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Lie Symmetry Analysis and Conservation Laws for the Time Fractional Biased Random Motion Equation

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ABSTRACT: In this paper, the Lie group method is applied to obtain the Lie group symmetries for the Riemann-Liouville time fractional Biased Random motion equation. These symmetries are employed to reduce the studied equation to a family of fractional ordinary differential equations in some particular cases with Erdélyi-Kober fractional operator. Moreover, some exact solutions and conserved quantities are given.

Key Words: Biased random motion equation, Fractional derivative, Lie symmetry, Symmetry reductions, Conservation laws.

Contents

1. Introduction

Fractional calculus is considered to be a generalization of differentiation to non-integer order. The emergence of this theory took place around 1695. Since that time, this field have received considerable attention owing to their applicability in different fields of sciences as engineering [1,2], chemistry [3], electrochemistry [4], biology [5,6], rheology [7], economics [8], electronics [9,10], dynamics [11,12], thermodynamics [13], vibration [14,10], viscoelasticity [15]. It has attracted the attention of many famous mathematicians who have contributed to the development of this theory. They proposed different definitions of fractional derivatives called fractional derivative approachs. Out of those derivatives, some ones are worth mentioning: Riemann-Liouville's approach [1,16], Weyl's approach [13], Grünwald-Letnikov approach [16], Caputo's approach [13].

Lie symmetry analysis is one of the most powerful and general approaches to study differential equations. By this method one can construct exact solutions, reduce the order, reduce the number of independent variables, etc. This approach has been presented by many authors and has been used in a wide range of physical and engineering models to study their invariance properties and construct their exact solutions, see for example [17,18]. The systematic method presented in this work is based on the extension formula established in [19] who derived the allowable extension formula by fractional differential equations where the considered fractional derivative is in the sense of Caputo and Riemann-Liouville and used by other authors [20,21,24,25,22,23].

The conservation law is a mathematical formulation of the statement that the total amount of a certain quantity remains unchanged during the evolution of a physical or biological system. Conservation

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laws play an important role in many problems appearing in mathematical physics and biologic. Noether [26] was the first to investigate symmetries to obtain conservation laws for a differential equation with integer order. She proved that if a Lagrangian admits a symmetry, then this symmetry is associated with a conserved quantity. Recently, Ibragimov [27] proposed a generalization of Noether's theorem in order to construct conservation laws of fractional differential equations.

Biased Random motion (BRM) model [28,29] is a study of attraction or repulsion between organisms

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(ku\frac{\partial u}{\partial x}\right), \tag{1.1}$$

where u is the population density, D is the diffusion coefficient and k is a measure of the tendency to move away from conspecifics when $k > 0$ and is a measure of the tendency to move towards conspecifics when $k < 0$. The strength of attraction (or avoidance) is ku and thus is a linear function of density. This model leads to clumping of organisms if the aggregation component dominates the random component and a distribution of animals that is bounded, implying that animals spread at a finite speed. Here we study the time fractional version of (BRM) equation. More precisely, by using the invariance properties we construct some exact solutions and conserved quantities.

This paper is organized as follows: First, in Section. (2) we recall definitions and some basic properties of the Riemann-Liouville derivative which are needed in the sequel. In section (3) we present the Lie symmetry analysis method for a fractional partial differential equation. Section (4) is devoted to construct Lie symmetry algebra of the time fractional Biased Random motion equation. Next, in section (5), we use symmetry reductions, we construct some exact solutions of the time fractional Biased Random motion equation. Finally, in section (6), based on the conservation theorem of Ibragimov, we construct some conserved quantities of the studied equation.

2. Preliminaries

In this section we recall some basic definitions and useful properties of fractional integrals and fractional derivatives.

Definition 2.1 *For $\alpha > 0$, the left-sided and right-sided fractional integrals of order α, respectively, are defined by:*

$$\begin{aligned} {}_0\mathcal{J}_t^\alpha f(t,x) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x)ds, \\ {}_t\mathcal{J}_T^\alpha f(t,x) &= \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1} f(s,x)ds, \end{aligned} \tag{2.1}$$

where $t > 0$, $\mathcal{J}_t^0 f(t,x) = f(t,x)$ and $\Gamma(\alpha)$ is the standard Euler gamma function.

Definition 2.2 *For $\alpha > 0$, the Riemann-Liouville fractional derivative of order α is defined by:*

$$\mathcal{D}_t^\alpha u(t,x) = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n, \\ \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t (t-s)^{n-\alpha-1}u(s,x)ds, & 0 \le n-1 < \alpha < n, \end{cases} \tag{2.2}$$

where $n \in \mathbb{N}$.

Properties For the suitable functions $f(t)$ and $g(t)$, the operators mentioned above satisfy :

1. $\mathcal{J}_t^\alpha\left(\mathcal{D}_t^\alpha f(t)\right) = f(t) - \sum_{r=0}^{n-1} \frac{f^{(r)}(0)}{r!}t^r, \quad n-1 < \alpha \le n,$
2. $\mathcal{J}_t^\alpha(f(t)+g(t)) = \mathcal{J}_t^\alpha(f(t)) + \mathcal{J}_t^\alpha(g(t)),$
3. $\mathcal{D}_t^\alpha(f(t)+g(t)) = \mathcal{D}_t^\alpha(f(t)) + \mathcal{D}_t^\alpha(g(t)),$
4. $\mathcal{J}_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma},$

