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On a Subclass of Bi-univalent Functions Affiliated with Bell and Gegenbauer Polynomials

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ABSTRACT: This research paper explores the development of a novel class of analytic bi-univalent functions, leveraging the Bell polynomials along with the Gegenbauer polynomials as a fundamental component for establishing the new subclass. Analytical techniques are employed to determine and evaluate the Maclaurin coefficients $|a_2|$ and $|a_3|$ and the Fekete-Szegö functional problem for functions belonging to the constructed class. We demonstrate that several new results can be derived by specializing the parameters in our main findings. The conclusions drawn from this research enrich the theoretical foundation of this field and open new avenues for mathematical inquiry and application.

Key Words: Bi-univalent functions, Gegenbauer(or ultraspherical) polynomials, Fekete-Szegö functional, Bell polynomials.

Contents

1. Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of all analytic functions f defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Consequently, every $f \in \mathcal{A}$ can be expressed in the form of Taylor-Maclaurin series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. In addition, let $f,\ g \in \mathcal{S}$, we say that the function f is subordinate to g, written as $f \prec g$, if there exists a Schwarzian function w that is analytic in $\mathbb{U}$ and satisfies

$$w(0) = 0 \text{ and } |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)).$$

If the function g is univalent in $\mathbb{U}$, then the following equivalence holds:

$$f(z) \prec g(z) \quad \text{if and only if} \quad f(0) = g(0),$$

and

$$f(\mathbb{U}) \subset g(\mathbb{U}).$$

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It is well known that for every function $f \in \mathcal{S}$, there exists an inverse denoted by f^{-1}, which is defined as follows

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f^{-1}(f(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function f is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad \log \frac{1}{1-z}, \quad \log \sqrt{\frac{1+z}{1-z}}.$$

In contrast, the Koebe function and the functions listed below does not belong to the set Σ

$$\frac{2z - z^2}{2} \text{ and } \frac{z}{1-z^2}.$$

Lewin [1] studied the class Σ and discovered that $|a_2| < 1.51$. Following that, Brannan and Clunie [2] claimed that $|a_2| < \sqrt{2}$. On the contrary, Netanyahu [3] demonstrated that $\max_{f \in \Sigma} |a_2| = 4/3$.

The problem of estimating the coefficient for each of $|a_n|\,(n \geq 3; n \in \mathbb{N})$ is presumably still an open problem. Brannan and Taha [4] presented the subclasses of the bi-univalent function class $\Sigma, \mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike, and bi-convex functions of order α $(0 < \alpha \leq 1)$ and the first two coefficients were estimated. These results are similar to the well-known subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex function of order α $(0 \leq \alpha < 1)$. Additional examples and details related to the class Σ, can be found in references [5,6,7,8].

Orthogonal polynomials have been widely studied since their discovery by Legendre in 1784 [9]. They have been used as a mathematical approach to solve ordinary differential equations associated with model problems under certain conditions. The advantages of orthogonal polynomials in modern mathematics and their application in physics and engineering cannot be ignored. In mathematics, orthogonal polynomials play a key role in approximation theory, differential integral equations, and mathematical statistics. Additionally, these polynomials have been instrumental in various applications, such as scattering theory, quantum mechanics, signal analysis, automatic control, and axially symmetric potential theory [10,11].

Amourah *et al.* [12] investigated the Gegenbauer polynomials, whose generating function $H_\alpha(x, \xi)$ is given by

$$H_\alpha(x, z) = \frac{1}{(1 - 2xz + z^2)^\alpha}, \tag{1.3}$$

where $-1 \leq x \leq 1$, and $z \in \mathbb{U}$. Also, since H_α is analytic in $\mathbb{U}$; hence it can be written in a power series expansion as follows

$$H_\alpha(x, z) = \sum_{n=0}^{\infty} C_n^\alpha(x) z^n,$$

where $C_n^\alpha(x)$ is a Gegenbauer polynomial of a degree n.

