Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.69209

On Cohomology of Lie Algebra Bundles of Finite Type

K. Kamalakshi, R. Rajendra and P. Siva Kota Reddy*

ABSTRACT: In this paper, radical bundle, nilradical bundle, annihilator and alternative bundle of a Lie algebra bundle of finite type are studied. Some results of vector bundles (more precisely, Lie algebra bundles) are generalised by weakening the condition of compactness on base space using finite type concept. Further, Whitehead's lemmas for semisimple Lie algebra bundles of finite type are proved.

Key Words: Bundles of finite type, Lie algebra bundle, ideal bundle, module bundle.

Contents

1 Introduction **1**

2 Preliminary results **2**

3 Cohomology of Lie algebra bundles of finite type **6**
3.1 Module bundle extensions 8
3.2 Lie algebra bundle extensions 8

1. Introduction

The concept of Lie algebra bundle is extensively explored in [3], [6], [2] and [4]. The relationship between semi-simple Lie algebra bundles and local triviality as well as the Whitehead lemma's on the cohomology of Lie algebra bundles have been studied in [7], [9] and [19]. However, the main conclusion depends on the fact that the base space is compact Hausdorff. We therefore examine the outcomes for finite type here by eliminating compactness condition on base space.

A Lie algebra bundle is a vector bundle $\xi = (\xi, p, X)$ together with a morphism $\theta : \xi \oplus \xi \to \xi$ which induces a Lie algebra structure on each fibre ξ_x. A locally trivial Lie algebra bundle is a vector bundle ξ in which each fibre is a Lie algebra and for each x in X there is an open set U in X containing x, a Lie algebra L and a homeomorphism $\phi : U \times L \to p^{-1}(U)$ such that for each x in U, $\phi_x : \{x\} \times L \to p^{-1}(x)$ is a Lie algebra isomorphism. A vector subbundle (η', q, X) of a ξ-module bundle (η, q, X) is said to be a submodule bundle of η, if the module structure $\rho : \xi \oplus \eta \to \eta$ maps $\xi \oplus \eta'$ into η'. A subalgebra bundle is an ideal bundle if each of its fibres is an ideal.

Following Seligman [21] and Lu Chaihui [18], R. Kumar created Lie algebra bundles for any field in terms of characteristic ideal bundles and discovered that the radical and nilradical bundles are characteristic ideal bundles in [15]. In [16], R. Kumar has developed the concept of module bundle and Lie algebra bundle extensions, proved the First and Second Whitehead Lemma's for semisimple Lie algebra bundles and also proved the complete reducibility of a module bundle and Levi decomposition for Lie algebra bundle (See also [17]).

The aim of this paper is to explore radical bundle, nilradical bundle, annihilator and alternative bundle of a Lie algebra bundle of finite type. Few results of Lie algebra bundles are generalised by weakening the condition of compactness on base space using finite type concept. Further, Whitehead's lemmas are are revisited for semisimple Lie algebra bundles of finite type. Throughout this paper, it is assumed that all underlying vector spaces are real and finite dimensional.

* Corresponding author

Submitted August 06, 2023. Published August 30, 2023
2010 *Mathematics Subject Classification:* 13D03, 17B20, 17B60, 17B62, 18Gxx, 55R25, 55R99.

2. Preliminary results

In this section, we give some preliminary definitions involving Lie algebra bundles and discuss some results. The definitions can be found in [6]-[17].

Definition 2.1 *Let $\xi = (\xi, p, X)$ and $\eta = (\eta, q, X)$ be two Lie algebra bundles over the same base space X. A Lie algebra bundle morphism $f : \xi \to \eta$ is a continuous map such that $p = qf$ and for each x in X, $f_x : \xi_x \to \eta_x$ is a Lie algebra homomorphism. We say that a morphism is an isomorphism if f is bijective and f^{-1} is a continuous map.*

Definition 2.2 *A section of a Lie algebra bundle (ξ, π, X) is a map $s : X \to \xi$ such that $p \circ s = id_X$. $\Gamma(\xi)$ denote the set of all sections of ξ.*

Definition 2.3 *A semi-simple Lie algebra bundle is a vector bundle ξ in which the morphism $\theta : \xi \oplus \xi \to \xi$ induces a semi-simple Lie algebra structure on each fibre ξ_x.*

Definition 2.4 *A vector bundle $\eta = (\eta, q, X)$ is said to be a ξ-module bundle if there exists a morphism $\rho : \xi \oplus \eta \to \eta$ inducing a ξ_x-module structure on each fibre η_x. By a representation of the Lie bundle (ξ, p, X) on a vector bundle (η, q, X) we mean a Lie bundle morphism $\rho : \xi \to End\eta$, where $End\eta$ is the Lie bundle whose fibres are the endomorphisms of the vector space $\eta_x, x \in X$.*

Definition 2.5 *Let η and η' be two ξ-module bundles. By a ξ-module bundle morphism $f : \eta \to \eta'$ we mean a vector bundle morphism in which for each $x \in X, f_x : \eta_x \to \eta'_x$ is a ξ_x module homomorphism.*

