - \beta_0(1 + \beta_1 \cos(2\pi(t+\phi)))\mu(I(t))S(t), \\ \dot{E}(t) &= \beta_0(1 + \beta_1 \cos(2\pi(t+\phi)))\mu(I(t))S(t) - (d_0 + \delta_0 + \varepsilon_0)E(t), \\ \dot{I}(t) &= \delta_0 E(t) - (d_0 + \kappa_0)I(t), \\ \dot{R}(t) &= \kappa_0 I(t) - d_0 R(t), \end{cases} \tag{5.3}$$

with positive initial condition $(S^0, E^0, I^0, R^0) \in \mathbb{R}^4_+$ where $\beta_1 = 0.8$. The basic reproduction number, $\mathcal{R}_0$, was approximated using the time-averaged system. We give some numerical results that confirm the asymptotic behaviour of the solution of (5.3). In Fig. 4, we give the results for the case where $\mathcal{R}_0 > 1$. The approximated solution of the given model (5.3) approaches asymptotically to a periodic solution with

persistence of the disease. In Fig. 5, we give the results for the case where $\mathcal{R}_0 < 1$. The approximated solution of the given model (5.3) approaches the disease-free trajectory $\mathcal{E}_0 = (S_{in0}, 0, 0, 0)$ once $\mathcal{R}_0 \leq 1$.

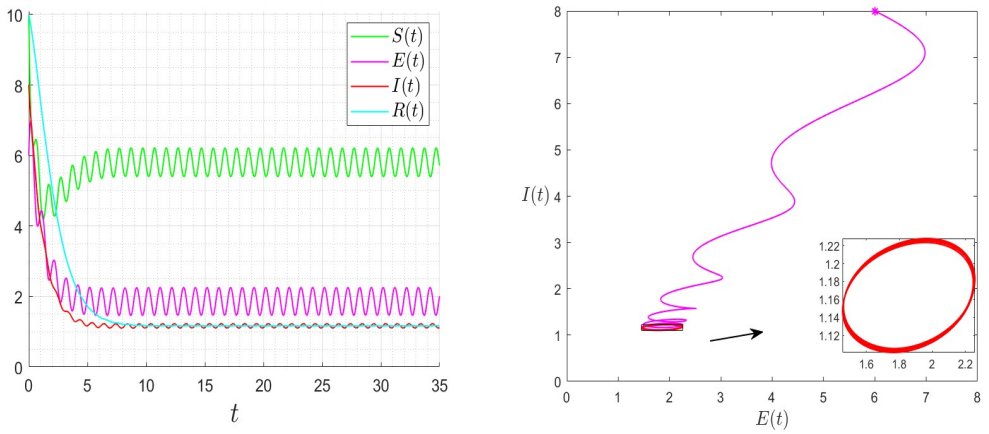

Figure 4: Behaviour of the solution of system (2.1) for $k = 2$ and $\mu_{max} = 0.8$ then $\mathcal{R}_0 \approx 2.778 > 1$.

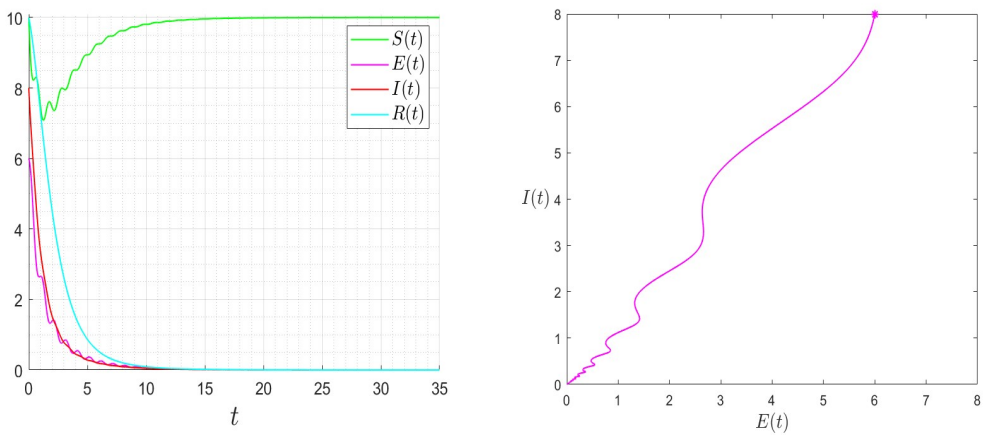

Figure 5: Behaviour of the solution of system (2.1) for $k = 4$ and $\mu_{max} = 0.4$ then $\mathcal{R}_0 \approx 0.694 < 1$.

5.3. The case of totally non-autonomous system

In a third step, we performed numerical simulations on the system (2.1) using classical Monod function to express the transmission rate where all parameters are T-periodic functions. Thus the model is given by

$$\begin{cases} \dot{S}(t) &= d(t)S_{in}(t) - d(t)S(t) - \beta(t)\mu(I(t))S(t), \\ \dot{E}(t) &= \beta(t)\mu(I(t))S(t) - (d(t) + \delta(t), \\ \dot{I}(t) &= \delta(t)E(t) - (d(t) + \kappa(t))I(t), \\ \dot{R}(t) &= \kappa(t)I(t) - d(t)R(t), \end{cases} \tag{5.4}$$

with positive initial condition $(S^0, E^0, I^0, R^0) \in \mathbb{R}^4_+$. Additional constants used for the numerical simulations in this step are given in Table 3. The basic reproduction number, $\mathcal{R}_0$, was approximated using the time-averaged system.