The Gegenbauer polynomials generate Legendre polynomials and Chebyshev polynomials when setting α as $1/2$ and 1; respectively, and they can also be defined by the following recurrence relations

$$C_n^\alpha(x) = \frac{1}{n}\left[2x(n + \alpha - 1)C_{n-1}^\alpha(x) - (n + 2\alpha - 2)C_{n-1}^\alpha(x)\right],$$

with the initial values

$$\begin{aligned} C_0^\alpha(x) &= 1 \\ C_1^\alpha(x) &= 2\alpha x \\ C_2^\alpha(x) &= 2\alpha(1+\alpha)x^2 - \alpha. \end{aligned} \tag{1.4}$$

The Bell polynomials, named after the mathematician Eric Temple Bell, play a crucial role in the study of set partitions and have significant connections to Stirling and Bell numbers. These polynomials provide a combinatorial framework for understanding the ways in which sets can be partitioned into non-empty subsets. Beyond their combinatorial importance, Bell polynomials appear in various mathematical applications, including their prominent use in Faà di Bruno's formula, which generalizes the chain rule for higher-order derivatives. Their versatility makes them valuable in fields such as combinatorics, probability theory, and mathematical analysis.

Amourah et al. in [13] has introduced the following convolution

$$\mathbb{F}_\tau f(z) = \mathbb{L}(\tau, z) * f(z) = z + \sum_{k=2}^{\infty} \frac{\tau^{k-1} e^{e^{\left(-\tau^2\right)+1}} B_k}{(k-1)!} a_k z^k, \quad z \in \mathbb{U}, \tag{1.5}$$

where $\mathbb{F}_\tau : \mathcal{A} \to \mathcal{A}$ is a linear operator, B_k are the bell numbers for $k \geq 2$ and $\tau > 0$.

In recent times, numerous researchers have been investigating the concept of bi-univalent functions linked to orthogonal polynomials. Some notable studies in this area include references [14,15,16,17,18,19,20,21,22,23,24,25]. However, when it comes to Gegenbauer polynomials, there is limited existing research on bi-univalent functions.

2. The class $\mathcal{B}_\Sigma^\alpha(x, \tau, \lambda, \mu, \delta)$

Recently, Yousef *et al.* [26] introduced the following class $\mathcal{M}_\Sigma^\alpha(x, \lambda, \mu, \delta)$ of analytic and bi-univalent functions defined below.

Definition 2.1 *For $\lambda \geq 1$, $\mu \geq 1$, $\delta \geq 0$, $0 \leq \alpha \leq 1$, $\zeta = \frac{2\lambda+\mu}{2\lambda+1}$, and $t \in (1/2, 1]$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{M}_\Sigma^\alpha(\lambda, \mu, \delta)$ if the following subordinations hold for all $z, w \in \mathbb{U}$:*

$$\mathfrak{Re}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \zeta\delta z f''(z)\right) > \alpha, \tag{2.1}$$

and

$$\mathfrak{Re}\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \zeta\delta z g''(w)\right) > \alpha, \tag{2.2}$$

where the function $f \in \Sigma$ defined by (1.1) , the function $g = f^{-1}$ given by (1.2).

Definition 2.2 *Let α be a nonzero real constant $\lambda \geq 1$, $\mu \geq 0$, $\delta \geq 0$, $\zeta = \frac{2\lambda+\mu}{2\lambda+1}$ and $x \in (1/2, 1]$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma^\alpha(x, \tau, \lambda, \mu, \delta)$ if the following subordinations hold for all $z, w \in \mathbb{U}$*

$$(1-\lambda)\left(\frac{\mathbb{F}_\tau f(z)}{z}\right)^{\mu} + \lambda\left(\mathbb{F}_\tau f(z)\right)'\left(\frac{\mathbb{F}_\tau f(z)}{z}\right)^{\mu-1} + \delta\zeta z\left(\mathbb{F}_\tau f(z)\right)'' \prec H_\alpha(x, z), \tag{2.3}$$

and

$$(1-\lambda)\left(\frac{\mathbb{F}_\tau g(w)}{w}\right)^{\mu} + \lambda\left(\mathbb{F}_\tau g(w)\right)'\left(\frac{\mathbb{F}_\tau g(w)}{w}\right)^{\mu-1} + \delta\zeta w\left(\mathbb{F}_\tau g(w)\right)'' \prec H_\alpha(x, w), \tag{2.4}$$

where the function $g = f^{-1}(w)$ is defined by (1.2) and H_α is the generating function of the Gegenbauer polynomial given by (1.3).

