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Existence and Ulam Hyers Mittag Leffler Stability Results of ψ-Hilfer Fuzzy Fractional Differential Equations

Elhoussain Arhrrabi*, M'hamed Elomari, Said Melliani and Lalla Saadia Chadli

ABSTRACT: In the current paper, we investigate a novel class of ψ-Hilfer type fuzzy fractional differential equations (FFDEs). Firstly, we convert the system under consideration into an analogous integral system. Secondly, by using Schauder and Banach fixed point theorems, the existence and uniqueness results of solutions for ψ-Hilfer FFDEs are then established. Additionally, with aid of generalized Grönwall inequality, we explore the Ulam–Hyers–Mittag-Leffler stability result of solution for the system under consideration.

Key Words: Fuzzy fractional differential equations, ψ-Hilfer derivative, Schauder fixed point theorem, Ulam–Hyers–Mittag-Leffler stability.

Contents

1. Introduction

Fuzzy fractional differential equations (FFDEs) are a generalization of classical fractional differential equations (FDEs) where the order of differentiation is a fuzzy number. Therefore, ψ-Hilfer fuzzy fractional differential equation is a subclass of FFDEs in which the Hilfer fractional derivative [10] is used. They have several applications in various fields, including physics, engineering, and finance (see [10,11]). They are useful for modeling complex systems where the behavior of the system is not fully understood, and where the parameters of the system may be uncertain or vary over time. By introducing fuzziness [25] into the differential equations, it becomes possible to capture this uncertainty and to obtain solutions that reflect the inherent uncertainty in the problem. Overall, ψ-Hilfer fuzzy fractional differential equations provide a powerful tool for modeling and analyzing complex systems with uncertainty, and they have important applications in many areas of science and engineering.
On the other hand, Ulam-Hyers stability [20,12,19,13,21,17,18] deals with the stability of solutions of functional equations under small perturbations of the function. So, Mittag-Leffler stability [23,1,14] is a type of stability concept for FDEs, which ensures that small perturbations in the initial conditions or coefficients result in small perturbations in the solution.
In summary, the theory of ψ-Hilfer fuzzy fractional differential equations is a relatively new area of research, and the existence and stability results for these equations are still being actively investigated.
Wang et al in [24] studied the existence and stability of solutions of Caputo type FFDEs with time-delays of the form

$$\begin{cases} {}^{C}\mathcal{D}^{\gamma}_{0^+}\mathbf{z}(u) = \mathfrak{g}\big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma)\big), & u \in [0,T], \\ \mathbf{z}(u) = \varphi(u), & u \in [-\sigma, 0], \end{cases} \tag{1.1}$$

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where ${}^{C}\mathcal{D}^{\gamma}_{0+}$ is the Caputo fractional derivative of order $0<\gamma<1$ and $\mathfrak{g}:\mathbf{I}\times\mathbf{E^n}\times\mathbf{E^n}\longrightarrow\mathbf{E^n}$ is continuous function, $\sigma\in\mathbb{R}^+$ represents the delay, $\varphi(u)$ is history function. They established existence results by Schauder's fixed point theorem and a hypothetical condition. Also they showed the uniqueness of the solution by using Banach contraction principle. In addition, with aid of generalized Grönwall inequality the Ulam-Hyers stability are discussed.

In [22] Vivek et al considered the following ψ-Hilfer FFDEs with time delay

$$\begin{cases} \mathcal{D}^{\gamma_1,\gamma_2;\psi}_{0^+}\mathbf{z}(u)=\mathfrak{h}\big(u,\mathbf{z}_u\big), & u\in[0,T],\\ \mathcal{I}^{1-\zeta;\psi}\mathbf{z}(0^+)=\mathbf{z}_0, & \\ \mathbf{z}(u)=\varphi(u), & u\in[-\sigma,0], \end{cases} \tag{1.2}$$

where $\zeta=\gamma_1+\gamma_2-\gamma_1\gamma_2$, $\mathcal{D}^{\gamma_1,\gamma_2;\psi}_{0^+}$ is the Hilfer fractional derivative of order $0<\gamma_1<1$ and type $0\le\gamma_2\le1$ and $\mathfrak{h}:[0,T]\times C([-\sigma,T],\mathbf{E^n})\longrightarrow\mathbf{E^n}$ is fuzzy function, $\varphi\in C([-\sigma,0],\mathbf{E^n})$. They studied the existence, uniqueness and finite-time stability of solution by applying standard theorems and a hypothetical conditions.

Liu et al [16] investigated sufficient conditions for the existence, uniqueness and Ulam–Hyers–Mittag-Leffler stability of solutions to a class of ψ-Hilfer fractional-order delay differential equations

$$\begin{cases} \mathcal{D}^{\gamma_1,\gamma_2;\psi}_{0^+}\mathbf{z}(u)=\mathfrak{k}\big(u,\mathbf{z}(u),\mathbf{z}(g(u))\big), & u\in[0,T],\\ \mathcal{I}^{1-\zeta;\psi}\mathbf{z}(0^+)=\mathbf{z}_0\in\mathbb{R}, \quad \zeta=\gamma_1+\gamma_2-\gamma_1\gamma_2, & \\ \mathbf{z}(u)=\varphi(u), & u\in[-\sigma,0], \end{cases} \tag{1.3}$$

where $0<\zeta<1$ and $\mathcal{D}^{\gamma_1,\gamma_2;\psi}_{0^+}$ is the ψ-Hilfer fractional derivative of order $0<\gamma_1<1$ and type $0\le\gamma_2\le1$ and $\mathfrak{k}:\mathbf{I}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given function. the results obtained are based on the Picard operator method and a generalized Grönwall inequality. Also Arhrrabi et al [2]- [9] studied several problems on FFSDEs.

In this study, we are concerned with a novel class of FFDEs with ψ-Hilfer derivative that are motivated by the aforementioned studies:

$$\begin{cases} \mathcal{D}^{\gamma_1,\gamma_2;\psi}_{0^+}\mathbf{z}(u)=\mathfrak{f}\big(u,\mathbf{z}(u),\mathbf{z}(u-\sigma)\big), & u\in\mathbf{I}:=[0,T],\\ \mathcal{I}^{1-\zeta;\psi}\mathbf{z}(0^+)=\mathbf{z}_0, \quad \zeta=\gamma_1+\gamma_2-\gamma_1\gamma_2, & \\ \mathbf{z}(u)=\varphi(u), & u\in[-\sigma,0], \end{cases} \tag{1.4}$$

where $0<\zeta<1$ and $\mathcal{D}^{\gamma_1,\gamma_2;\psi}_{0^+}$ is the ψ-Hilfer fractional derivative of order $0<\gamma_1<1$ and type $0\le\gamma_2\le1$ and $\mathfrak{f}:\mathbf{I}\times\mathbf{E^n}\times\mathbf{E^n}\longrightarrow\mathbf{E^n}$ is continuous function, $\sigma\in\mathbb{R}^+$ represents the delay, $\varphi(u)$ is history function. The novelties and main contributions of this manuscript are:

- Although the technique used to analyze the presence and stability of different systems is similar, there are many changes in the methods of proof.