5. $\mathcal{D}_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}$,
6. $\mathcal{D}_t^{\alpha} c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}$, where c is a constant,
7. $\mathcal{D}_t^{\alpha} f(t) = D_t^n \mathcal{J}_t^{n-\alpha} f(t), \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}$,
8. $\mathcal{D}_t^{\alpha}[f(t)g(t)] = \sum_{n=0}^{\infty} \left(\begin{array}{c} \alpha \\ n \end{array} \right) \mathcal{D}_t^{\alpha-n} f(t) \mathcal{D}_t^n g(t), \quad \alpha > 0$,
   where the generalized binomial coefficient is given by:
$$\left(\begin{array}{c} \alpha \\ n \end{array} \right) = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.$$

3. Lie symmetry method for fractional partial differential equations

Consider the following time fractional partial differential equation:
$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = F\left(t, x, u, u_x, u_{xx}, \ldots\right). \tag{3.1}$$
We consider a one-parameter group of point transformations:
$$\begin{aligned} \tilde{t} &= t + \epsilon\tau(t,x,u) + o\left(\epsilon^2\right), \\ \tilde{x} &= x + \epsilon\xi(t,x,u) + o\left(\epsilon^2\right), \\ \tilde{u} &= u + \epsilon\eta(t,x,u) + o\left(\epsilon^2\right), \end{aligned} \tag{3.2}$$
where ϵ is the group parameter, τ, ξ and η are infinitesimal functions.
The associated infinitesimal generator of the group (3.2) is given by:
$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{3.3}$$
The vectors field X generates a symmetry of the equation (3.1) if and only if the invariance criterion
$$X^{(\alpha)}(\Delta)\Big|_{\Delta=0} = 0, \tag{3.4}$$
is satisfied, where $\Delta = \frac{\partial^{\alpha} u}{\partial t^{\alpha}} - F$ and $X^{(\alpha)}$ is the $\alpha-$ prolongation of X given by:
$$X^{(\alpha)} = X + \eta^x \frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{\alpha,t}\frac{\partial}{\partial\left(\partial_t^{\alpha} u\right)}, \tag{3.5}$$
where η^x, η^{xx} and $\eta^{\alpha,t}$ are the extended infinitesimals given by:
$$\begin{aligned} \eta^x =& D_x(\eta) - u_x\left(D_x\xi\right) - u_t\left(D_x\tau\right) \\ =& \eta_x + \left(\eta_u - \xi_x\right)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\ \eta^{xx} =& D_x\left(\eta^x\right) - u_{xx}\left(D_x\xi\right) - u_{xt}\left(D_x\tau\right) \\ =& \eta_{xx} + \left(2\eta_{xu} - \xi_{xx}\right)u_x - \tau_{xx}u_t + \left(\eta_{uu} - 2\xi_{xu}\right)u_x^2 - 2\tau_{xu}u_x u_t \\ &- \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t + \left(\eta_u - 2\xi_x\right)u_{xx} - 2\tau_x u_{xt} \\ &- 3\xi_u u_x u_{xx} - \tau_u u_{xx}u_t - 2\tau_u u_x u_{xt}, \\ \eta^{\alpha,t} =& D_t^{\alpha}(\eta) + \xi D_t^{\alpha}\left(u_x\right) - D_t^{\alpha}\left(\xi u_x\right) + D_t^{\alpha}\left(D_t(\tau)u\right) - D_t^{\alpha+1}(\tau_u) + \tau D_t^{\alpha+1}(u), \end{aligned} \tag{3.6}$$
with D_t^{α} denotes the total time fractional derivative.
The α th-extended infinitesimals $\eta^{\alpha,t}$ can be rewritten as:
$$\begin{aligned} \eta^{\alpha,t} =& \frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} + \left(\eta_u - \alpha D_t(\tau)\right)\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - u\frac{\partial^{\alpha}\eta_u}{\partial t^{\alpha}} + \mu \\ &+ \sum_{n=1}^{\infty}\left[\left(\begin{array}{c} \alpha \\ n \end{array} \right)\frac{\partial^n \eta_u}{\partial t^n} - \left(\begin{array}{c} \alpha \\ n+1 \end{array} \right) D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(u) \\ &- \sum_{n=1}^{\infty}\left(\begin{array}{c} \alpha \\ n \end{array} \right) D_t^n(\xi) D_t^{\alpha-n}\left(u_x\right), \end{aligned} \tag{3.7}$$

where

$$\mu=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\begin{pmatrix}\alpha\\n\end{pmatrix}\begin{pmatrix}n\\m\end{pmatrix}\begin{pmatrix}k\\r\end{pmatrix}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}\left[u^{k-r}\right]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial^k u}. \tag{3.8}$$

The invariance condition should arrive at $\tau(t,x,u)|_{t=0}=0$.

4. Lie symmetry analysis for the time fractional Biased Random motion equation

The time fractional Biased Random motion (FBRM) equation [28,29] is given by:

$$\frac{\partial^\alpha u}{\partial t^\alpha}-D\frac{\partial^2 u}{\partial x^2}-ku\frac{\partial^2 u}{\partial x^2}-k\left(\frac{\partial u}{\partial x}\right)^2=0, \tag{4.1}$$

According to the Lie symmetry analysis, we retrieve the invariance criterion to be of the form:

$$\eta^{\alpha,\mathrm{t}}+(-D-ku)\eta^{xx}-2ku_x\eta^x-ku_{xx}\eta=0. \tag{4.2}$$