By setting the values of the parameters λ, μ and δ, we establish some subclasses of the class $\mathcal{B}_{\Sigma}^{\alpha}(x,\lambda,\mu,\delta)$, as shown below.

Definition 2.3 *The function* $f \in {}^{1}\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\mu) := \mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\mu,0)$ *iff it satisfies the following subordination*

$$(1-\lambda)\left(\frac{\mathbb{F}_{\tau}f(z)}{z}\right)^{\mu}+\lambda\left(\mathbb{F}_{\tau}f(z)\right)'\left(\frac{\mathbb{F}_{\tau}f(z)}{z}\right)^{\mu-1}\prec H_{\alpha}(x,z),$$

and

$$(1-\lambda)\left(\frac{\mathbb{F}_{\tau}g(w)}{w}\right)^{\mu}+\lambda\left(\mathbb{F}_{\tau}g(w)\right)'\left(\frac{\mathbb{F}_{\tau}g(w)}{w}\right)^{\mu-1}\prec H_{\alpha}(x,w).$$

Definition 2.4 *The function* $f \in {}^{2}\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\delta) := \mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,1,\delta)$ *iff it satisfies the following subordination*

$$(1-\lambda)\left(\frac{\mathbb{F}_{\tau}f(z)}{z}\right)+\lambda\left(\mathbb{F}_{\tau}f(z)\right)'+\delta\zeta z\left(\mathbb{F}_{\tau}f(z)\right)''\prec H_{\alpha}(x,z),$$

and

$$(1-\lambda)\left(\frac{\mathbb{F}_{\tau}g(w)}{w}\right)+\lambda\left(\mathbb{F}_{\tau}g(w)\right)'+\delta\zeta w\left(\mathbb{F}_{\tau}g(w)\right)''\prec H_{\alpha}(x,w).$$

Definition 2.5 *The function* $f \in {}^{3}\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda) := \mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,1,0)$ *iff it satisfies the following subordination:*

$$(1-\lambda)\left(\frac{\mathbb{F}_{\tau}f(z)}{z}\right)+\lambda\left(\mathbb{F}_{\tau}f(z)\right)'\prec H_{\alpha}(x,z),$$

and

$$(1-\lambda)\left(\frac{\mathbb{F}_{\tau}g(w)}{w}\right)+\lambda\left(\mathbb{F}_{\tau}g(w)\right)'\prec H_{\alpha}(x,w).$$

Definition 2.6 *The function* $f \in {}^{4}\mathcal{B}_{\Sigma}^{\alpha}(x,\tau) := \mathcal{B}_{\Sigma}^{\alpha}(x,\tau,1,1,0)$ *iff it satisfies the following subordination:*

$$\left(\mathbb{F}_{\tau}f(z)\right)'\prec H_{\alpha}(x,z),$$

and

$$\left(\mathbb{F}_{\tau}g(w)\right)'\prec H_{\alpha}(x,w).$$

Next, we state the following lemmas that we shall use to establish the desired bounds in our study. Let $\mathcal{P}=\{p:\mathbb{U}\to\mathbb{C}\mid p$ is analytic function, such that $\mathfrak{Re}(p)>0\}$, and of the form

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots.$$

Lemma 2.1 *([27]) If* $p\in\mathcal{P}$ *, then*

$$|p_n|\leq 2, n\in\mathbb{N}. \tag{2.5}$$

Lemma 2.2 *([28]) Let* $\omega(z)=\sum_{n=1}^{\infty}\omega_n z^n,\ \ z\in\mathbb{U}$. *Then*

$$|\omega_1|\leq 1,\quad |\omega_n|\leq 1-|\omega_1|^2\quad for\quad n\in\mathbb{N}\backslash\{1\}.$$

3. Main Results

This research paper employs Gegenbauer polynomial expansions to calculate approximations for the initial coefficients for a subset of bi-univalent functions, denoted as $\mathcal{B}_{\Sigma}^{\alpha}(x,\lambda,\mu,\delta)$ associated with Bell polynomials. Additionally, we address the Fekete-Szegö problem for functions belonging to this class.