- Existence and uniqueness results for the considered system are investigated by applying Schauder and Banach fixed point theorems under the weaker non-Lipschitz condition.

- We investigate the Ulam–Hyers–Mittag-Leffler stability of FFDEs, which in fact advances the understanding of Ulam–Hyers–Mittag-Leffler stability in fuzzy space.

The rest of the paper is organized as follows. In Section 2, We introduce some essential definitions and lemmas. The existence and uniqueness results for FFDEs are given in Section 3. Afterwards, in Section 4 Ulam–Hyers–Mittag-Leffler stability result of system under consideration is established. Section 5 includes an example to demonstrate the usefulness of our findings. The last section is where you come to a conclusion.

2. Preliminaries

The definitions and lemmas that are utilized throughout this papeer will be introduced in this part.

Definition 2.1 *[24] The set of fuzzy subsets of $\mathbb{R}^n$ is denoted by $\mathbf{E^n} := \Big\{\Upsilon : \mathbb{R}^n \longrightarrow [0,1]\Big\}$ which satisfies:*

(i) Υ is upper semicontinous on $\mathbb{R}^n$,

(ii) Υ is fuzzy convex, i.e, for $0 \leq \lambda \leq 1$

$$\Upsilon\big(\lambda z_1 + (1-\lambda)z_2\big) \geq min\left\{\Upsilon(z_1), \Upsilon(z_2)\right\}, \ \forall z_1, z_2 \in \mathbb{R}^n,$$

(iii) $[\Upsilon]^0 = \overline{\{z \in \mathbb{R}^n : \Upsilon(z) > 0\}}$ is compact,

(iv) Υ is normal, i.e $\exists z_0 \in \mathbb{R}^n$ such that $\Upsilon(z_0) = 1$.

Remark 2.1 $\mathbf{E^n}$ is called the space of fuzzy number.

Definition 2.2 *[24] The p-level set of $\Upsilon \in \mathbf{E^n}$ is defined by:*
For $p \in (0,1]$, we have $[\Upsilon]^p = \{z \in \mathbb{R}^n | \Upsilon(z) \geq p\}$ and for $p = 0$ we have $[\Upsilon]^0 = \overline{\{z \in \mathbb{R}^n | \Upsilon(z) > 0\}}$.

Remark 2.2 From Definition 2.1, it follows that the p-level set $[\Upsilon]^p$ of Υ, is a nonempty compact interval and $[\Upsilon]^p = [\underline{\Upsilon}(p), \overline{\Upsilon}(p)]$. Moreover, $len([\Upsilon]^p) = \overline{\Upsilon}(p) - \underline{\Upsilon}(p)$.

Definition 2.3 *[24] For addition and scalar multiplication in fuzzy set space $\mathbf{E^n}$, we have*

$$[\Upsilon_1 + \Upsilon_2]^p = [\Upsilon_1]^p + [\Upsilon_2]^p = \{z_1 + z_2 \mid z_1 \in [\Upsilon_1]^p, z_2 \in [\Upsilon_2]^p\},$$

and

$$[\alpha\Upsilon]^p = \alpha[\Upsilon]^p = \{\alpha z \mid z \in [\Upsilon]^p\},$$

for all $p \in [0,1]$.

Definition 2.4 *[24] The Hausdorff distance is given by*

$$\begin{aligned}\mathbf{D}_\infty\big(\Upsilon_1, \Upsilon_2\big) &= \sup_{0 \leq p \leq 1} \left\{|\underline{\Upsilon}_1(p) - \underline{\Upsilon}_2(p)|, |\overline{\Upsilon}_1(p) - \overline{\Upsilon}_2(p)|\right\}, \\ &= \sup_{0 \leq p \leq 1} \mathcal{D}_H\big([\Upsilon_1]^p, [\Upsilon_2]^p\big).\end{aligned}$$

Remark 2.3 $\mathbf{E^n}$ is complet metric space with the above definition (see [24]) and we have the following properties of $\mathbf{D}_\infty$:

$$\mathbf{D}_\infty\big(\Upsilon_1 + \Upsilon_3, \Upsilon_2 + \Upsilon_3\big) = \mathbf{D}_\infty\big(\Upsilon_1, \Upsilon_2\big),$$
$$\mathbf{D}_\infty\big(\lambda\Upsilon_1, \lambda\Upsilon_2\big) = |\lambda|\mathbf{D}_\infty\big(\Upsilon_1, \Upsilon_2\big),$$
$$\mathbf{D}_\infty\big(\Upsilon_1, \Upsilon_2\big) \leq \mathbf{D}_\infty\big(\Upsilon_1, \Upsilon_3\big) + \mathbf{D}_\infty\big(\Upsilon_3, \Upsilon_2\big),$$

for all $\Upsilon_1, \Upsilon_2, \Upsilon_3 \in \mathbf{E^n}$ and $\lambda \in \mathbb{R}^n$.

Definition 2.5 *[24] Let Υ_1, $\Upsilon_2 \in \mathbf{E^n}$, if there exists $\Upsilon_3 \in \mathbf{E^n}$ such that $\Upsilon_1 = \Upsilon_2 + \Upsilon_3$, then Υ_3 is called the Hukuhara difference of Υ_1 and Υ_2 noted by $\Upsilon_1 \ominus \Upsilon_2$.*

Definition 2.6 *[22] The generalized Hukuhara difference (gH-difference) of $\Upsilon_1, \Upsilon_2 \in \mathbf{E^n}$ is defined as follows:*

$$\Upsilon_1 \ominus_{gH} \Upsilon_2 = \Upsilon_3 \Leftrightarrow \begin{cases} (i) \quad \Upsilon_1 = \Upsilon_2 + \Upsilon_3, \ \ if \quad len([\Upsilon_1]^p) \geq len([\Upsilon_2]^p). \\ \\ (ii) \quad \Upsilon_2 = \Upsilon_1 + (-1)\Upsilon_3, \ \ if \quad len([\Upsilon_2]^p) \geq len([\Upsilon_1]^p). \end{cases}$$

Definition 2.7 *[24] Let a fuzzy function $\Upsilon : [a,b] \longrightarrow \mathbf{E^n}$. If for every $p \in [0,1]$, the function $u \longmapsto len[\Upsilon(u)]^p$ is increasing (decreasing) on $[a,b]$, then Υ is called increasing (decreasing) on $[a,b]$.*

Remark 2.4 If Υ is increasing or decreasing, then we say that Υ is monotone on $[a,b]$.