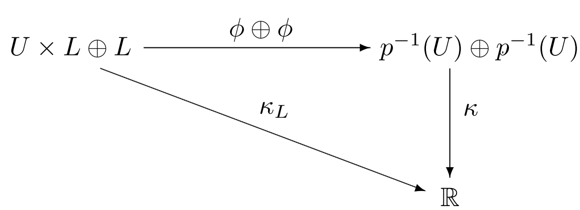
Definition 2.6 *A ξ-module bundle (η, q, X) is said to be simple if it has no proper non-zero submodule bundles.*

Definition 2.7 *A Lie algebra bundle ξ over an arbitrary space X is of finite type if there is a finite partition S of unity on X (that is, a finite set S of non-negative continuous functions on X whose sum is 1) such that the restriction of the bundle to the set $\{x \in X \mid f(x) \neq 0\}$ is a trivial Lie algebra bundle for each f in S.*

Remark 2.1 *Let ξ be a Lie algebra bundle of finite type over X. Then there exists $\{f_{\alpha_i}\}_{i=1}^{n}$ such that $\xi|_{U_{\alpha_i}}$ is a trivial bundle, where $U_{\alpha_i} = \{x \in X \mid f_{\alpha_i}(x) \neq 0\}$. We observe that $\{U_{\alpha_i}\}$ covers X.*

Theorem 2.1 *Every Lie bundle of finite type admits a non-degenerate killing metric.*

Proof: Let $\xi = (\xi, p, X)$ be a Lie bundle of finite type. Define $\kappa : \xi \oplus \xi \to \mathbb{R}$ such that for all $x \in X$, $\kappa|_{\xi_x \times \xi_x}$ is the canonical killing form of ξ_x. That is, $\kappa(u, v) = Tr(adu \circ adv)$. Now, since ξ is a Lie bundle of finite type, for all $x \in X$, there exists neighbourhood U of x and a Lie bundle isomorphism, $\phi : U \times L \to p^{-1}(U)$. Also the following diagram is commutative:



That is $\kappa(\phi \oplus \phi) = \kappa_L$. Since $\kappa_L(l_1, l_2) = \kappa(T(l_1), T(l_2))$, if $T : L \to p^{-1}(x)$ is any isomorphism, then κ is continuous being a bilinear form and $\phi \oplus \phi$ is a Lie bundle isomorphism. □

Lemma 2.1 *If ξ is any Lie algebra bundle of finite type over an arbitrary space X, then there exists a Hermitian metric on ξ.*

Proof: Since ξ is a Lie algebra bundle of finite type, there exists a finite partition of unity $\{f_{\alpha_i}\}_{i=1}^n$ such that $\xi|_{U_{\alpha_i}}$ is a trivial bundle, where $U_{\alpha_i} = \{x \in X \mid f_{\alpha_i}(x) \neq 0\}$. A metric on a vector space V defines a metric on the product bundle $X \times V$. Hence metrics exist on trivial bundles. Let h_{α_i} be a metric on $\xi|_{U_{\alpha_i}}$. Define

$$k_{\alpha_i}(x) = \begin{cases} f_{\alpha_i}(x)h_{\alpha_i}(x), & \text{for } x \in U_{\alpha_i}; \\ 0, & \text{otherwise.} \end{cases}$$

Then k_{α_i} is a section of $Herm(\xi)$ and is positive semi-definite. But for any $x \in X$ there exists α_i such that $f_{\alpha_i}(x) > 0$ (since $\sum f_{\alpha_i} = 1$) and so $x \in U_{\alpha_i}$. Hence for this α_i, $k_{\alpha_i}(x)$ is positive definite. Hence $\sum_{\alpha_i=1}^n k_{\alpha_i}(x)$ is positive definite for all $x \in X$. Therefore, $k = \sum k_{\alpha_i}$ is a Hermitian metric on ξ. □

Corollary 2.1 *Suppose that*

$$0 \longrightarrow \xi' \xrightarrow{\phi'} \xi \xrightarrow{\phi''} \xi'' \longrightarrow 0$$

is an exact sequence of Lie algebra bundles of finite type over X. Then there exists an isomorphism $\xi \cong \xi' \oplus \xi''$.

Proof: By Lemma 2.1, ξ can be given a metric and $\xi \cong \xi' \oplus (\xi')^\perp$. But $(\xi')^\perp \cong \xi''$. □

Lemma 2.2 *If ξ is any Lie algebra bundle of finite type, there exists an epimorphism $\phi : X \times C^m \to \xi$ for some integer m.*

Corollary 2.2 *If ξ is any Lie algebra bundle of finite type, there exists a bundle ζ of finite type such that $\xi \oplus \zeta$ is trivial.*