Theorem 3.1 *Assume that the function $f\in\Sigma$, in **Definition** 2.2, is a member of the class $\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\mu,\delta)$. Then*

$$|a_2|\le \frac{4\alpha x\sqrt{2|\alpha|x}}{\sqrt{\left|\left[5\tau^2(2\alpha x)^2(\mu+2\lambda)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2+\frac{12\zeta\delta}{\mu+2\lambda}\right)-8\tau^2(\alpha(1+\alpha)x^2-\alpha)(\mu+\lambda+2\zeta\delta)^2e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}\right|}}, \tag{3.1}$$

and

$$|a_3|\le \frac{\alpha^2x^2}{\tau^2(\mu+\lambda+2\zeta\delta)^2e^{2e^{1-\tau^2}}}+\frac{4|\alpha|x}{5\tau^2(\mu+2\lambda+6\zeta\delta)e^{e^{1-\tau^2}}}. \tag{3.2}$$

Proof: If f belongs to the class $\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\mu,\delta)$, then from the Definition 2.2, we can find two functions analytic in $\mathbb{U}$, namely ω and v, satisfying $\omega(0)=0=v(0)$ and for all $z,w\in\mathbb{U}$, $|\omega(z)|<1$, $|v(w)|<1$, and

$$(1-\lambda)\left(\frac{\mathbb{F}_\tau f(z)}{z}\right)^{\mu}+\lambda\left(\mathbb{F}_\tau f(z)\right)'\left(\frac{\mathbb{F}_\tau f(z)}{z}\right)^{\mu-1}+\delta\zeta z\left(\mathbb{F}_\tau f(z)\right)''=G_\alpha(x,\omega(z)), \tag{3.3}$$

and

$$(1-\lambda)\left(\frac{\mathbb{F}_\tau g(w)}{w}\right)^{\mu}+\lambda\left(\mathbb{F}_\tau g(w)\right)'\left(\frac{\mathbb{F}_\tau g(w)}{w}\right)^{\mu-1}+\delta\zeta w\left(\mathbb{F}_\tau g(w)\right)''=G_\alpha(x,v(w)). \tag{3.4}$$

From equating (3.3) and (3.4), we obtain

$$\begin{aligned}(1-\lambda)\left(\frac{\mathbb{F}_\tau f(z)}{z}\right)^{\mu}+\lambda\left(\mathbb{F}_\tau f(z)\right)'\left(\frac{\mathbb{F}_\tau f(z)}{z}\right)^{\mu-1}+\delta\zeta z\left(\mathbb{F}_\tau f(z)\right)''\\=1+C_1^{\alpha}(x)c_1z+\left[C_1^{\alpha}(x)c_2+C_2^{\alpha}(x)c_1^2\right]z^2+\dots,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}(1-\lambda)\left(\frac{\mathbb{F}_\tau g(w)}{w}\right)^{\mu}+\lambda\left(\mathbb{F}_\tau g(w)\right)'\left(\frac{\mathbb{F}_\tau g(w)}{w}\right)^{\mu-1}+\delta\zeta w\left(\mathbb{F}_\tau g(w)\right)''\\=1+C_1^{\alpha}(x)d_1w+\left[C_1^{\alpha}(x)d_2+C_2^{\alpha}(x)d_1^2\right]w^2+\dots,\end{aligned} \tag{3.6}$$

where

$$\omega(z)=\sum_{j=1}^{\infty}c_jz^j,\quad\text{and}\quad v(w)=\sum_{j=1}^{\infty}d_jw^j. \tag{3.7}$$

Referring to Lemma 2.2 and (3.7), we have

$$|c_j|\le 1\quad\text{and}\quad |d_j|\le 1\quad\text{for all}\quad j\in\mathbb{N}. \tag{3.8}$$