Let $C\big([a,b],\mathbf{E^n}\big)$ be the set of all continuous fuzzy functions and $AC\big([a,b],\mathbf{E^n}\big)$ the set of all absolutely continuous fuzzy functions on $[a,b]$ with value in $\mathbf{E^n}$ and $AC^1\big([a,b],\mathbf{E^n}\big)$ the set of all absolutely continuously differentiable fuzzy functions on $[a,b]$ with value in $\mathbf{E^n}$. For $\zeta \in (0,1)$, let $C_{\zeta;\psi}\big([a,b],\mathbf{E^n}\big)$ be the space of continuous functions defined by

$$C_{\zeta;\psi} := \{\mathbf{z} \in (a,b] \longrightarrow \mathbf{E^n} : \quad (\psi(u)-\psi(a))^{1-\zeta}\mathbf{z}(u) \in C[a,b]\}.$$

Denote by $L\big([a,b],\mathbf{E^n}\big)$ the set of all fuzzy functions $\mathbf{z} : [a,b] \longrightarrow \mathbf{E^n}$ such that $t \mapsto \mathbf{D}_\infty[\mathbf{z}(tu,\hat{0}]$ belong to $L^1[a,b]$.

Definition 2.8 *[22] The ψ-Riemann–Liouville fractional integral of order $\gamma_1 > 0$ of a continuous function is defined by*

$$\mathcal{I}_{a+}^{\gamma_1;\psi}\mathbf{z}(u) = \frac{1}{\Gamma(\gamma_1)}\int_a^u (\psi(u)-\psi(v))^{\gamma_1-1}\mathbf{z}(v)dv.$$

Definition 2.9 *[22] The ψ-Riemann–Liouville fractional derivative of order $\gamma_1 > 0$ of a continuous function $\mathbf{z}$ is given by*

$$\begin{aligned}\mathcal{D}_{a+}^{\gamma_1;\psi}\mathbf{z}(u) &: = D^n\mathcal{I}_{a^+}^{n-\gamma_1;\psi}\mathbf{z}(u),\\ &= \frac{1}{\Gamma(n-\gamma_1)}\left(\frac{1}{\psi'(v)}\frac{d}{du}\right)^n\int_a^u \psi'(v)(\psi(u)-\psi(v))^{n-\gamma_1-1}\mathbf{z}(v)dv,\end{aligned}$$

where $n = [\gamma_1]+1$.

For $x \in L\big([a,b],\mathbf{E^n}\big)$, we define the ψ-Hilfer fractional integral of order γ_1 of the fuzzy function $\mathbf{z}$:

$$\mathbf{z}_{\gamma_1;\psi}(u) := \mathcal{I}_{a^+}^{\gamma_1;\psi}\mathbf{z}(u) = \frac{1}{\Gamma(\gamma_1)}\int_a^u \psi'(v)(\psi(u)-\psi(v))^{\gamma_1-1}\mathbf{z}(v)dv, \quad u \geq a.$$

Since $[\mathbf{z}(u)]^p = \big[\underline{\mathbf{z}}(u,p),\overline{\mathbf{z}}(u,p)\big]$, we can define the fuzzy ψ-Hilfer fractional integral of fuzzy function $\mathbf{z}$ based on lower and upper functions

$$[\mathcal{I}_{a^+}^{\gamma_1;\psi}\mathbf{z}(u)]^p = \big[\mathcal{I}_{a^+}^{\gamma_1;\psi}\underline{\mathbf{z}}(u,p),\mathcal{I}_{a^+}^{\gamma_1;\psi}\overline{\mathbf{z}}(u,p)\big], \qquad u \geq a.$$

Where

$$\mathcal{I}_{a^+}^{\gamma_1;\psi}\underline{\mathbf{z}}(u,p) = \frac{1}{\Gamma(\gamma_1)}\int_a^u \psi'(v)(\psi(u)-\psi(v))^{\gamma_1-1}\underline{\mathbf{z}}(v,p)dv,$$

and

$$\mathcal{I}_{a^+}^{\gamma_1;\psi}\overline{\mathbf{z}}(u,p) = \frac{1}{\Gamma(\gamma_1)}\int_a^u \psi'(v)(\psi(u)-\psi(v))^{\gamma_1-1}\overline{\mathbf{z}}(v,p)dv.$$

It follows that the operator $\mathbf{z}_{\gamma_1;\psi}(t)$ is linear and bounded from $C\big([a,b],\mathbf{E^n}\big)$ to $C\big([a,b],\mathbf{E^n}\big)$ and we have

$$K \leq \frac{\|\mathbf{z}\|}{\Gamma(\gamma_1)}\int_a^u \psi'(v)(\psi(u)-\psi(v))^{\gamma_1-1}dv = \frac{\|\mathbf{z}\|}{\Gamma(\gamma_1+1)}(\psi(u)-\psi(a))^{\gamma_1},$$

where $\|\mathbf{z}\| = \sup_{a\leq u\leq b}\mathbf{D}_\infty\big(\mathbf{z}(u),\hat{0}\big)$.

Definition 2.10 *[22] The fuzzy ψ-Hilfer fractional derivative of order γ_1 and parameter γ_2 of a function $\mathbf{z} \in C_{1-\zeta;\psi}[a,b]$ is defined by*

$$\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\mathbf{z}(u) = \mathcal{I}_{a^+}^{\gamma_2(n-\gamma_1)}\left(\frac{1}{\psi'(u)}\frac{d}{du}\right)^n\mathcal{I}_{a^+}^{(1-\gamma_2)(n-\gamma_1)}\mathbf{z}(u),$$

if the gH-derivative $\mathbf{z}'_{(1-\gamma_1);\psi}(t)$ exists, where $n-1 < \gamma_1 < n$, and $0 \leq \gamma_2 \leq 1$.

Lemma 2.1 *[22] Let $\gamma_1, \gamma_2 > 0$ and $\zeta > 0$. Then*

$$\mathcal{I}_{a^+}^{\gamma_1;\psi}\mathcal{I}_{a^+}^{\gamma_2;\psi}\mathbf{z}(u) = \mathcal{I}_{a^+}^{\gamma_1+\gamma_2;\psi}\mathbf{z}(u),$$

$$\mathcal{I}_{a^+}^{\gamma_1;\psi}\big(\psi(u)-\psi(a)\big)^{\zeta-1} = \frac{\Gamma(\zeta)}{\Gamma(a+\zeta)}\big(\psi(u)-\psi(a)\big)^{\gamma_1+\zeta-1}.$$

In the following lemma, we give the composition of $\mathcal{I}_{a^+}^{\gamma_1;\psi}$ with $\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}$.