Proof: By Lemma 2.1, we have a bundle ζ such that $\xi \oplus \zeta$ is trivial. It remains to prove that ζ is of finite type. Let $\xi = (\xi, \pi_1, X)$, $\zeta = (\zeta, \pi_2, X)$ and $\xi \oplus \zeta \cong X \times L$. Since ξ is of finite type there exists an finite partition of unity $\{f_{\alpha_i}\}_{i=1}^n$ such that $\xi|_{U_{\alpha_i}}$ is a trivial bundle, where $U_{\alpha_i} = \{x \in X | f_{\alpha_i}(x) \neq 0\}$. That is, $\pi_1^{-1}(U_{\alpha_i}) \cong U_{\alpha_i} \times L$. Since $\xi \oplus \zeta \cong X \times L$, we have $U_{\alpha_i} \times L \cong \{(x, s) \mid \pi_1(x) = \pi_2(s), x \in \xi\}$ and $s \in \zeta\}|_{U_{\alpha_i}}$. But $U_{\alpha_i} \times L \cong \{x \in \xi \mid \pi_1(x) \in U_{\alpha_i}\}$. Thus $\{x \in \xi \mid \pi_1(x) \in U_{\alpha_i}\} \cong \{(x, s) \mid \pi_1(x) = \pi_2(s)\}$, where $s \in \zeta$. So, we have $U_{\alpha_i} \times L \cong \pi_2^{-1}(U_{\alpha_i})$. □

Definition 2.8 *Let ξ be a Lie algebra bundle over X. A morphism $D : \xi \to \xi$ is a derivation of ξ, if D_x is linear on ξ_x such that $D_x[u, v] = [D_x u, v] + [u, D_x v]$ for all $u, v \in \xi_x$. We denote all derivations of ξ by $D(\xi)$.*

Definition 2.9 *An ideal bundle μ of a Lie algebra bundle ξ is called a characteristic ideal bundle if $D(\mu) \subseteq \mu$, for all $D \in D(\xi)$.*

Definition 2.10 *We define the derived series of a Lie algebra bundle ξ by $\xi^0 = \xi$, $\xi^{(1)} = [\xi, \xi]$, $\xi^{(2)} = [\xi^{(1)}, \xi^{(1)}]$, ..., $\xi^{(k)} = [\xi^{(k-1)}, \xi^{(k-1)}]$. We say that ξ is solvable if $\xi^{(n)} = 0$ for some positive integer n.*

Definition 2.11 *The central series of a Lie algebra bundle ξ is defined by $\xi^0 = \xi$, $\xi^1 = [\xi, \xi]$, $\xi^2 = [\xi, \xi^1]$, ..., $\xi^k = [\xi, \xi^{k-1}]$. We say that ξ is nilpotent if $\xi^n = 0$ for some positive integer n.*

Definition 2.12 *Let ξ be a locally trivial Lie algebra bundle and $\Phi : U \times L \to \bigcup_{x \in U} \xi_x$ be a locally triviality of ξ, where L is a Lie algebra. Let R be the the radical of L and $\mathcal{R}_x$ be the radical of ξ_x. Then $\Phi|_{U \times R} : U \times R \to \bigcup_{x \in U} \mathcal{R}_x$ is an isomorphism. The radical bundle $\mathcal{R}$ of ξ is the bundle $\mathcal{R} = \bigcup_{x \in U} \mathcal{R}_x$.*

Definition 2.13 *Let ξ be a locally trivial Lie algebra bundle and $\Phi : U \times L \to \bigcup_{x \in U} \xi_x$ be a locally triviality of ξ where L is a Lie algebra. Let N be the the nilradical of L and η_x be the nilradical of ξ_x. Then $\Phi|_{U \times N} : U \times N \to \bigcup_{x \in U} \eta_x$ is an isomorphism. The nilradical bundle η of ξ is the bundle $\eta = \bigcup_{x \in U} \eta_x$,*
.

Proposition 2.1 *The radical bundle and nilradical bundle of a Lie algebra bundle of finite type is also of finite type.*

Proof: Let ξ be a Lie algebra bundle of finite type. Let $\{f_1, f_2, \ldots, f_n\}$ be a finite partition of unity on the base space X and $U_i = \{x \in X \mid f_i(x) \neq 0\}$. Then for all i, $\xi|_{U_i}$ is trivial. Let $\phi_i : U_i \times L_i \to \bigcup_{x \in U_i} \xi_x$ be the isomorphism. Then $\psi_i : U_i \times R(L_i) \to \bigcup_{x \in U_i} \mathcal{R}_x$ is an isomorphism ($\mathcal{R}_x = R(\xi_x)$). Hence the radical bundle is of finite type. Similarly, we can prove that nilradical bundle is also of finite type. □

Definition 2.14 *Given a vector bundle* ξ, *we define* $Alt^1(\xi) = \bigcup_{x \in X} Alt^1(\xi_x)$, *where* $Alt^1(\xi_x)$ *is the vector space of all alternating bilinear maps of* ξ_x *onto itself.*

Remark 2.2 *If* ξ *is a vector bundle of finite type, then* $Alt^1(\xi)$ *is also of finite type. For, let* $\{f_1, f_2, \ldots, f_n\}$ *be a finite partition of unity,* $U_i = \{x \in X \mid f_i(x) \neq 0\}$ *and* $U_i \times V \cong \bigcup_{x \in U_i} \xi_x$. *Then* $U_i \times Alt^1(V) \cong \bigcup_{x \in U_i} Alt^1(\xi_x)$.