Hence, from equations (3.5) and (3.6), we get

$$2\tau(\mu+\lambda+2\zeta\delta)e^{e^{1-\tau^2}}a_2=C_1^{\alpha}(x)c_1, \tag{3.9}$$

$$\frac{5\tau^2}{2}(\mu+2\lambda)\left(\frac{4}{5}(\mu-1)e^{e^{1-\tau^2}}a_2^2+\left(1+\frac{6\zeta\delta}{\mu+2\lambda}\right)a_3\right)e^{e^{1-\tau^2}}=C_1^{\alpha}(x)c_2+C_2^{\alpha}(x)c_1^2, \tag{3.10}$$

$$-2\tau(\mu+\lambda+2\zeta\delta)e^{e^{1-\tau^2}}a_2=C_1^{\alpha}(x)d_1, \tag{3.11}$$

and

$$\frac{5\tau^2}{2}(\mu+2\lambda)\left(\left[\frac{4}{5}(\mu-1)e^{e^{1-\tau^2}}+2+\frac{12\zeta\delta}{\mu+2\lambda}\right]a_2^2-\left(1+\frac{6\zeta\delta}{\mu+2\lambda}\right)a_3\right)e^{e^{1-\tau^2}}=C_1^{\alpha}(x)d_2+C_2^{\alpha}(x)d_1^2. \quad (3.12)$$

It follows from (3.9) and (3.11) that

$$c_1=-d_1, \quad (3.13)$$

and

$$8\tau^2(\mu+\lambda+2\zeta\delta)^2e^{2e^{1-\tau^2}}a_2^2=\left(C_1^{\alpha}(x)\right)^2(c_1^2+d_1^2). \quad (3.14)$$

Adding (3.10) and (3.12) yield

$$5\tau^2(\mu+2\lambda)e^{e^{1-\tau^2}}\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2+\frac{12\zeta\delta}{\mu+2\lambda}\right)a_2^2=C_1^{\alpha}(x)(c_2+d_2)+C_2^{\alpha}(x)(c_1^2+d_1^2). \quad (3.15)$$

Substituting the value of $\left(c_1^2+d_1^2\right)$ from (3.14) in the right hand side of (3.15), we deduce that

$$\left[5\tau^2(\mu+2\lambda)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2+\frac{12\zeta\delta}{\mu+2\lambda}\right)-\frac{8\tau^2C_2^{\alpha}(x)(\mu+\lambda+2\zeta\delta)^2e^{e^{1-\tau^2}}}{\left(C_1^{\alpha}(x)\right)^2}\right]e^{e^{1-\tau^2}}a_2^2=C_1^{\alpha}(x)(c_2+d_2). \quad (3.16)$$

Now, using (1.4), (3.8) and (3.16), we conclude that

$$|a_2|\le\frac{4\alpha x\sqrt{2|\alpha|x}}{\sqrt{\left|\left[5\tau^2(2\alpha x)^2(\mu+2\lambda)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2+\frac{12\zeta\delta}{\mu+2\lambda}\right)-8\tau^2(\alpha(1+\alpha)x^2-\alpha)(\mu+\lambda+2\zeta\delta)^2e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}\right|}}. \quad (3.17)$$

Moreover, if we subtract (3.12) from (3.10), we have

$$5\tau^2(\mu+2\lambda+6\zeta\delta)e^{e^{1-\tau^2}}\left(a_3-a_2^2\right)=C_1^{\alpha}(x)\left(c_2-d_2\right)+C_2^{\alpha}(x)\left(c_1^2-d_1^2\right). \quad (3.18)$$

Then, in view of (3.13) and (3.14), the equation (3.18) becomes

$$a_3=\frac{\left(C_1^{\alpha}(x)\right)^2}{8\tau^2(\mu+\lambda+2\zeta\delta)^2e^{2e^{1-\tau^2}}}\left(c_1^2+d_1^2\right)+\frac{C_1^{\alpha}(x)}{5\tau^2(\mu+\lambda+2\zeta\delta)}\left(c_2-d_2\right). \quad (3.19)$$

Thus, applying (1.4), we conclude that

$$|a_3|\le\frac{\alpha^2x^2}{\tau^2(\mu+\lambda+2\zeta\delta)^2e^{2e^{1-\tau^2}}}+\frac{4|\alpha|x}{5\tau^2(\mu+2\lambda+6\zeta\delta)e^{e^{1-\tau^2}}}. \quad (3.20)$$

Hence, the proof of the theorem is complete. □

The next result regarding the Fekete–Szegö functional problem for functions in the class $\mathcal{B}_{\Sigma}^{\alpha}(x,\lambda,\mu,\delta)$.