After some algebraic calculations, the invariance criterion condition leads to the system of determining equations:

$$\begin{aligned}
&\tau_x=\tau_u=\xi_u=\xi_t=0,\\
&-2k\eta_x+(-D-ku)(2\eta_{xu}-\xi_{xx})=0,\\
&-2k\left(\eta_u-\xi_x\right)+(-D-ku)(\eta_{uu}-2\xi_{xu})+k\left(\eta_u-\alpha D_t(\tau)\right)=0,\\
&-k\eta+(-D-ku)\left(\eta_u-2\xi_x\right)+(D+ku)\left(\eta_u-\alpha D_t(\tau)\right)=0,\\
&\frac{\partial^\alpha\eta}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}+(-D-ku)\eta_{xx}=0,\\
&\begin{pmatrix}\alpha\\n\end{pmatrix}\frac{\partial^n\eta_u}{\partial t^n}-\begin{pmatrix}\alpha\\n+1\end{pmatrix}D_t^{n+1}(\tau)=0,\quad n=0,1,2...
\end{aligned} \tag{4.3}$$

The solution of the above system is obtained to be of the form:

$$\tau=at,\quad \xi=bx+c,\quad \eta=(2b-\alpha a)u+\frac{D(2b-\alpha a)}{k}, \tag{4.4}$$

where a,b and c are arbitrary constants.
Then, the Lie symmetry algebra of FBRM equation is spanned by the following vector fields:

$$\begin{aligned}
X_1&=x\frac{\partial}{\partial x}+\left(2u+\frac{2D}{k}\right)\frac{\partial}{\partial u},\\
X_2&=\frac{\partial}{\partial x},\\
X_3&=t\frac{\partial}{\partial t}+\left(-\alpha u-\frac{\alpha D}{k}\right)\frac{\partial}{\partial u}.
\end{aligned} \tag{4.5}$$

According to the commutator operator $[X_i,X_j]=X_iX_j-X_jX_i$, we get the following commutators:

$$[X_1,X_2]=-X_2,\ \ [X_2,X_1]=X_2,\ \ [X_1,X_3]=[X_2,X_3]=0. \tag{4.6}$$

5. Lie symmetry reductions and exact solutions

We construct similarity variables and similarity solutions to reduce the FBRM equation.
Case 1. Reduction with X_1
The similarity variable corresponding to X_1 can be obtained by solving the following characteristic equation:

$$\frac{dt}{0}=\frac{dx}{x}=\frac{du}{2u+\frac{2D}{k}}. \tag{5.1}$$

The invariants are given by:

$$u(\xi, x) = x^2\phi(\xi) - \frac{D}{k}, \tag{5.2}$$

with $\xi = t$, and $\phi(\xi)$ is a function of ξ.
Inserting these into (4.1), it yields the reduced equation,

$$\left(D_t^{\alpha}\right)(\phi) - \frac{D}{kx^2\Gamma(1-\alpha)}\xi^{-\alpha} - 6\phi^2 = 0. \tag{5.3}$$

By deriving the equation (5.3) with respect to x we obtain $D = 0$, so the equation (5.3) becomes:

$$\left(D_t^{\alpha}\right)(\phi) - 6\phi^2 = 0. \tag{5.4}$$

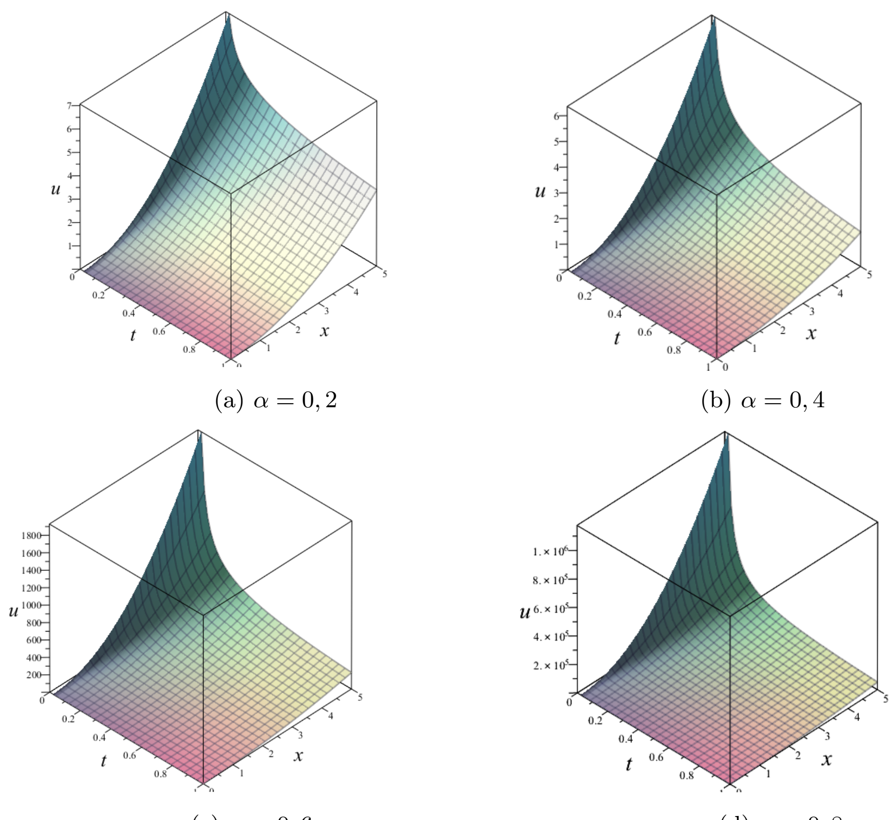
The solution of the obtained equation (5.4) is given by:

$$\phi = \frac{\Gamma(1-\alpha)}{6\Gamma(1-2\alpha)}t^{-\alpha}. \tag{5.5}$$

Consequently, a solution of FBRM equation is obtained to be of the form:

$$u(t, x) = \frac{\Gamma(1-\alpha)}{6\Gamma(1-2\alpha)}x^2t^{-\alpha}. \tag{5.6}$$

Next, to see the effect of the order α on the solution, we draw graphical representations for this solution displayed in Figure. 1.