Parameter	S_{in1}	d_1	β_1	δ_1	κ_1
Value	0.75	-0.6	0.8	-0.6	0.4

Table 3: Additional parameters for numerical simulations of the totally non-autonomous system.

We give some numerical results that confirm the asymptotic behaviour of the solution of (5.4). In Fig. 6, we give the results for the case where $\mathcal{R}_0 > 1$. The approximated solution of the given model (5.4) approaches asymptotically to a periodic solution with persistence of the disease.
In Fig. 7, we give the results for the case where $\mathcal{R}_0 < 1$. The approximated solution of the given model (5.4) approaches the disease-free periodic trajectory $\mathcal{E}_0(t) = (S^*(t), 0, 0, 0)$ once $\mathcal{R}_0 \leq 1$.

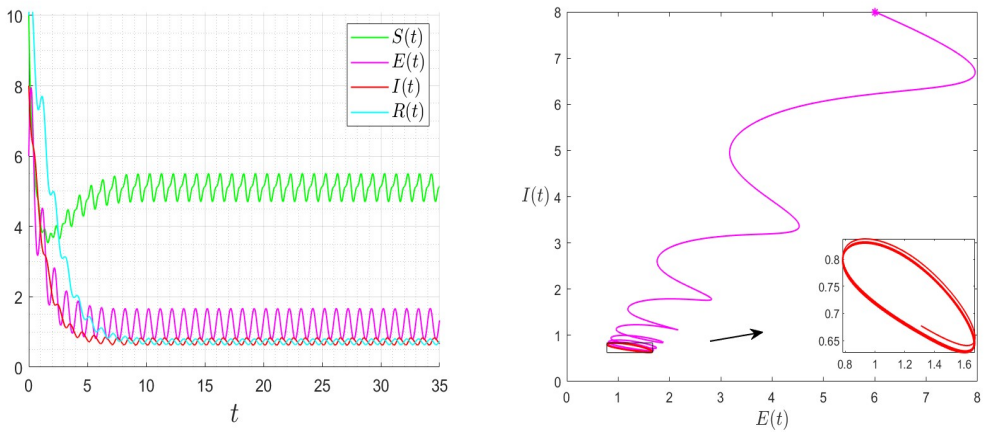

Figure 6: Behaviour of the solution of system (2.1) for $k = 2$ and $\mu_{max} = 0.8$ then $\mathcal{R}_0 \approx 2.778 > 1$.

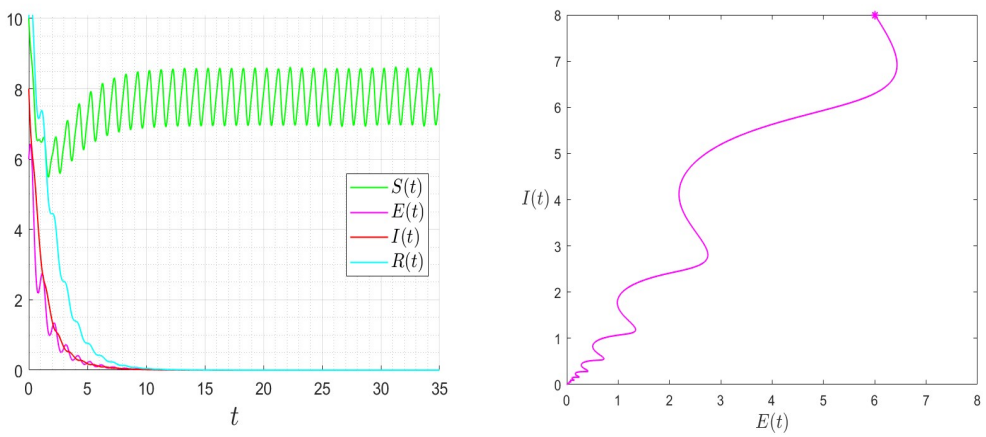

Figure 7: Behaviour of the solution of system (2.1) for $k = 2$ and $\mu_{max} = 0.3$ then $\mathcal{R}_0 \approx 0.694 < 1$.

6. Conclusion

In this work, we proposed an extension of the SEIR epidemic model given in [25] and describing epidemic diseases is a seasonal environment. In the first step we studied the case of autonomous system where all parameters are supposed to be constants. In the second step, we considered the non-autonomous

system and we give some basic results and we defined the basic reproduction number, $\mathcal{R}_0$. We show that $\mathcal{R}_0$-value compared to the unit is the threshold value between uniform persistence and extinction of the considered disease. More precisely, we showed that if $\mathcal{R}_0$ is less than 1, then the disease free periodic solution is globally asymptotically stable and if $\mathcal{R}_0$ is greater than 1, then the disease persists. Finally, we gave some numerical examples that supports the theoretical findings, including the autonomous system, the partially non-autonomous system and the full non-autonomous system. It is deduced that if the system is autonomous, the trajectories converge to one of the equilibrium of the system (3.1) according to theorems 3 and 4. However, if at least one of the model parameters is periodic, the trajectories converge to a limit cycle according to theorems 6 and 7.

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