Lemma 2.2 *[22] Let $\gamma_1 > 0$ and let $\mathfrak{f} \in C_{\zeta;\psi}\big([a,b],\mathbb{R}^n\big)$, $\mathcal{I}_{a^+}^{(1-\gamma_1);\psi}\mathfrak{f}(t) \in C^1_{\zeta;\psi}\big([a,b],\mathbb{R}^n\big)$. Then*

$$\mathcal{I}_{a^+}^{\gamma_1;\psi}(\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\mathfrak{f})(u) = \mathfrak{f}(u) - \frac{\mathcal{I}_{a^+}^{(1-\gamma_1);\psi}\mathfrak{f}(a)}{\Gamma(\gamma_1)}(\psi(u)-\psi(a))^{\gamma_1-1}.$$

Lemma 2.3 *Let $0 < \gamma_1 \leq 1$ and $0 \leq \zeta \leq 1$. If $\mathbf{z} \in C_{\zeta;\psi}\big([a,b],\mathbf{E^n}\big)$ and $\mathcal{I}_{a^+}^{1-\gamma_1;\psi}\mathbf{z} \in C^1_{\zeta;\psi}[a,b]$, then we have*

$$\mathcal{I}_{a^+}^{\gamma_1;\psi}\Big(\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\mathbf{z}\Big)(u) = \mathbf{z}(u) \ominus_{gH} \frac{\mathcal{I}_{a^+}^{1-\gamma_1;\psi}\mathbf{z}(a)}{\Gamma(\gamma_1)}(\psi(u)-\psi(a))^{\gamma_1-1},$$

if $\mathbf{z}$ is increasing, and we have

$$\mathcal{I}_{a^+}^{\gamma_1;\psi}\Big(\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\mathbf{z}\Big)(u) = -\frac{\mathcal{I}_{a^+}^{1-\gamma_1;\psi}\mathbf{z}(a)}{\Gamma(\gamma_1)}(\psi(u)-\psi(a))^{\gamma_1-1} \ominus_{gH} \big(-\mathbf{z}(u)\big),$$

if $\mathbf{z}$ is decreasing and provided that the mentioned Hukuhara differences exist.

Proof: By using Lemma 2.2, we have for case of $\mathbf{z}$ is increasing:

$$\begin{aligned}\mathcal{I}_{a^+}^{\gamma_1;\psi}\Big(\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\mathbf{z}\Big)(u,p) &= \Big[\mathcal{I}_{a^+}^{\gamma_1;\psi}\big(\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\underline{\mathbf{z}}\big)(u,p), \mathcal{I}_{a^+}^{\gamma_1;\psi}\big(\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\overline{\mathbf{z}}\big)(u,p)\Big],\\ &= \left[\underline{\mathbf{z}}(u,p) - \frac{\underline{\mathbf{z}}_{1-\gamma_1;\psi}(a)}{\Gamma(\gamma_1)}(\psi(u)-\psi(a))^{\gamma_1-1}, \overline{\mathbf{z}}(u,p) - \frac{\overline{\mathbf{z}}_{1-\gamma_1;\psi}(a)}{\Gamma(\gamma_1)}(\psi(u)-\psi(a))^{\gamma_1-1}\right].\end{aligned}$$

And for case of $\mathbf{z}$ is decreasing, we have:

$$\begin{aligned}\mathcal{I}_{a^+}^{\gamma_1;\psi}\Big(\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\mathbf{z}\Big)(u,p) &= \Big[\mathcal{I}_{a^+}^{\gamma_1;\psi}\big(\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\overline{\mathbf{z}}\big)(u,p), \mathcal{I}_{a^+}^{\gamma_1;\psi}\big(\mathcal{D}_{a^+}^{\gamma_1,\gamma_2;\psi}\underline{\mathbf{z}}\big)(u,p)\Big],\\ &= \left[\overline{\mathbf{z}}(u,p) - \frac{\overline{\mathbf{z}}_{1-\gamma_1;\psi}(a)}{\Gamma(\gamma_1)}(\psi(u)-\psi(a))^{\gamma_1-1}, \underline{\mathbf{z}}(u,p) - \frac{\underline{\mathbf{z}}_{1-\gamma_1;\psi}(a)}{\Gamma(\gamma_1)}(\psi(u)-\psi(a))^{\gamma_1-1}\right],\end{aligned}$$

for all $p \in [0,1]$ which complete the proofs. □

3. Existance and Uniqueness Results

The fuzzy fractional differential equations with ψ-Hilfer fractional derivative given in system (1.4) are discussed in this section.

Lemma 3.1 *[22] Let $\mathfrak{f} : \mathbf{I} \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is continuous function, $\varphi \in C\big([-\sigma,0],\mathbf{E^n}\big)$. Then a monotone fuzzy function $\mathbf{z} \in C\big([-\sigma,T],\mathbf{E^n}\big)$ is a solution of (1.4) if and only if for $u \in \mathbf{I}$, $\mathbf{z}$ satisfies the following integral equation*

$$\mathbf{z}(u) \ominus_{gH} \frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} = \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)ds, \tag{3.1}$$

and $\mathbf{z}(u) = \varphi(u)$, $u \in [-\sigma,0]$ and $u \longmapsto \mathcal{I}^{1-\zeta;\psi}\mathfrak{f}\big(u,\mathbf{z}(u),\mathbf{z}(u-\sigma)\big)$ is increasing on $\mathbf{I}$.

We make the following hypotheses concerning the coefficients of the system under consideration:

($\mathcal{H}1$) For all $\phi_1, \phi_2,\ \nu_1, \nu_2 \in \mathbf{E^n}$ and for all $u \in \mathbf{I}$, we have

$$\mathbf{D}_\infty\left[\mathfrak{f}(u,\phi_1,\nu_1), \mathfrak{f}(u,\phi_2,\nu_2)\right] \leq H\big(u, \mathbf{D}_\infty[\phi_1,\phi_2], \mathbf{D}_\infty[\nu_1,\nu_2]\big),$$

where $H : \mathbf{I} \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is a monotone increasing, continuous and concave function with $H(u,0,0) = 0$ and $H(u, \mathbf{z}(u), \mathbf{z}(u)) = kH(u, \mathbf{z}(u))$, k is a constant.

($\mathcal{H}2$) For any $\phi, \nu \in \mathbf{E^n}$, we suppose that there exists a function $h \in C(\mathbf{I}, \mathbf{E^n})$ such that

$$H\big(u, \mathbf{D}_\infty[\phi,\nu]\big) \leq h(u)\mathbf{D}_\infty[\phi,\nu].$$

We will now use the Schauder's fixed point Theorem to demonstrate our result.