(a) $\alpha = 0,2$ (b) $\alpha = 0,4$

(c) $\alpha = 0,6$ (d) $\alpha = 0,8$

Figure 1: Representation of the solution (5.6) for different values of α.

Remark 5.1 It can be shown that the function u is intensively increases where the α-parameter goes to 1.

Case 2. Reduction with X_2

The corresponding characteristic equation for this vector field is:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}. \tag{5.7}$$

So, the invariants are

$$u = \phi(\xi), \quad \text{where} \quad \xi = t. \tag{5.8}$$

Using of invariants (5.8) and the equation (4.1) then, we get:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = 0. \tag{5.9}$$

That yields,

$$u(t,x) = \frac{c}{\Gamma(\alpha)} t^{\alpha-1}, \tag{5.10}$$

as an exact solution of the studied equation (4.1).

Case 3. Reduction with X_3

The similarity variable corresponding to X_3 can be obtained by solving the following characteristic equation:

$$\frac{dt}{t} = \frac{dx}{0} = \frac{du}{-\alpha u - \frac{\alpha D}{k}}. \tag{5.11}$$

We get:

$$u(t,x) = t^{-\alpha}\varphi(x) - \frac{D}{k}, \tag{5.12}$$

where $\varphi(x)$ is a function of x.
Substituting expression (5.12) into (4.1) yields the reduced equation:

$$\frac{\varphi}{\Gamma(1-\alpha)} - \frac{D}{k\Gamma(1-\alpha)} t^\alpha - k\varphi\varphi_{xx} - k\varphi_x^2 = 0. \tag{5.13}$$

Taking in mind that $\varphi = \varphi(x)$, we get $D = 0$, hence the above equation becomes:

$$\frac{\varphi}{\Gamma(1-\alpha)} - k\varphi\varphi_{xx} - k\varphi_x^2 = 0. \tag{5.14}$$

Solving the reduced equation, we obtain:

$$\varphi(x) = \frac{1}{2}C_1 \left[(C_1)^2 \left(\frac{1}{k\Gamma(1-\alpha)} \right)^2 + \frac{2C_1}{k\Gamma(1-\alpha) e^{\frac{C_2+x}{C_1}}} + \frac{1}{e^{2\frac{C_2+2x}{C_1}}} \right] e^{\frac{C_2+x}{C_1}}, \tag{5.15}$$

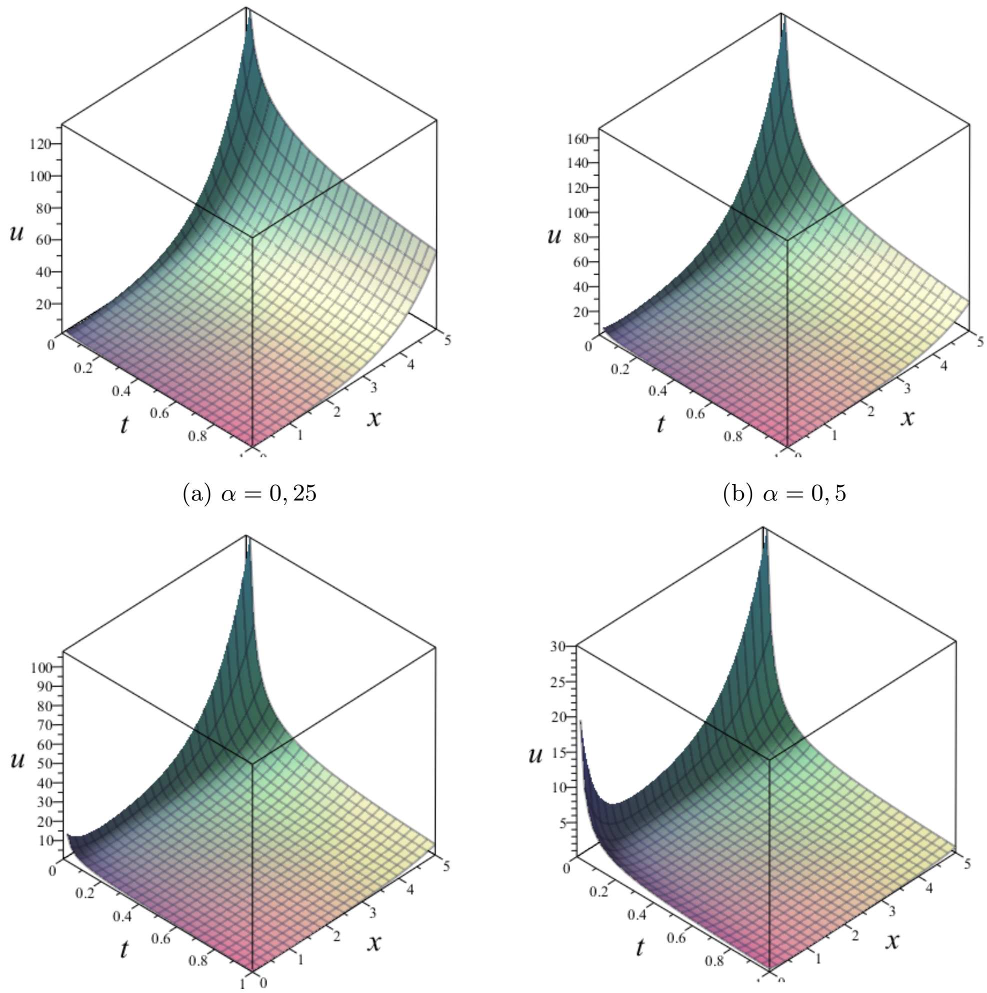
where C_1 and C_2 are arbitrary constants.
Hence, an exact solution obtained in this case is given by:

$$u(t,x) = t^{-\alpha}\frac{1}{2}C_1 \left[(C_1)^2 \left(\frac{1}{k\Gamma(1-\alpha)} \right)^2 + \frac{2C_1}{k\Gamma(1-\alpha) e^{\frac{C_2+x}{C_1}}} + \frac{1}{e^{2\frac{C_2+2x}{C_1}}} \right] e^{\frac{C_2+x}{C_1}}. \tag{5.16}$$

To see the effect of the order α on the solution, we present a graph for this solution in Figure. 2.

Remark 5.2 Note that when the α-parameter goes to 1, the function u intensively decreases.

Case 4. Reduction with $X_{1,3} = X_1 + \lambda X_3$, $(\lambda \neq \{0, \frac{2}{\alpha}\})$



(a) $\alpha = 0,25$ (b) $\alpha = 0,5$

(c) $\alpha = 0,75$ (d) $\alpha = 0,9$

Figure 2: Representation of the solution (5.16) for $C_2 = 0$, $C_1 = k = 1$ and different values of α .

We have:

$$X_{1,3} = x\frac{\partial}{\partial x} + \lambda t\frac{\partial}{\partial t} + (2-\alpha\lambda)\left(u + \frac{D}{k}\right)\frac{\partial}{\partial u}.$$

The similarity variables corresponding to $X_{1,3}$ can be obtained by solving the following characteristic equation:

$$\frac{dt}{\lambda t} = \frac{dx}{x} = \frac{du}{(2-\alpha\lambda)\left(u+\frac{D}{k}\right)}. \tag{5.17}$$

Integrating the above system leads to:

$$u = t^{\frac{2}{\lambda}-\alpha}\varphi(\xi) - \frac{D}{k}, \text{ and } \xi = xt^{\frac{-1}{\lambda}}. \tag{5.18}$$

Theorem 5.1 *Similarity transformation $u = t^{\frac{2}{\lambda}-\alpha}\varphi(\xi) - \frac{D}{k}$, where $\xi = xt^{\frac{-1}{\lambda}}$ converts the time fractional biased random motion equation to a nonlinear fractional ordinary differential equation (NFODE) given by:*

$$\left(\mathcal{P}_{\lambda}^{1-2\alpha+\frac{2}{\lambda},\alpha}\varphi\right)(\xi) - \frac{D}{k\Gamma(1-\alpha)}t^{\alpha-\frac{2}{\lambda}} - 2Dt^{\alpha-\frac{2}{\lambda}}\varphi_{\xi\xi} + k\varphi_{\xi}^2 + k\varphi\varphi_{\xi} = 0, \tag{5.19}$$

where $\mathcal{P}_{\sigma}^{\beta,\alpha}\varphi$ is the Erdélyi-Kober fractional operator defined by:

$$\begin{aligned} \left(\mathcal{P}_{\sigma}^{\beta,\alpha}\varphi\right)(\xi) &= \prod_{j=0}^{n-1}\left(\beta + j - \frac{1}{\sigma}\xi\frac{d}{d\xi}\right)\left(\mathcal{K}_{\sigma}^{\beta+\alpha,n-\alpha}\varphi\right)(\xi), \\ \textit{and} \quad n &= \begin{cases} \alpha, & \alpha \in \mathbb{N}, \\ [\alpha]+1, & \alpha \notin \mathbb{N}, \end{cases} \end{aligned} \tag{5.20}$$

with

$$\begin{cases} \left(\mathcal{K}_{\sigma}^{\beta,\alpha}\varphi\right)(\xi) = \frac{1}{\Gamma(\alpha)}\int_1^{\infty}(\mu-1)^{\alpha-1}\mu^{-(\beta+\alpha)}\varphi\left(\xi\mu^{\frac{1}{\sigma}}\right)d\mu, & \alpha > 0, \\ \left(\mathcal{K}_{\sigma}^{\beta,\alpha}\varphi\right)(\xi) = \varphi(\xi). & \alpha = 0, \end{cases} \tag{5.21}$$

is the Erdélyi-Kober integral operator.

Proof: Using the Riemann-Liouville fractional derivative definition for similarity transformation $u = t^{\frac{2}{\lambda}-\alpha}\varphi(\xi) - \frac{D}{k}$ with $\xi = xt^{\frac{-1}{\lambda}}$,
we have:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^n}{\partial t^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\left(s^{\frac{2}{\lambda}-\alpha}\varphi\left(xs^{\frac{-1}{\lambda}}\right) - \frac{D}{k}\right)ds\right). \tag{5.22}$$

Let $\mu = \frac{t}{s}$, then $ds = -\frac{t}{\mu^2}d\mu$.
Thus,

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^n}{\partial t^n}\left(\frac{t^{n+\frac{2}{\lambda}-2\alpha}}{\Gamma(n-\alpha)}\int_1^{\infty}(\mu-1)^{n-\alpha-1}\mu^{-\left(n+1+\frac{2}{\lambda}-2\alpha\right)}\varphi\left(\xi\mu^{\frac{1}{\lambda}}\right)d\mu\right) - \frac{D}{k\Gamma(1-\alpha)}t^{-\alpha}, \tag{5.23}$$