Proposition 2.2 *If* ξ *is a Lie algebra bundle of finite type, then* $D(\xi)$ *is a Lie algebra bundle of finite type.*

Proof: Let $\{f_1, f_2, \ldots, f_n\}$ be finite partition of unity, $U_i = \{x \in X \mid f_i(x) \neq 0\}$ and $\xi|_{U_i}$ is trivial for all i. The topology on $D(\xi)$ is defined by taking open sets $V \subseteq D(\xi)$ such that $V \cap (D(\xi|_U))$ is open in $D(\xi|_U)$ for all open sets U in X for which $\xi|_U$ is a trivial Lie algebra bundle. Let $\phi : U_i \times L_i \to \bigcup_{x \in U_i} \xi_x$ be the isomorphism. Define a morphism

$$U \times D(L_i) \to \textstyle\bigcup_{x \in U_i} D(\xi_x) \quad \text{by} \quad D \to \phi D \phi^{-1}.$$

Then $\phi_x D \phi_x^{-1}$ is a derivation of ξ_x. Thus $\phi_x D \phi_x^{-1}$ is a derivation of ξ_x when D is a derivation of L_i. Hence $D(\xi)$ is a Lie algebra bundle of finite type. □

Theorem 2.2 *Let* ξ *be a Lie algebra bundle of finite type. Then there exists a Lie algebra bundle* ξ' *of finite type containing* ξ *as a Lie algebra bundle such that every derivation of* ξ *is the restriction of inner derivation of* ξ'.

Proof: Let D be any derivation of ξ. Consider $\xi^* = \bigcup_{x \in X} \xi_x^*$, where ξ_x^* is a Lie algebra bundle obtained by embedding Lie algebra ξ_x. Let $\{f_1, f_2, \ldots, f_n\}$ be a finite partition of unity, $U_i = \{x \in X \mid f_i(x) \neq 0\}$ and $U_i \times L_i \cong \xi|_{U_i}$ for all i. Let $\phi : U_i \times L_i \to \bigcup_{x \in U_i} \xi_x$ be the isomorphism. Define

$$\phi^* : U_i \times L_i^* \to \bigcup_{x \in U_i} \xi_x^* \quad \text{by} \quad \phi^*(a, (u, \alpha)) = \phi((a, u), \alpha),$$

where α is in the underlying field of L_i. Define topology on $\bigcup_{x \in X} \xi_x^*$ such that ϕ^* is a homeomorphism. Hence ξ^* is a Lie algebra bundle.

Consider $\xi \oplus \xi^*$ as the vector bundle underlying ξ' and define multiplication in ξ'_x by the formula $[(u, a), (v, b)] = ([u, v] + aD(v) - bD(u), 0)$. Then ξ' is a Lie algebra bundle. For, ξ being Lie algebra bundle we have $\phi : U_i \times L_i \to \bigcup_{x \in U_i} \xi_x$. Then we get $\phi^* : U_i \times L_i^* \to \bigcup_{x \in U_i} \xi_x^*$. Define

$$\psi : U_i \times (L_i \oplus L_i^*) \to \bigcup_{x \in U_i} (\xi_x \oplus \xi_x^*)$$

by

$$\psi(x, u, u^*) = \phi(x, u) \oplus \phi(x, u^*).$$

Clearly, ψ is a homeomorphism. We observe that ξ' is a Lie algebra bundle of finite type. We identify the subalgebra bundle $(\xi, 0)$ with ξ. Let $w = (0, 1)$. Consider the inner derivation, $ad_w : \xi' \to \xi'$ by $ad_w(s) = [w, s]$ for $s \in \xi'_x$. When $u \in \xi$, we get

$$ad_w(u, 0) = [w, (u, 0)] = [(0, 1), (u, 0)] = ([0, u] + 1D(u) - 0D(0), 0) = (D(u), 0).$$

Therefore, the restriction of the inner derivation ad_w to ξ is equal to the original derivation. □

Lemma 2.3 *[12, Lemma 4.1] Let $(X \times V, p, X)$ be a trivial vector bundle and $(X \times L, p_L, X)$ be a trivial completely semisimple Lie algebra bundle. If $\phi : X \times V \to X \times L$ is a vector bundle monomorphism such that for each $x \in X$, $\phi_x(V)$ is a characteristic ideal in L, then there exists a finite open partition $\bigcup X_i = X$ such that $\phi_x(V) = \phi_y(V)$, for all x, y in X_i.*

Lemma 2.4 *Let ξ be a completely semisimple Lie algebra bundle of finite type and η be a vector subbundle of ξ such that $D_x(\eta_x) \subseteq \eta_x$, for all $D \in D(\xi)$. Then η is also a characteristic ideal bundle of finite type.*