Theorem 3.2 *If* $f\in\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\mu,\delta)$, *then*

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\frac{4\alpha x}{5\tau^2(2\lambda+\mu+6\delta\zeta)e^{e^{1-\tau^2}}} & \text{if}\quad 0\le|h(\eta)|\le\frac{1}{5\tau^2(2\lambda+\mu+6\delta\zeta)e^{e^{1-\tau^2}}}\\ 4x\,|\alpha|\,|h(\eta)| & \text{if}\quad |h(\eta)|\ge\frac{1}{5\tau^2(2\lambda+\mu+6\delta\zeta)e^{e^{1-\tau^2}}},\end{cases}$$

where

$$h(\eta)=\frac{4\alpha^2x^2(1-\eta)}{\left[5\tau^2(2\alpha x)^2(\mu+2\lambda)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2+\frac{12\zeta\delta}{\mu+2\lambda}\right)-8\tau^2(\alpha(1+\alpha)x^2-\alpha)(\mu+\lambda+2\zeta\delta)^2e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}}.$$

Proof: If f in the class $\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\mu,\delta)$, then from (3.16) and (3.18) we have

$$\begin{aligned} a_3-\eta a_2^2 &= a_2^2+\frac{C_1^{\alpha}(x)(c_2-d_2)}{5\tau^2(\mu+2\lambda+6\zeta\delta)e^{e^{1-\tau^2}}}-\eta a_2^2 \\ &= (1-\eta)a_2^2+\frac{C_1^{\alpha}(x)(c_2-d_2)}{5\tau^2(\mu+2\lambda+6\zeta\delta)e^{e^{1-\tau^2}}} \\ &= \frac{(1-\eta)\big(C_1^{\alpha}(x)\big)^3(c_2+d_2)}{\left[5\tau^2(C_1^{\alpha}(x))^2(\mu+2\lambda)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2+\frac{12\zeta\delta}{\mu+2\lambda}\right)-8\tau^2C_2^{\alpha}(x)(\mu+\lambda+2\zeta\delta)^2e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}} \\ &\quad+\frac{C_1^{\alpha}(x)(c_2-d_2)}{5\tau^2(\mu+2\lambda+6\zeta\delta)e^{e^{1-\tau^2}}} \\ &= C_1^{\alpha}(x)\left(\left[h(\eta)+\frac{1}{5\tau^2(\mu+2\lambda+6\zeta\delta)e^{e^{1-\tau^2}}}\right]c_2+\left[h(\eta)-\frac{1}{5\tau^2(\mu+2\lambda+6\zeta\delta)e^{e^{1-\tau^2}}}\right]d_2\right), \end{aligned}$$

and

$$h(\eta)=\frac{(1-\eta)\big(C_1^{\alpha}(x)\big)^2}{\left[5\tau^2(C_1^{\alpha}(x))^2(\mu+2\lambda)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2+\frac{12\zeta\delta}{\mu+2\lambda}\right)-8\tau^2C_2^{\alpha}(x)(\mu+\lambda+2\zeta\delta)^2e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}}.$$

Then, in view of (1.4), we conclude that

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\frac{4\alpha x}{5\tau^2(2\lambda+\mu+6\delta\zeta)e^{e^{1-\tau^2}}} & \text{if} \quad 0\le|h(\eta)|\le\frac{1}{5\tau^2(2\lambda+\mu+6\delta\zeta)e^{e^{1-\tau^2}}}\\ 4x\,|\alpha|\,|h(\eta)| & \text{if} \quad |h(\eta)|\ge\frac{1}{5\tau^2(2\lambda+\mu+6\delta\zeta)e^{e^{1-\tau^2}}},\end{cases}$$

this completes the proof of Theorem 3.2.