Theorem 3.1 *Suppose that $\mathfrak{f} : \mathbf{I} \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is continuous and satisfying the hypotheses ($\mathcal{H}1$) and ($\mathcal{H}2$). Then, there exist at least a solution $\mathbf{z}(u)$ to the system (1.4).*

Proof: Consider the operator $\mathfrak{L} : C\big(\mathbf{I}, \mathbf{E^n}\big) \longrightarrow C\big(\mathbf{I}, \mathbf{E^n}\big)$ defined as follows

$$\mathfrak{L}(\mathbf{z}(u)) = \frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)ds,$$

To prove this result, we divide the subsequent proof into two steps.
Step 1: We will prove that $\mathfrak{L}$ is completely continuous. For this, let us prove that:
ⓐ- $\mathfrak{L}$ is continuous. Indeed, for any integer $n \geq 1$, define $\mathbf{z}_n(u) = \varphi(u)$ for all $u \in [-\sigma, 0]$. For all $u \in \mathbf{I}$

$$\mathfrak{L}(\mathbf{z}_n(u)) = \frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathfrak{f}\big(s,\mathbf{z}_n(s),\mathbf{z}_n(s-\sigma)\big)ds. \tag{3.2}$$

By using the properties of the metric $\mathbf{D}_\infty$ and hypotheses ($\mathcal{H}1$) and ($\mathcal{H}2$), we have

$$\begin{aligned}
\mathbf{D}_\infty\big[\mathfrak{L}(\mathbf{z}_n(u)), \mathfrak{L}(\mathbf{z}(u))\big] &= \mathbf{D}_\infty\Bigg[\frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}_n(s),\mathbf{z}_n(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}}ds,\\
&\qquad \frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}}ds\Bigg],\\
&= \mathbf{D}_\infty\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}_n(s),\mathbf{z}_n(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}}ds, \frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}}ds\Bigg],\\
&\leq \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathbf{D}_\infty\big[\mathfrak{f}\big(s,\mathbf{z}_n(s),\mathbf{z}_n(s-\sigma)\big), \mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)\big]ds,\\
&\leq \frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}H\Big(s, \mathbf{D}_\infty\big[\mathbf{z}_n(s),\mathbf{z}(s)\big], \mathbf{D}_\infty\big[\mathbf{z}_n(s-\sigma),\mathbf{z}(s-\sigma)\big]\Big),
\end{aligned}$$

or, by using the definition of $\mathbf{D}_\infty$, we have

$$\begin{aligned}
\mathbf{D}_\infty[\mathbf{z}_n(s-\sigma), \mathbf{z}(s-\sigma)] &= \sup_{0\leq r\leq 1}\max_{0\leq s\leq u}\{|\, \underline{\mathbf{z}_n}(s-\sigma,r) - \underline{\mathbf{z}}(s-\sigma,r)\,|, |\, \overline{\mathbf{z}_n}(s-\sigma,r) - \overline{\mathbf{z}}(s-\sigma,r)\,|\},\\
&= \sup_{0\leq r\leq 1}\max_{-\sigma\leq \mu\leq u-\sigma}\{|\, \underline{\mathbf{z}_n}(\mu,r) - \underline{\mathbf{z}}(\mu,r)\,|, |\, \overline{\mathbf{z}_n}(\mu,r) - \overline{\mathbf{z}}(\mu,r)\,|\},\\
&\leq \sup_{0\leq r\leq 1}\max_{-\sigma\leq \mu\leq 0}\{|\, \underline{\mathbf{z}_n}(\mu,r) - \underline{\mathbf{z}}(\mu,r)\,|, |\, \overline{\mathbf{z}_n}(\mu,r) - \overline{\mathbf{z}}(\mu,r)\,|\}\\
&\qquad + \sup_{0\leq r\leq 1}\max_{0\leq \mu\leq u-\sigma}\{|\, \underline{\mathbf{z}_n}(\mu,r) - \underline{\mathbf{z}}(\mu,r)\,|, |\, \overline{\mathbf{z}_n}(\mu,r) - \overline{\mathbf{z}}(\mu,r)\,|\},\\
&\leq \sup_{0\leq r\leq 1}\max_{0\leq s\leq u}\{|\, \underline{\mathbf{z}_n}(s,r) - \underline{\mathbf{z}}(s,r)\,|, |\, \overline{\mathbf{z}_n}(s,r) - \overline{\mathbf{z}}(s,r)\,|\} = \mathbf{D}_\infty[\mathbf{z}_n(s), \mathbf{z}(s)].
\end{aligned}$$

Then, using the hypothesis ($\mathcal{H}1$), we get

$$\mathbf{D}_\infty\big[\mathfrak{L}(\mathbf{z}_n(u)),\mathfrak{L}(\mathbf{z}(u))\big] \leq \frac{k\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)} H\Big(s,\mathbf{D}_\infty\big[\mathbf{z}_n(s),\mathbf{z}(s)\big]\Big),$$

Since H is continuous, we can conclude that $\mathbf{D}_\infty\big[\mathfrak{L}(\mathbf{z}_n(u)),\mathfrak{L}(\mathbf{z}(u))\big] \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, $\mathfrak{L}$ is continuous.
ⓑ- We prove that there exists a positive constant ξ_1 and for all $\varsigma_1 > 0$ satisfying for all $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_1} := \Big\{\mathbf{z}(u) \in C\big([-\sigma,T],\mathbf{E^n}\big)|\mathbf{D}_\infty[\mathbf{z}(u),\hat{0}] \leq \varsigma_1\Big\}$ one has $\mathbf{D}_\infty\big[\mathfrak{L}\big(\mathbf{z}(u)\big),\hat{0}\big] \leq \xi_1$. In fact, for all $u \in \mathbf{I}$ and $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_1}$, we have

$$\begin{aligned}\mathbf{D}_\infty\big[\mathfrak{L}\big(\mathbf{z}(u)\big),\hat{0}\big] &= \mathbf{D}_\infty\Bigg[\frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)ds,\hat{0}\Bigg],\\ &\leq \frac{\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}\mathbf{D}_\infty\big[\mathbf{z}_0,\hat{0}\big] + \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big),\hat{0}\Big]ds,\\ &\leq \frac{\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}\mathbf{D}_\infty\big[\mathbf{z}_0,\hat{0}\big] + \frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big),\hat{0}\Big].\end{aligned}$$

Since the function $\mathfrak{f}$ is continuous, there is exist a constant $N_\mathfrak{f} > 0$ such that $\mathbf{D}_\infty\big[\mathfrak{f}(u,\varphi,\phi),\hat{0}\big] \leq N_\mathfrak{f}$. Then

$$\mathbf{D}_\infty\big[\mathfrak{L}\big(\mathbf{z}(u)\big),\hat{0}\big] \leq \frac{\big(\psi(T)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}\mathbf{D}_\infty\big[\mathbf{z}_0,\hat{0}\big] + \frac{N_\mathfrak{f}\big(\psi(T)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)} := \xi_1.$$