Based on the last equation with the definition of Erdélyi-Kober fractional integral operator, one can get:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^n}{\partial t^n}\left(t^{n+\frac{2}{\lambda}-2\alpha}\left(\mathcal{K}_{\lambda}^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right) - \frac{D}{k\Gamma(1-\alpha)}t^{-\alpha}. \tag{5.24}$$

In addition, we have:

$$\begin{aligned} &\frac{\partial^n}{\partial t^n}\left(t^{n+\frac{2}{\lambda}-2\alpha}\left(\mathcal{K}_{\lambda}^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right) \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n+\frac{2}{\lambda}-2\alpha}\left(\mathcal{K}_{\lambda}^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right)\right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\left(n+\frac{2}{\lambda}-2\alpha\right)t^{n+\frac{2}{\lambda}-2\alpha-1}\left(\mathcal{K}_{\lambda}^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi) + t^{n+\frac{2}{\lambda}-2\alpha}\frac{\partial}{\partial t}\left(\mathcal{K}_{\lambda}^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right]. \end{aligned} \tag{5.25}$$

Applying the chain rule, we get:

$$t^{n+\frac{2}{\lambda}-2\alpha}\frac{\partial}{\partial t}\left(\mathcal{K}_{\lambda}^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi) = -\frac{1}{\lambda}t^{n+\frac{2}{\lambda}-2\alpha-1}\xi\frac{\partial}{\partial\xi}\left(\left(\mathcal{K}_{\lambda}^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right), \tag{5.26}$$

and thus,

$$\begin{aligned} &\frac{\partial^n}{\partial t^n}\left(t^{n+\frac{2}{\lambda}-2\alpha}\left(\mathcal{K}_{\lambda}^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right) \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n+\frac{2}{\lambda}-2\alpha-1}\left(n+\frac{2}{\lambda}-2\alpha-\frac{1}{\lambda}\xi\frac{\partial}{\partial\xi}\right)\left(\mathcal{K}_{\lambda}^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right]. \end{aligned} \tag{5.27}$$

Repeating the same procedure for $(n-1)$ times, we obtain:

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\left(t^{n+\frac{2}{\lambda}-2\alpha}\left(\mathcal{K}_\lambda^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right)\\
&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n+\frac{2}{\lambda}-2\alpha-1}\left(n+\frac{2}{\lambda}-2\alpha-\frac{1}{\lambda}\xi\frac{\partial}{\partial\xi}\right)\left(\mathcal{K}_\lambda^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right]\\
&\vdots\\
&=t^{\frac{2}{\lambda}-2\alpha}\prod_{j=0}^{n-1}\left(1+\frac{2}{\lambda}-2\alpha+j+\frac{-1}{\lambda}\xi\frac{d}{d\xi}\right)\left(\mathcal{K}_\lambda^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi).
\end{aligned} \tag{5.28}$$

Using the definition of Erdélyi-Kober fractional differential operator, we get:

$$\frac{\partial^n}{\partial t^n}\left(t^{n+\frac{2}{\lambda}-2\alpha}\left(\mathcal{K}_\lambda^{1-\alpha+\frac{2}{\lambda},n-\alpha}\varphi\right)(\xi)\right)=t^{\frac{2}{\lambda}-2\alpha}\left(\mathcal{P}_\lambda^{1-2\alpha+\frac{2}{\lambda},\alpha}\varphi\right)(\xi). \tag{5.29}$$

By calculation of u_x and u_{xx} we find that the time fractional Biased Random motion equation reduces into a nonlinear fractional ordinary differential equation (NFODE) given by:

$$\left(\mathcal{P}_\lambda^{1-2\alpha+\frac{2}{\lambda},\alpha}\varphi\right)(\xi)-\frac{D}{k\Gamma(1-\alpha)}t^{\alpha-\frac{2}{\lambda}}-2Dt^{\alpha-\frac{2}{\lambda}}\varphi_{\xi\xi}+k\varphi_\xi^2+k\varphi\varphi_\xi=0.$$

□

Case 5. Reduction with $X_{2,3}=X_3+\lambda X_2 \quad \lambda\neq 0$.

We have:

$$X_{2,3}=t\frac{\partial}{\partial t}+\lambda\frac{\partial}{\partial x}+\left(-\alpha u-\frac{\alpha D}{k}\right)\frac{\partial}{\partial u}.$$

The similarity variables corresponding to $X_{2,3}$ can be obtained by solving the following characteristic equation:

$$\frac{dx}{\lambda}=\frac{dt}{t}=\frac{du}{-\alpha u-\frac{\alpha D}{k}}. \tag{5.30}$$

That yields

$$u=t^{-\alpha}\varphi(\xi)-\frac{D}{k},\quad\text{and}\quad \xi=t^\lambda e^{-x}. \tag{5.31}$$

Theorem 5.2 *Similarity transformation $u=t^{-\alpha}\varphi(\xi)-\frac{D}{k}$, where $\xi=t^\lambda e^{-x}$ converts the time fractional Biased random motion equation to a nonlinear fractional ordinary differential equation (NFODE) given by:*

$$\left(\mathcal{P}_{\frac{-1}{\lambda}}^{1-2\alpha,\alpha}\varphi\right)(\xi)-\frac{D}{k\Gamma(1-\alpha)}t^\alpha-kt^\lambda e^{-x}\varphi\varphi_\xi-kt^{2\lambda-\alpha}e^{-2x}\varphi\varphi_{\xi\xi}-kt^{2\lambda}e^{-2x}\varphi_\xi^2=0, \tag{5.32}$$

where $\mathcal{P}_\sigma^{\beta,\alpha}\varphi$ is the Erdélyi-Kober fractional operator defined as in (5.20).