Proof: Since $D_x(\eta_x) \subseteq \eta_x$ for all $D \in D(\xi)$, η_x is an ideal. Let $\{f_1, f_2, \ldots, f_n\}$ be finite partition of unity and $\xi|_{U_i}$ is trivial for all i, where $U_i = \{x \in X \mid f_i(x) \neq 0\}$. Let $\phi : p^{-1}(U_i) \to U_i \times L$ be the isomorphism. We shall prove that $\phi|_\eta : p_\eta^{-1}(U_i) \to U_i \times L'$ is an isomorphism for a Lie algebra L'. Since η is a vector subbundle of ξ, we can choose suitable neighborhood U_i of x such that

$$U_i \times V \xrightarrow{\psi} p_\eta^{-1}(U_i) \xrightarrow{i} p^{-1}(U_i) \xrightarrow{\phi} U_i \times L,$$

where ψ is a vector bundle isomorphism, i is the inclusion map. Then $\phi \circ i \circ \psi : U_i \times V \to U_i \times L$ is continuous. For each $x \in U_i$, $(\phi \circ i \circ \psi)_x(V) = (\phi \circ i)_x(\psi_x(V)) = (\phi \circ i)_x(p_\eta^{-1}(x)) = \psi_x(p_\eta^{-1}(x))$ is a characteristic ideal in L, as $p_\eta^{-1}(x) = \eta_x$ is a characteristic ideal in ξ and ϕ_x takes characteristic ideals to characteristic ideals. Then by Lemma 2.3, there exists an open partition $\bigcup U_{ij} = U_i$ such that for all $x \in U_{ij}$, $(\phi \circ i \circ \psi)_x(V)$ depends only on j and not on x. Hence $(\phi \circ i)_x(p_\eta^{-1}(x)) = (\phi \circ i)_x(p_\eta^{-1}(y))$ for all $x, y \in U_{ij}$. Therefore we can write $(\phi \circ i)_x(p_\eta^{-1}(x)) = L'$, for all $x \in U_{ij}$. Then the morphism $\phi \circ i = \phi|_\eta : p_\eta^{-1}(U_i) \to U_i \times L'$ is a Lie algebra bundle isomorphism. □

Definition 2.15 *Let η be a characteristic ideal bundle of a Lie algebra bundle ξ. We define the annhilator of η in ξ by*

$$Ann(\eta) = \bigcup_{x \in X} Ann(\eta_x),$$

where $Ann(\eta_x) = \{y \in \xi_x \mid [y, \eta_x]_x = 0\}$.

Proposition 2.3 *If η is a characteristic ideal bundle of ξ and ξ is a completely semisimple Lie algebra bundle of finite type, then $Ann(\eta)$ is also a characteristic ideal bundle of finite type.*

Proof: We have $Ann(\eta)$ is a characteristic ideal bundle of ξ [12, Lemma 4.4]. It is clear that $Ann(\eta_x)$ is a characteristic ideal of ξ_x. Let $\{f_1, f_2, \ldots, f_n\}$ be a finite partition of unity, $U_i = \{x \in X \mid f_i(x) \neq 0\}$ and for all i, $\phi : U_i \times L \to \bigcup_{x \in U_i} \xi_x$ be the isomorphism. Then by the proof of Lemma 2.4, $\phi|_{U_i \times L'} : U_i \times L' \to \bigcup_{x \in U_i} \eta_x$ is an isomorphism and $\phi_x : L' \to \eta_x$ is a Lie algebra isomorphism. Then

$$\Phi : U_i \times Ann(L') \to \bigcup_{x \in U_i} Ann(\eta_x)$$

is also an isomorphism, as ϕ_x takes annihilators to annihilators. □

3. Cohomology of Lie algebra bundles of finite type

We begin with the definition of cohomology of Lie algebra bundles (See [9], [16] and [17]).

Definition 3.1 *Let ξ be a Lie bundle and η a ξ-module bundle with representation $\rho : \xi \to End\eta$. Let us identify $M^0(\xi, \eta)$ with $\Gamma(\eta)$. Given $A \in M^0(\xi, \eta)$, consider the function $f : \xi \to \eta$ given by $f(x) = \rho(x)(A(s))$ if $x \in \xi_s$. The function f is a morphism being the combination of the following maps:*

$$\xi \xrightarrow{(Id,p)} \xi \times X \xrightarrow{(Id,A)} \xi \oplus \eta \xrightarrow{\rho} \eta$$

$$x \to (x, p(x) = s) \to (x, A(s)) \to \rho(x)(A(s)).$$

Hence $f \in M^1(\xi, \eta)$ and we define a linear map $\delta^0 : M^0(\xi, \eta) \to M^1(\xi, \eta)$ by $\delta^0(A) = f$. Consider $f \in M^n(\xi, \eta)$ and let $g : \xi^{n+1} \to \eta$ be defined as

$$g(x_0, x_1, \ldots, x_n) = \sum_i (-1)^i \rho(x_i) f(x_0, \ldots, \hat{x_i}, \ldots, x_n) + \sum_{i<j} (-1)^{i+j} f(\theta(x_i, x_j), x_0, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_n),$$

where $\hat{x_i}$ means that the i^{th} term is omitted. Then $g \in M^{n+1}(\xi, \eta)$ and we define $\delta^n : M^n(\xi, \eta) \to M^{n+1}(\xi, \eta)$ by $\delta^n(f)g$. Then $\{M^n(\xi, \eta), \delta^n\}$ forms a cochain complex and we define n^{th} cohomology group of ξ with coefficients in η by $H^n(\xi, \eta) = ker\delta^n / Im\delta^{n-1}$. We call δ^n, the differentials.