Therefore, for every $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_1}$, we have $\mathbf{D}_\infty\big[\mathfrak{L}\big(\mathbf{z}(u)\big),\hat{0}\big] \leq \xi_1$, this implies that $\mathfrak{L}(\mathbf{B}_{\varsigma_1}) \subseteq \mathbf{B}_{\xi_1}$.
ⓒ- $\mathfrak{L}$ maps bounded set into equi-continuous set. Indeed, for each $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_2}$ and $u_1,u_2 \in \mathbf{I}$ such that $0 \leq u_1 < u_2 \leq T$, we have

$$\begin{aligned}\mathbf{D}_\infty\big[\mathfrak{L}\big(\mathbf{z}(u_1)\big),\mathfrak{L}\big(\mathbf{z}(u_2)\big)\big] &= \mathbf{D}_\infty\Bigg[\frac{\mathbf{z}_0\big(\psi(u_1)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^{u_1}\frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u_1)-\psi(s)\big)^{1-\gamma_1}}ds,\\ &\qquad \frac{\mathbf{z}_0\big(\psi(u_2)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^{u_2}\frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u_2)-\psi(s)\big)^{1-\gamma_1}}ds\Bigg],\\ &\leq \mathbf{D}_\infty\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_0^{u_1}\frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u_1)-\psi(s)\big)^{1-\gamma_1}}ds, \frac{1}{\Gamma(\gamma_1)}\int_0^{u_2}\frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u_2)-\psi(s)\big)^{1-\gamma_1}}ds\Bigg],\\ &\leq \mathbf{D}_\infty\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_0^{u_1}\frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u_1)-\psi(s)\big)^{1-\gamma_1}}ds, \frac{1}{\Gamma(\gamma_1)}\int_0^{u_1}\frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u_2)-\psi(s)\big)^{1-\gamma_1}}ds\\ &\qquad + \frac{1}{\Gamma(\gamma_1)}\int_{u_1}^{u_2}\frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u_2)-\psi(s)\big)^{1-\gamma_1}}ds\Bigg],\\ &\leq \frac{1}{\Gamma(\gamma_1)}\int_0^{u_1}\psi'(s)\Big|\big(\psi(u_1)-\psi(s)\big)^{\gamma_1-1} - \big(\psi(u_2)-\psi(s)\big)^{\gamma_1-1}\Big|\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big),\hat{0}\Big]ds\\ &\qquad + \frac{1}{\Gamma(\gamma_1)}\int_{u_1}^{u_2}\psi'(s)\big(\psi(u_2)-\psi(s)\big)^{\gamma_1-1}\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big),\hat{0}\Big]ds,\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\big(\psi(u_1)-\psi(0)\big)^{\gamma_1}+\big(\psi(u_2)-\psi(u_1)\big)^{\gamma_1}-\big(\psi(u_2)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big),\hat{0}\Big]\\
&\qquad\qquad +\frac{\big(\psi(u_2)-\psi(u_1)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big),\hat{0}\Big],\\
&\leq \frac{\big(\psi(u_1)-\psi(0)\big)^{\gamma_1}+2\big(\psi(u_2)-\psi(u_1)\big)^{\gamma_1}-\big(\psi(u_2)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big),\hat{0}\Big],\\
&\leq \frac{\big(\psi(u_1)-\psi(0)\big)^{\gamma_1}+2\big(\psi(u_2)-\psi(u_1)\big)^{\gamma_1}-\big(\psi(u_2)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}N_{\mathfrak{f}}.
\end{aligned}
$$

We have $\Theta := \frac{\big(\psi(u_1)-\psi(0)\big)^{\gamma_1}+2\big(\psi(u_2)-\psi(u_1)\big)^{\gamma_1}-\big(\psi(u_2)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}$ is independent of $\mathbf{z}(u)$ and $\Theta \longrightarrow 0$ as $u_2 \longrightarrow u_1$. Then, we obtain

$$\mathbf{D}_\infty\big[\mathfrak{L}\big(\mathbf{z}(u_1)\big),\mathfrak{L}\big(\mathbf{z}(u_2)\big)\big] \longrightarrow 0.$$

It means that $\mathfrak{L}(\mathbf{B}_{\varsigma_2})$ is equi-continuous. Then, according to Arzela-Ascoli Theorem, $\mathfrak{L}$ is completely continuous.

Step 2: In this step, we will demonstrate that there is a closed, convex and bounded subset $\mathbf{B}_\xi = \Big\{\mathbf{z}(u) \in C\big([-\sigma,T],\mathbf{E^n}\big) | \mathbf{D}_\infty[\mathbf{z}(u),\hat{0}] \leq \xi\Big\}$ such that $\mathfrak{L}(\mathbf{B}_\xi) \subseteq \mathbf{B}_\xi$. We know that $\mathbf{B}_\xi$ is a closed, convex and bounded subset of $C\big([-\tau,T],\mathbf{E^n}\big)$ for all $\xi > 0$. Suppose that for all $\xi > 0$, $\exists \mathbf{z}_\xi(u) \in \mathbf{B}_\xi$ such that $\mathfrak{L}(\mathbf{z}_\xi(u)) \notin \mathcal{B}_\xi$, that is $\mathbf{D}_\infty\big[\mathfrak{L}(\mathbf{z}_\xi(u)),\hat{0}\big] > \xi$. Then

$$
\begin{aligned}
\xi &< \mathbf{D}_\infty\big[\mathfrak{L}(\mathbf{z}_\xi(u)),\hat{0}\big]\\
&= \mathbf{D}_\infty\Bigg[\frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}+\frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathfrak{f}\big(s,\mathbf{z}_\xi(s),\mathbf{z}_\xi(s-\sigma)\big)ds,\hat{0}\Bigg],\\
&\leq \frac{\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}\mathbf{D}_\infty\big[\mathbf{z}_0,\hat{0}\big]+\frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{z}_\xi(s),\mathbf{z}_\xi(s-\sigma)\big),\hat{0}\Big]ds,\\
&\leq \frac{\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}\mathbf{D}_\infty\big[\mathbf{z}_0,\hat{0}\big]+\frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{z}_\xi(s),\mathbf{z}_\xi(s-\sigma)\big),\hat{0}\Big],\\
&\leq \frac{\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}\mathbf{D}_\infty\big[\mathbf{z}_0,\hat{0}\big]+\frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}L_{\mathfrak{f}}.
\end{aligned}
$$