4. Consequences and Corollaries

By referring to the Definition (2.3) (considering $\delta=0$), Definition (2.4) (considering $\lambda=1$), Definition 2.5 (considering $\mu=1$ and $\delta=0$), and Definition 2.6 (considering $\mu=1,\delta=0$ and $\lambda=1$), and from Theorems 3.1 and 3.2 we deduce the next consequences, respectively. Setting $\delta=0$, we obtain the following corollary.

Corollary 4.1 *If $f\in{}^1\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\mu)$, then*

$$|a_2|\le\frac{4\alpha x\sqrt{2|\alpha|x}}{\sqrt{\left|\left[5\tau^2(2\alpha x)^2(\mu+2\lambda)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2\right)-8\tau^2(\alpha(1+\alpha)x^2-\alpha)(\mu+\lambda)^2e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}\right|}}, \tag{4.1}$$

$$|a_3|\le\frac{\alpha^2x^2}{\tau^2(\mu+\lambda)^2e^{2e^{1-\tau^2}}}+\frac{4|\alpha|x}{5\tau^2(\mu+2\lambda)e^{e^{1-\tau^2}}}, \tag{4.2}$$

and

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\frac{4\alpha x}{5\tau^2(2\lambda+\mu)e^{e^{1-\tau^2}}} & \textit{if} \quad 0\le|h_1(\eta)|\le\frac{1}{5\tau^2(2\lambda+\mu)e^{e^{1-\tau^2}}}\\ 4x\,|\alpha|\,|h_1(\eta)| & \textit{if} \quad |h_1(\eta)|\ge\frac{1}{5\tau^2(2\lambda+\mu)e^{e^{1-\tau^2}}},\end{cases}$$

where

$$h_1(\eta)=\frac{4\alpha^2x^2(1-\eta)}{\left[5\tau^2(2\alpha x)^2(\mu+2\lambda)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2\right)-8\tau^2(\alpha(1+\alpha)x^2-\alpha)(\mu+\lambda)^2e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}}.$$

Next, setting $\mu = 1$ yields the following consequence.

Corollary 4.2 *[13] If $f \in {}^2\mathcal{B}_\Sigma^\alpha(x,\tau,\lambda,\delta)$, then*

$$|a_2| \leq \frac{4\alpha x\sqrt{2|\alpha|x}}{\sqrt{\left|\left[5\tau^2(2\alpha x)^2(1+2\lambda)\left(2+\frac{12\zeta\delta}{\mu+2\lambda}\right) - 8\tau^2(\alpha(1+\alpha)x^2-\alpha)(1+\lambda+2\zeta\delta)^2 e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}\right|}}, \quad (4.3)$$

and

$$|a_3| \leq \frac{\alpha^2x^2}{\tau^2(1+\lambda+2\zeta\delta)^2 e^{2e^{1-\tau^2}}} + \frac{4|\alpha|x}{5\tau^2(1+2\lambda+6\zeta\delta)e^{e^{1-\tau^2}}}, \quad (4.4)$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{4\alpha x}{5\tau^2(1+2\lambda+6\delta\zeta)e^{e^{1-\tau^2}}} & \text{if} \quad 0 \leq |h_2(\eta)| \leq \frac{1}{5\tau^2(1+2\lambda+6\delta\zeta)e^{e^{1-\tau^2}}} \\ 4x\,|\alpha|\,|h_2(\eta)| & \text{if} \quad |h_2(\eta)| \geq \frac{1}{5\tau^2(1+2\lambda+6\delta\zeta)e^{e^{1-\tau^2}}}, \end{cases}$$

where

$$h_2(\eta) = \frac{4\alpha^2x^2(1-\eta)}{\left[5\tau^2(2\alpha x)^2(1+2\lambda)\left(2+\frac{12\zeta\delta}{1+2\lambda}\right) - 8\tau^2(\alpha(1+\alpha)x^2-\alpha)(1+\lambda+2\zeta\delta)^2 e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}}.$$

Now, setting $\lambda = 1$, and $\delta = 0$, we have the following consequence.