Proof: As the previous case, using the Riemann-Liouville fractional derivative definition for similarity transformation $u=t^{-\alpha}\varphi(\xi)-\frac{D}{k}$, with $\xi=t^\lambda e^{-x}$,
we get,

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}\left(s^{-\alpha}\varphi\left(s^\lambda e^{-x}\right)-\frac{D}{k}\right)ds\right). \tag{5.33}$$

Let $\mu=\frac{t}{s}$, then $ds=-\frac{t}{\mu^2}d\mu$.
Hence,

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left(\frac{t^{n-2\alpha}}{\Gamma(n-\alpha)}\int_1^\infty(\mu-1)^{n-\alpha-1}\mu^{-(n+1-2\alpha)}\varphi\left(\xi\mu^{-\lambda}\right)d\mu\right)-\frac{D}{k\Gamma(1-\alpha)}t^{-\alpha}. \tag{5.34}$$

Based on the last equation with the definition of Erdélyi-Kober fractional integral operator, one can get:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left(t^{n-2\alpha}\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right) - \frac{D}{k\Gamma(1-\alpha)}t^{-\alpha}. \tag{5.35}$$

Additionally, we have:

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\left(t^{n-2\alpha}\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right)\\
&= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-2\alpha}\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right)\right]\\
&= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[(n-2\alpha)\,t^{n-2\alpha-1}\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi) + t^{n-2\alpha}\frac{\partial}{\partial t}\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right].
\end{aligned} \tag{5.36}$$

Applying the chain rule, we find:

$$t^{n-2\alpha}\frac{\partial}{\partial t}\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi) = \lambda t^{n-2\alpha-1}\xi\frac{\partial}{\partial \xi}\left(\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right), \tag{5.37}$$

and thus,

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\left(t^{n-2\alpha}\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right)\\
&= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-2\alpha-1}\left(n-2\alpha+\lambda\xi\frac{\partial}{\partial\xi}\right)\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right].
\end{aligned} \tag{5.38}$$

Repeating the same procedure for $(n-1)$ times, we obtain:

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\left(t^{n-2\alpha}\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right)\\
&= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-2\alpha-1}\left(n-2\alpha+\lambda\xi\frac{\partial}{\partial\xi}\right)\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right]\\
&\vdots\\
&= t^{-2\alpha}\prod_{j=0}^{n-1}\left(1-2\alpha+j+\lambda\xi\frac{d}{d\xi}\right)\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi).
\end{aligned} \tag{5.39}$$

According to the definition of Erdélyi-Kober fractional differential operator, we get:

$$\frac{\partial^n}{\partial t^n}\left(t^{n-2\alpha}\left(\mathcal{K}^{1-\alpha,n-\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi)\right) = t^{-2\alpha}\left(\mathcal{P}^{1-2\alpha,\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi). \tag{5.40}$$

By calculation of u_x and u_{xx} we find that the time fractional biased random motion equation reduces into a nonlinear fractional ordinary differential equation (NFODE) given by

$$\left(\mathcal{P}^{1-2\alpha,\alpha}_{\frac{-1}{\lambda}}\varphi\right)(\xi) - \frac{D}{k\Gamma(1-\alpha)}t^{\alpha} - kt^{\lambda}e^{-x}\varphi\varphi_\xi - kt^{2\lambda-\alpha}e^{-2x}\varphi\varphi_{\xi\xi} - kt^{2\lambda}e^{-2x}\varphi_\xi^2 = 0.$$

□

6. Conservation laws of the time fractional Biased Random motion equation

A conserved vector $C = (C^t, C^x)$ for the FBRM equation satisfies the following conservation equation:

$$D_t\left(C^t\right) + D_x\left(C^x\right)|_{(4.1)} = 0. \tag{6.1}$$

According to the conservation theorem [27], the formal Lagrangian for equation (4.1) is given by:

$$\mathcal{L} = v(t,x)\left(\frac{\partial^\alpha u}{\partial t^\alpha} - D\frac{\partial^2 u}{\partial x^2} - ku\frac{\partial^2 u}{\partial x^2} - k\left(\frac{\partial u}{\partial x}\right)^2\right), \tag{6.2}$$

in which v is a new dependent variable.
And its adjoint equation is defined by:

$$\frac{\delta \mathcal{L}}{\delta u} = 0, \tag{6.3}$$

where $\frac{\delta}{\delta u}$ is the Euler-Lagrange operator which is presented by the expression

$$\begin{aligned}\frac{\delta}{\delta u} =& \frac{\partial}{\partial u} + \left(D_t^\alpha\right)^* \frac{\partial}{\partial\left(D_t^\alpha u\right)} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} \\ &+ \sum_{k=3}^{\infty}(-1)^k D_{i_1}\dots D_{i_k}\frac{\partial}{\partial u_{t_1\dots i_k}},\end{aligned} \tag{6.4}$$

where

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots \tag{6.5}$$

and $(D_t^\alpha)^*$ is the adjoint operator of D_t^α, and D_x is the total derivative with respect to x.
In view of the Riemann-Liouville fractional differential operators, we have:

$$\left(D_t^\alpha\right)^* = (-1)^n \mathcal{P}_T^{n-\alpha}\left(D_t^n\right), \tag{6.6}$$

where

$$\mathcal{P}_T^{n-\alpha} f(t,x) = \frac{1}{\Gamma(n-\alpha)}\int_t^T \frac{f(\tau,x)}{(\tau-t)^{1+\alpha-\pi}}d\tau, \quad n = [\alpha]+1 \tag{6.7}$$