Taking limit as $\xi \longrightarrow +\infty$, we obtain that $\frac{\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}\mathbf{D}_\infty\big[\mathbf{z}_0,\hat{0}\big]+\frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}L_{\mathfrak{f}} \longrightarrow +\infty$ which is in contradiction with $\frac{\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}\mathbf{D}_\infty\big[\mathbf{z}_0,\hat{0}\big]+\frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}L_{\mathfrak{f}}$ is bounded. Therefore, for every positive constant ξ, we obtain $\mathfrak{L}(\mathbf{B}_\xi) \subseteq \mathbf{B}_\xi$. By means of Schauder's fixed point Theorem implying that there is at least one solution to the system (1.4). □

For the uniqueness result, we have the following theorem:

Theorem 3.2 *Assume that the hypotheses* ($\mathcal{H}1$)-($\mathcal{H}2$) *holds. If*

$$\sup_{0\leq u\leq T} h(u) \leq \frac{\Gamma(\gamma_1+1)}{\big(\psi(u)-\psi(0)\big)^{\gamma_1}k},$$

then the solution of system (1.4) *is unique.*

Proof: We know that $\mathbf{z}(u)$ is a solution of system (1.4) if

$$\mathfrak{L}(\mathbf{z}(u)) = \frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)}+\frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)ds,$$

hold. If $\mathbf{z}(u) \in C\big([-\sigma, T], \mathbf{E^n}\big)$ is a fixed point of $\mathfrak{L}$ which define as in Theorem 3.1, therefore $\mathbf{z}(u)$ is the solution of system (1.4). Let $\mathbf{z}_1(u), \mathbf{z}_2(u) \in C\big([-\sigma, T], \mathbf{E^n}\big)$ and for $u \in [-\sigma, 0]$, $\mathbf{z}_1(u) = \mathbf{z}_2(u) = \varphi(u)$. For all $u \in \mathbf{I}$, we have

$$\begin{aligned}
\mathbf{D}_\infty\big[\mathfrak{L}(\mathbf{z}_1(u)), \mathfrak{L}(\mathbf{z}_2(u))\big] &= \mathbf{D}_\infty\Bigg[\frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s, \mathbf{z}_1(s), \mathbf{z}_1(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}} ds, \\
&\qquad \frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s, \mathbf{z}_2(s), \mathbf{z}_2(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}} ds\Bigg], \\
&\le \mathbf{D}_\infty\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s, \mathbf{z}_1(s), \mathbf{z}_1(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}} ds, \frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s, \mathbf{z}_2(s), \mathbf{z}_2(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}} ds\Bigg], \\
&\le \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathbf{D}_\infty\Big[\mathfrak{f}(s, \mathbf{z}_1(s), \mathbf{z}_1(s-\sigma)), \mathfrak{f}(s, \mathbf{z}_2(s), \mathbf{z}_2(s-\sigma))\Big] ds, \\
&\le \frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)} H\Big(s, \mathbf{D}_\infty\big[\mathbf{z}_1(s), \mathbf{z}_2(s)\big], \mathbf{D}_\infty\big[\mathbf{z}_1(s-\sigma), \mathbf{z}_2(s-\sigma)\big]\Big), \\
&\le \frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1} k}{\Gamma(\gamma_1+1)} \sup_{0\le u\le T} H\Big(u, \mathbf{D}_\infty\big[\mathbf{z}_1(u), \mathbf{z}_2(u)\big]\Big), \\
&\le \frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1} k}{\Gamma(\gamma_1+1)} \sup_{0\le u\le T} h(u)\mathbf{D}_\infty\big[\mathbf{z}_1(u), \mathbf{z}_2(u)\big].
\end{aligned}$$

Since $\sup\limits_{0\le u\le T} h(u) \le \frac{\Gamma(\gamma_1+1)}{\big(\psi(u)-\psi(0)\big)^{\gamma_1} k}$, we have

$$\mathbf{D}_\infty\big[\mathfrak{L}(\mathbf{z}_1(u)), \mathfrak{L}(\mathbf{z}_2(u))\big] \le \mathbf{D}_\infty\big[\mathbf{z}_1(u), \mathbf{z}_2(u)\big].$$

Based on the Banach contraction principle, $\mathfrak{L}$ has an unique fixed point $\mathbf{z}(u)$.

□

4. Stability Result

In this part, Ulam–Hyers–Mittag-Leffler stability principles for the system (1.4) are provided.

Definition 4.1 *[16] The system (1.4) is said to be Ulam–Hyers–Mittag-Leffler stable with respect to* $\mathbf{E}_{\gamma_1}\big((\psi(u)-\psi(0))^{\gamma_1}\big)$ *if there exist a constant* $\omega_{\mathbf{E}_{\gamma_1}} > 0$ *such that for each* $\varepsilon > 0$ *and solution* $\mathbf{z}(u) \in C\big([-\sigma, T], \mathbf{E^n}\big)$ *of the following inequality*

$$\mathbf{D}_\infty\Big[\mathcal{D}_{0^+}^{\gamma_1,\gamma_2;\psi}\mathbf{z}(u), \mathfrak{f}\big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma)\big)\Big] \le \varepsilon, \qquad u \in [-\sigma, T] \tag{4.1}$$

there exist a solution $\mathbf{v}(u) \in C\big([-\sigma, T], \mathbf{E^n}\big)$ *of system (1.4), such that*

$$\mathbf{D}_\infty\Big[\mathbf{z}(u), \mathbf{v}(u)\Big] \le \omega_{\mathbf{E}_{\gamma_1}}\varepsilon\mathbf{E}_{\gamma_1}\big((\psi(u)-\psi(0))^{\gamma_1}\big), \qquad u \in [-\sigma, T].$$

Remark 4.1 $\mathbf{z}(u) \in C\big([-\sigma, T], \mathbf{E^n}\big)$ is a solution of (4.1) if and only if $\exists \phi \in C\big([-\sigma, T], \mathbf{E^n}\big)$ such that

(i)- $\mathbf{D}_\infty\big[\phi(u), \hat{0}\big] \le \varepsilon\mathbf{E}_{\gamma_1}\big((\psi(u)-\psi(0))^{\gamma_1}\big)$, $u \in [-\sigma, T]$,

(ii)- For $u \in \mathbf{I}$,

$$\mathbf{z}(u) \ominus_{gH} \frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} = \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\Big[\mathfrak{f}\big(s, \mathbf{z}(s), \mathbf{z}(s-\sigma)\big) + \phi(s)\Big] ds.$$

Now, we prove the Ulam–Hyers–Mittag-Leffler stability result.