Proposition 3.1 *Let ξ be a Lie algebra bundle of finite type over any arbitrary space X. Let*

$$0 \to \eta_1 \xrightarrow{\alpha} \eta_2 \xrightarrow{\beta} \eta_3 \to 0$$

be a short exact sequence of ξ-module bundles of finite type over X. Then there exists a connecting map, $\omega_n : H^n(\xi, \eta_3) \to H^{n+1}(\xi, \eta_1)$ such that

$$0 \to H^0(\xi, \eta_1) \xrightarrow{\alpha_0} H^0(\xi, \eta_2) \xrightarrow{\beta_0} H^0(\xi, \eta_3) \xrightarrow{\omega_0} H^1(\xi, \eta_1) \to \cdots \to H^n(\xi, \eta_1) \xrightarrow{\alpha_n} H^n(\xi, \eta_2) \xrightarrow{\beta_n} H^n(\xi, \eta_3) \xrightarrow{\omega_n} H^{n+1}(\xi, \eta_1) \to \cdots$$

is exact, where α_n and β_n are given by

$$\alpha_n(f + Im\delta_1^{n-1}) = \alpha f + Im\delta_2^{n-1} \text{ and } \beta_n(f + Im\delta_2^{n-1}) = \beta f + Im\delta_3^{n-1}.$$

Proof: The sequence $0 \to \eta_1 \xrightarrow{\alpha} \eta_2 \xrightarrow{\beta} \eta_3 \to 0$ being an exact sequence ξ-module bundles, is an exact sequence vector bundles of finite type. Hence there exists an isomophism $\eta_2 = \eta_1 + \eta_3$, by Corollary 2.1. This gives a splitting vector bundle morphism, $\gamma : \eta_3 \to \eta_2$. Now we can define

$$\omega_n : H^n(\xi, \eta_3) \to H^{n+1}(\xi, \eta_1)$$

by

$$\omega_n(f + Im\delta_3^{n-1}) = \alpha^{-1}\delta_2^n(\gamma f) + Im\delta_1^n.$$

Then ω_n is well-defined and the sequence,

$$0 \to H^0(\xi, \eta_1) \xrightarrow{\alpha_0} H^0(\xi, \eta_2) \xrightarrow{\beta_0} H^0(\xi, \eta_3) \xrightarrow{\omega_0} H^1(\xi, \eta_1) \to \cdots$$

is exact. □

Remark 3.1 *Let* $Hom(\xi^p, \eta) = \bigcup_{x \in X} Hom(\xi_x^p, \eta_x)$ *be the vector bundle, where* $Hom(\xi_x^p, \eta_x)$ *is the vector space of all* p *skew-symmetric maps from* ξ_x^p *to* η_x*. We can define the vector bundle morphisms*

$$\Delta^n : Hom(\xi^n, \eta) \to Hom(\xi^{n+1}, \eta)$$

by

$$\Delta^n(f)(x_0, \dots, x_n) = \sum_i (-1)^i \rho_x(x_i) f(x_0, \dots, \hat{x_i}, \dots x_n) + \sum_{i<j} (-1)^{i+j} f(\theta(x_i, x_j), x_0 \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_n),$$

if $f \in Hom(\xi_x^n, \eta_x)$*. These morphisms* Δ^n *induce the mappings*

$$\delta^n : \Gamma(Hom(\xi^n, \eta)) \to \Gamma(Hom(\xi^{n+1}, \eta)).$$

Thus the cohomology defined in definition is the space of sections of the fibre bundle whose fibres are the cohomology of the fibres. That is, $H^n(\xi, \eta) = \Gamma(\bigcup_{x \in X} H^n(\xi_x, \eta_x))$*.*

In view of the above remark, we have have the following results:

Proposition 3.2 *The cohomology groups* $H^n(\xi, \eta), n \geq 1$*, vanish, when* ξ *is semisimple Lie algebra bundle of finite type and* η *is a* ξ*-module bundle of finite type over an arbitrary space* X*.*

Proposition 3.3 *Let* η *be a vector bundle of finite type and* η' *be a subbundle of* η*. Then the quotient bundle* $\eta/\eta' = \bigcup_{x \in X} \eta_x/\eta'_x$ *is also a vector bundle of finite type.*

Proof: Let $\{f_1, f_2, \dots, f_n\}$ be finite partition of unity, $U_i = \{x \in X \mid f_i(x) \neq 0\}$ and for all i, $U_i \times L \cong \eta|_{U_i}$. We observe that η' is also finite type and $U_i \times L' \cong \eta'|_{U_i}$ where L' is a subspace of L. Let $\phi : U_i \times L \to \eta|_{U_i}$ and $\phi' = \phi|_{L'} : U_i \times L' \to \eta'|_{U_i}$ be the isomorphisms. Now for any $s \in U_i$, define

$$\psi_s : s \times L/L' \to \eta_s/\eta'_s$$

by

$$\psi_s(s, l + L') = \phi(s, l) + \phi'(s, L') = \phi(s, l) + \eta'_s, \text{ for all } l \in L.$$

Then ψ_s is an isomorphism and $\psi : U_i \times L/L' \to \bigcup_{s \in U_i} \eta_s/\eta'_s$ is also an isomorphism. Hence the result follows. □