Combining the equation (6.2) and equation (6.3), we obtain:

$$\left(D_t^\alpha\right)^*(v) - Dv_{xx} - kuv_{xx} = 0. \tag{6.8}$$

Since Ibragimov's method for deriving the conservation laws of fractional partial differential equations closely mirrors that of standard partial differential equations, the adjoint equation follows a similar structure to the case of integer order. Therefore, we have:

$$X^{(\alpha)}(\mathcal{L}) + D_t(\tau)\mathcal{L} + D_x(\xi)\mathcal{L} = W_i\frac{\delta\mathcal{L}}{\delta u} + D_t\left(C^t\right) + D_x\left(C^x\right), \tag{6.9}$$

where W_i are the characteristic functions given by:

$$W_i = \eta_i - \tau_i u_t - \xi_i u_x. \tag{6.10}$$

For the Riemann-Liouville time fractional derivative, the components C^t and C^x of the conserved vectors C are given by [27]:

$$\begin{aligned}C^t =& \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}\left(W_i\right) D_t^k \frac{\partial\mathcal{L}}{\partial\left(D_t^\alpha u\right)} - (-1)^n\mathcal{J}\left(W_i, D_t^n\frac{\partial\mathcal{L}}{\partial\left(D_t^\alpha u\right)}\right), \\ C^x =& W_i\frac{\delta\mathcal{L}}{\delta u_x} + D_x\left(W_i\right)\frac{\delta\mathcal{L}}{\delta u_{xx}},\end{aligned} \tag{6.11}$$

where $\mathcal{J}$ is defined by:

$$\mathcal{J}(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(\tau,x)g(\theta,x)}{(\theta-\tau)^{1+\alpha-n}}\mathrm{d}\theta\mathrm{d}\tau. \tag{6.12}$$

Then,

$$C^t = D_t^{\alpha-1}(W_i)\frac{\partial \mathcal{L}}{\partial(D_t^\alpha V)} + \mathcal{J}\left(W_i, D_t\left(\frac{\partial \mathcal{L}}{\partial(D_t^\alpha V)}\right)\right), \tag{6.13}$$

and

$$C^x = W_i\left(\frac{\partial \mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right)\right) + D_x(W_i)\frac{\partial \mathcal{L}}{\partial u_{xx}}. \tag{6.14}$$

Based on the Lie point symmetry generators (4.5), we obtain the following conserved vectors for (4.1):
Case 1. For the generator $X_1 = x\frac{\partial}{\partial x} + \left(2u + \frac{2D}{k}\right)\frac{\partial}{\partial u}$.
We obtain the corresponding Lie characteristic function:

$$W_1 = \eta_1 - \tau_1 u_t - \xi_1 u_x = 2u + \frac{2D}{k} - xu_x. \tag{6.15}$$

Then, the conserved quantities are as follows:

$$C_1^t = vD_t^{\alpha-1}\left(2u + \frac{2D}{k} - xu_x\right) + \mathcal{J}\left(2u + \frac{2D}{k} - xu_x, v_t\right), \tag{6.16}$$

and

$$C_1^x = \left(2u + \frac{2D}{k} - xu_x\right)(-kvu_x + Dv_x + kuv_x) + (u_x - xu_{xx})(-Dv - kuv). \tag{6.17}$$

Case 2. For the generator $X_2 = \frac{\partial}{\partial x}$.
Here, we obtain the corresponding Lie characteristic function:

$$W_2 = -u_x. \tag{6.18}$$

Hence, the conserved vectors in this case are given by:

$$C_2^t = vD_t^{\alpha-1}(-u_x) + \mathcal{J}(-u_x, v_t), \tag{6.19}$$

and

$$C_2^x = kvu_x^2 - Du_xv_x - kuu_xv_x + Dvu_{xx} + kuvu_{xx}. \tag{6.20}$$

Case 3. For the generator $X_3 = t\frac{\partial}{\partial t} + \left(-\alpha u - \frac{\alpha D}{k}\right)\frac{\partial}{\partial u}$, we obtain the corresponding Lie characteristic function:

$$W_3 = -\alpha u - \frac{\alpha D}{k} - tu_t. \tag{6.21}$$

Consequently, the conserved vectors are obtained to be of the form:

$$C_3^t = vD_t^{\alpha-1}\left(-\alpha u - \frac{\alpha D}{k} - tu_t\right) + \mathcal{J}\left(-\alpha u - \frac{\alpha D}{k} - tu_t, v_t\right), \tag{6.22}$$

and

$$C_3^x = \left(-\alpha u - \frac{\alpha D}{k} - tu_t\right)(-kvu_x + Dv_x + kuv_x) + (-\alpha u_x - tu_{tx})(-Dv - kuv). \tag{6.23}$$

7. Conclusion

In this work, we have shown that the Lie symmetry analysis could be of great interest to reduce fractional partial differential equations to fractional ordinary differential equations. This work has investigated some properties of invariance to derive exact solutions and conserved quantities and it can be used to other linear and nonlinear fractional differential equations. In addition, this work will open up new possibilities for the future in studying and develop the Lie symmetry analysis for differential equations involving different approach of fractional derivatives.

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