Theorem 4.1 *Assume that the hypotheses $(\mathcal{H}1)$ and $(\mathcal{H}2)$ holds. Then, the system (1.4) is Ulam-Hyers-Mittag-Leffler stable.*

Proof: Let $\mathbf{z}(u)$ be the solution of the system (4.1) and $\mathbf{v}(u)$ be the solution of the proposed system (1.4). Note that for $u \in [-\sigma, 0]$, we have $\mathbf{D}_\infty\big[\mathbf{v}(u), \mathbf{z}(u)\big] = 0$.
For $u \in \mathbf{I}$, we have

$$\begin{aligned}
\mathbf{D}_\infty\big[\mathbf{v}(u), \mathbf{z}(u)\big] &= \mathbf{D}_\infty\Bigg[\frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{v}(s),\mathbf{v}(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}}ds, \frac{\mathbf{z}_0\big(\psi(u)-\psi(0)\big)^{\zeta-1}}{\Gamma(\zeta)} \\
&\quad + \frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}}ds + \frac{1}{\Gamma(\gamma_1)}\int_0^u \frac{\psi'(s)\phi(s)}{\big(\psi(u)-\psi(s)\big)^{1-\gamma_1}}ds\Bigg], \\
&\leq \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathbf{D}_\infty\Big[\mathfrak{f}\big(s,\mathbf{v}(s),\mathbf{v}(s-\sigma)\big), \mathfrak{f}\big(s,\mathbf{z}(s),\mathbf{z}(s-\sigma)\big)\Big]ds \\
&\quad + \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\mathbf{D}_\infty\Big[\phi(s), \hat{0}\Big]ds, \\
&\leq \frac{1}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}H\Big(s, \mathbf{D}_\infty\big[\mathbf{v}(s),\mathbf{z}(s)\big], \mathbf{D}_\infty\big[\mathbf{v}(s-\sigma),\mathbf{z}(s-\sigma)\big]\Big)ds \\
&\quad + \frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}\varepsilon\mathbf{E}_{\gamma_1}\big((\psi(u)-\psi(0))^{\gamma_1}\big), \\
&\leq \frac{k}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\sup_{0\leq u\leq T} H\Big(u, \mathbf{D}_\infty\big[\mathbf{v}(u),\mathbf{z}(u)\big]\Big)ds \\
&\quad + \frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}\varepsilon\mathbf{E}_{\gamma_1}\big((\psi(u)-\psi(0))^{\gamma_1}\big), \\
&\leq \frac{k}{\Gamma(\gamma_1)}\int_0^u \psi'(s)\big(\psi(u)-\psi(s)\big)^{\gamma_1-1}\sup_{0\leq u\leq T} h(u)\mathbf{D}_\infty\big[\mathbf{v}(s),\mathbf{z}(s)\big]ds \\
&\quad + \frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}\varepsilon\mathbf{E}_{\gamma_1}\big((\psi(u)-\psi(0))^{\gamma_1}\big).
\end{aligned}$$

So, the generalized Gronwall inequality implies that

$$\mathbf{D}_\infty\big[\mathbf{v}(u), \mathbf{z}(u)\big] \leq \Delta\varepsilon\mathbf{E}_{\gamma_1}\big((\psi(u)-\psi(0))^{\gamma_1}\big)\mathbf{E}_{\gamma_1}\Big(\Theta(u)\big(\psi(u)-\psi(0)\big)^{\gamma_1}\Big),$$

where

$$\Delta = \frac{\big(\psi(u)-\psi(0)\big)^{\gamma_1}}{\Gamma(\gamma_1+1)}\varepsilon \qquad \text{and} \qquad \Theta(u) = k\sup_{0\leq u\leq T} h(u).$$

Therefore

$$\mathbf{D}_\infty\big[\mathbf{v}(u), \mathbf{z}(u)\big] \leq \omega_{\mathbf{E}_{\gamma_1}}\varepsilon\mathbf{E}_{\gamma_1}\big((\psi(u)-\psi(0))^{\gamma_1}\big),$$

where $\omega_{\mathbf{E}_{\gamma_1}} = \Delta\mathbf{E}_{\gamma_1}\Big(\Theta(u)\big(\psi(u)-\psi(0)\big)^{\gamma_1}\Big)$.
Hence, from Definition 4.1, the system (1.4) is Ulam–Hyers–Mittag-Leffler stable. □

5. Example

In this section, we provide an illustration of the results from the previous part.

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{3},\frac{1}{4};u^2}\mathbf{z}(u) = \frac{\mathbf{z}(u-2)}{1+u^2(u-1)} + \frac{1}{4}\sin(\mathbf{z}(u-2)), \qquad u \in (0,1], \\ \mathcal{I}^{1-\zeta;u^2}\mathbf{z}(0^+) = \mathbf{z}_0, \\ \mathbf{z}(u) = (u+2, 1.3, -u-2), \quad u \in [-2,0], \end{cases} \tag{5.1}$$

where $\gamma_1 = \frac{1}{3}$, $\gamma_2 = \frac{1}{4}$. Then, $\zeta = \frac{1}{2}$, $\psi(u) = u^2$ and $\mathfrak{f} = \frac{\mathbf{z}(u-2)}{1+u^2(u-1)} + \frac{1}{4}\sin(\mathbf{z}(u-2))$. The verification demonstrates that all assumptions in Theorem 3.1 are met in full. Then, the problem (5.1) has a unique solution on $[-2,1]$. Also, we can verify that system (5.1) satisfies all assumptions in Theorem 4.1. Then, system (5.1) is Ulam–Hyers–Mittag-Leffler stable.

6. Conclusion

This research has examined a class of ψ-Hilfer fuzzy fractional differential equations with time delay. The Schauder's fixed point theorem is employed under non-Lipschitz conditions to demonstrate the existence result. In addition, Banach fixed point theorem are used to arrive at uniqueness result. Finally, by using the generalized Grönwall inequality Ulam–Hyers–Mittag-Leffler stability result for the main system is provided.

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Elhoussain Arhrrabi,
Laboratory of Applied Mathematics and Scienti
c Calculus,
Sultan Moulay Sliman University,
Morocco.
E-mail address: arhrrabi.elhoussain@gmail.com

and

M'hamed Elomari,
Laboratory of Applied Mathematics and Scienti
c Calculus,
Sultan Moulay Sliman University,
Morocco.
E-mail address: m.elomari@usms.ma

and

Said Melliani,
Laboratory of Applied Mathematics and Scienti
c Calculus,
Sultan Moulay Sliman University,
Morocco.
E-mail address: s.melliani@usms.ma

and

Lalla Saadia Chadli,
Laboratory of Applied Mathematics and Scienti
c Calculus,
Sultan Moulay Sliman University,
Morocco.
E-mail address: sa.chadli@yahoo.fr