Remark 3.2 *Let* ξ *be a Lie algebra bundle and* η_1, η_2 *be* ξ*-module bundles with representation* ρ_1, ρ_2*, respectively. Consider* $Hom(\eta_1, \eta_2)$ *which can be made into* ξ*-module by defining* $\rho^* : \xi \oplus Hom(\eta_1, \eta_2) \to Hom(\eta_1, \eta_2)$ *by* $\rho^*(x, f) = g$ *for* $x \in \xi_s, f \in Hom((\eta_1)_s, (\eta_2)_s)$*, where* $g \in Hom((\eta_1)_s, (\eta_2)_s)$ *is given by* $g(a) = \rho_2(x)(f(a)) - f(\rho_1(x)(a))$*.*

Theorem 3.1 *Let* ξ *be a semisimple Lie bundle of finite type. Then every module bundle* η *of finite type over* ξ *is completely reducible.*

Proof: Let η' be a submodule of η. Then $0 \to \eta' \xrightarrow{i} \eta \xrightarrow{\pi} \eta/\eta' \to 0$ is an exact sequence of ξ-module bundle, where i is the inclusion map and π is the canonical projection. Then by Remark 3.2,

$$0 \to Hom(\eta/\eta', \eta') \xrightarrow{\pi_*} Hom(\eta, \eta') \xrightarrow{i_*} Hom(\eta', \eta') \to 0$$

is an exact sequence of ξ-module bundles over X. Then by using [9, Proposition 3.2 and Theorem 4.3], we have that η is completely reducible. □

Theorem 3.2 (The First Whitehead Lemma) *If* ξ *is a semisimple Lie algebra bundle of finite type, then for any* ξ*-module bundle of finite type* η*,* $H^1(\xi, \eta) = 0$*.*

Proof: Since ξ is of finite type, X has a partition of unity, using which we can get the result by following the techniques of [16,17]. □

3.1. Module bundle extensions

We recall the following definitions from [16,17].

Definition 3.2 *Let η and η' be ξ-module bundles. If $\eta \subset E$ and the sequence $0 \to \eta \xrightarrow{i} E \xrightarrow{\pi} \eta' \to 0$ of ξ-module bundles is exact, then we say that the pair (E, π) is an extension of η by η'.*

Definition 3.3 *Two extensions (E_1, π_1) and (E_2, π_2) of η by η' are called equivalent if there is an isomorphism σ of E_1 onto E_2 such that $\pi_2 \sigma = \pi_1$.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \eta & \xrightarrow{i} & E_1 & \xrightarrow{\pi_1} & \eta' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & \eta & \xrightarrow{i} & E_2 & \xrightarrow{\pi_2} & \eta' & \longrightarrow & 0 \end{array}$$

We denote the class of all extensions equivalent to (E, π) by $\{(E, \pi)\}$. Let $M = \bigcup_{x \in X} M_x$, where M_x is the vector space of all linear maps of η'_x into η_x. Then M is a vector bundle. Now we make M into ξ-module bundle by defining $(u.f)q = u.f(q) - f(u.q)$ for all $u \in \xi_x, f \in \eta_x$ and $q \in \eta'_x$.

Lemma 3.1 *Let ξ be a lie algebra bundle of finite type and η be ξ-module bundle of finite type over an arbitrary space X. An extension of η by η' determines a unique element of $H^1(\xi, M)$.*

Proof: Consider an extension $(E, \pi) : 0 \to \eta \xrightarrow{i} E \xrightarrow{\pi} \eta' \to 0$. Then by Corollary 2.1, there is a vector bundle splitting morphism $\gamma : \eta' \to E$ such that $\gamma \circ \pi = Id_E$. Define $f : \xi \to M$ by $f(u)(q) = u.\gamma(q) - \gamma(u.q)$ for all $u \in \xi_x$ and $q \in \eta'_x$. Then $f \in Z^1(\xi, M)$. Let γ_1, γ_2 be two maps such as above with corresponding f_1, f_2 in $Z^1(\xi, M)$. Define $S : X \to M$ by $S(x)q = \gamma_{1x}(q) - \gamma_{2x}(q)$. Then $S \in \Gamma(M)$ and $\partial S = f_1 - f_2$. Thus $f_1 - f_2 \in B^1(\xi, M)$. □

The following results are straightforward (analogous to the results in [16,17]) and are stated without proof:

Lemma 3.2 *Let ξ be a Lie algebra bundle of finite type. Every element of $Z^1(\xi, M)$ determines an extension of η by η'.*

Proposition 3.4 *Let ξ be a Lie algebra bundle of finite type. Then there is a bijection between the first cohomology group $H^1(\xi, M)$ and $Ext(\eta, \eta')$, the collection of all equivalence classes of extensions of the ξ- module bundle of finite type η by η'.*

Theorem 3.3 *The following properties of a Lie algebra bundle of finite type are equivalent:*

1. *ξ is semisimple,*
2. *$H^1(\xi, \eta) = 0$ for any ξ- module bundle η of finite type,*
3. *any representation ξ is completely reducible.*