Corollary 4.3 *If $f \in {}^3\mathcal{B}_\Sigma^\alpha(x,\tau,\mu)$, then*

$$|a_2| \leq \frac{4\alpha x\sqrt{2|\alpha|x}}{\sqrt{\left|\left[5\tau^2(2\alpha x)^2(\mu+2)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2\right) - 8\tau^2(\alpha(1+\alpha)x^2-\alpha)(\mu+1)^2 e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}\right|}}, \quad (4.5)$$

$$|a_3| \leq \frac{\alpha^2x^2}{\tau^2(\mu+1)^2 e^{2e^{1-\tau^2}}} + \frac{4|\alpha|x}{5\tau^2(\mu+2)e^{e^{1-\tau^2}}}, \quad (4.6)$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{4\alpha x}{5\tau^2(\mu+2)e^{e^{1-\tau^2}}} & \text{if} \quad 0 \leq |h_3(\eta)| \leq \frac{1}{5\tau^2(\mu+2)e^{e^{1-\tau^2}}} \\ 4x\,|\alpha|\,|h_3(\eta)| & \text{if} \quad |h_3(\eta)| \geq \frac{1}{5\tau^2(\mu+2)e^{e^{1-\tau^2}}}, \end{cases}$$

where

$$h_3(\eta) = \frac{4\alpha^2x^2(1-\eta)}{\left[5\tau^2(2\alpha x)^2(\mu+2)\left(\frac{8}{5}(\mu-1)e^{e^{1-\tau^2}}+2\right) - 8\tau^2(\alpha(1+\alpha)x^2-\alpha)(\mu+1)^2 e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}}.$$

Finally, sitting $\mu = 1$, $\delta = 0$, and $\lambda = 1$, we obtain our last consequence.

Corollary 4.4 *If $f \in {}^4\mathcal{B}_\Sigma^\alpha(x,\tau)$, then*

$$|a_2| \leq \frac{4\alpha x\sqrt{2|\alpha|x}}{\sqrt{\left|\left[30\tau^2(2\alpha x)^2 - 32\tau^2(\alpha(1+\alpha)x^2-\alpha)e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}\right|}}, \quad (4.7)$$

$$|a_3| \leq \frac{\alpha^2x^2}{4\tau^2 e^{2e^{1-\tau^2}}} + \frac{4|\alpha|x}{15\tau^2 e^{e^{1-\tau^2}}}, \quad (4.8)$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{4\alpha x}{15\tau^2 e^{e^{1-\tau^2}}} & \text{if} \quad 0 \leq |h_4(\eta)| \leq \frac{1}{15\tau^2 e^{e^{1-\tau^2}}} \\ 4x\,|\alpha|\,|h_4(\eta)| & \text{if} \quad |h_4(\eta)| \geq \frac{1}{15\tau^2 e^{e^{1-\tau^2}}}, \end{cases}$$

where

$$h_4(\eta) = \frac{4\alpha^2 x^2(1-\eta)}{\left[30\tau^2(2\alpha x)^2 - 32\tau^2(\alpha(1+\alpha)x^2-\alpha)e^{e^{1-\tau^2}}\right]e^{e^{1-\tau^2}}}.$$

5. Conclusions

In the current investigation, we have established a new comprehensive subclass $\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\mu,\delta)$ of normalized bi-univalent analytic functions that involve Gegenbauer polynomials and a Bell polynomial series. First, we have provided the best estimates for the first initial Taylor–Maclaurin coefficients, $|a_2|$ and $|a_3|$, and then we solved the Fekete–Szegö inequality problem. Moreover, by setting appropriate values of the parameters δ, μ, and λ, we obtain similar findings for the subclasses ${}^1\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\mu)$, ${}^2\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda,\delta)$, ${}^3\mathcal{B}_{\Sigma}^{\alpha}(x,\tau,\lambda)$, and ${}^4\mathcal{B}_{\Sigma}^{\alpha}(x,\tau)$. The results presented in the present work will lead to many different results for the subclasses of Legendre polynomials $\mathcal{B}_{\Sigma}^{1/2}(x,\tau,\lambda,\mu,\delta)$ and Chebyshev polynomials of the second kind $\mathcal{B}_{\Sigma}^{1}(x,\tau,\lambda,\mu,\delta)$.

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