3.2. Lie algebra bundle extensions

In [16,17], the abelian extension of a Lie algebra bundle is defined, which motivates the following definitions:

Definition 3.4 *Let ξ and ζ be Lie algebra bundles of finite type and η a ξ-module bundle of finite type with representation $\rho : \xi \to End\, \eta$. Let $0 \to \eta \xrightarrow{\alpha} \zeta \xrightarrow{\beta} \xi \to 0$ be an exact sequence of Lie algebra bundles, where η is considered as an abelian Lie algebra bundle. Since ξ and ζ are Lie algebra bundles of finite type, there exists vector bundle morphism $\gamma : \xi \to \zeta$. Now for any $u \in \xi_x$, $a \in \eta_x$, the element $[\gamma(u), \alpha(a)]_\zeta \in ker\beta = Im\alpha$, we can define $\hat{\rho} : \xi \oplus \eta \to \eta$ by $\hat{\rho}(u, a) = \alpha^{-1}([\gamma(u), \alpha(a)]_\zeta)$. The defined*

$\hat{\rho}$ is independent of splitting morphism γ and defines a ξ-module bundle structure on η. If the induced module structure $\hat{\rho}$ coincides with the original module structure ρ, the extension

$$(\zeta, \beta) : 0 \to \eta \xrightarrow{\alpha} \zeta \xrightarrow{\beta} \xi \to 0$$

is called an extension of the Lie algebra bundle ξ by the ζ-module bundle η.

Definition 3.5 *Two extensions (ζ_1, β_1) and (ζ_2, β_2) are said to be equivalent, if there is an isomorphism σ of ζ_1 onto ζ_2 such that $\beta_2\sigma = \beta_1$.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \eta & \xrightarrow{i} & \zeta_1 & \xrightarrow{\beta_1} & \xi & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & \eta & \xrightarrow{i} & \zeta_2 & \xrightarrow{\beta_2} & \xi & \longrightarrow & 0 \end{array}$$

We denote the classes of all extensions equivalent to (ζ, β) by $\{(\zeta, \beta)\}$.

Proposition 3.5 *Let ξ be a Lie algebra bundle of finite type over an arbitrary space X. Then there is a bijection between the second cohomology group $H^2(\xi, \eta)$ of ξ with coefficients in ξ-module bundle of finite type η and $Ext(\xi, \eta)$, the collections of all equivalence classes of all extensions of η by ξ.*

Theorem 3.4 (Second Whitehead Lemma) *If ξ is a semisimple Lie algebra bundle of finite type, then $H^2(\xi, \eta) = 0$, for any ξ-module bundle η of finite type.*

Proof: Let $0 \to \eta \xrightarrow{\alpha} \zeta \xrightarrow{\beta} \xi \to 0$ be any Lie algebra bundle extension of ξ by η. Since ξ is a Lie algebra bundle of finite type, there exists a vector bundle splitting morphism $\gamma : \xi \to \zeta$ such that $\beta\gamma = Id_\xi$ (by Corollary 2.1). Then γ induces a ξ-module structure on ζ as follows: Consider $\bar{\rho} : \xi \oplus \zeta \to \zeta$ by $\bar{\rho}(u, a) = [\gamma(u), a]_\zeta$, for all $u \in \xi_x, a \in \zeta_x$. Now

$$\beta(\gamma[u, v] - [\gamma(u), \gamma(v)]_\zeta) = [u, v] - [\beta\gamma(u), \beta\gamma(v)] = 0.$$

We note that $\gamma[u, v] = [\gamma(u), \gamma(v)]_\zeta + \kappa(u, v)$ for all $u, v \in \xi_x$ and $\kappa : \xi \oplus \xi \to \eta$ is a 2-cochain. Thus $\gamma[u, v] - [\gamma(u), \gamma(v)]_\zeta \in ker\beta = Im\alpha$. But $Im\alpha \cong \eta$. Hence $\bar{\rho}$ gives a ξ-module structure on ζ. Since η is ξ-submodule of ζ and ξ is semisimple, by Theorem 3.3 there exists η' such that $\zeta = \eta \oplus \eta'$ and ξ is isomorphic to η' follows from the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \eta & \xrightarrow{\alpha} & \eta \oplus \eta' & \xrightarrow{\beta} & \xi & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & \eta & \xrightarrow{i} & \eta \oplus \xi & \xrightarrow{\pi} & \xi & \longrightarrow & 0 \end{array}$$

Thus given extension is equivalent to the null extension. Hence the result. □

Acknowledgments

We thank Dr. Ranjitha Kumar, VIT Bhopal University for his timely help and suggestions.

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K. Kamalakshi,
Department of Mathematics
Field Marshal K.M. Cariappa College (A Constituent College of Mangalore University)
Madikeri - 571201, India.
E-mail address: kamalakshiaithal@yahoo.com

and

R. Rajendra,
Department of Mathematics
Field Marshal K.M. Cariappa College (A Constituent College of Mangalore University)
Madikeri - 571201, India.
E-mail address: rrajendrar@gmail.com

and

P. Siva Kota Reddy,
Department of Mathematics
JSS Science and Technology University
Mysuru - 570 006, India
E-mail address: pskreddy@jssstuniv.in