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Study on conformable fractional neutral evolution equations with nonlocal-delay conditions

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ABSTRACT: In this paper, we study a class of conformable fractional neutral evolution equations with nonlocal-delay conditions. The existence and uniqueness for mild solutions are proved by fixed point theorems of Krasnoselskii and Banach contraction mapping, with some conditions. In the end, we give an example of applications.

Key Words: Fractional derivative, neutral evolution equations, nonlocal Cauchy problem.

Contents

1. Introduction

Many authors try to replace the classical derivative by a fractional derivative in differntial equations [8,9,12], because the model of many phenomena with memory with fractional differential become more precesily and advanced than the classical model in various fields of science and engineering [2,3,4,5]. The conformable was a new fractional derivative introduced by Khalil and al. [10], after Abdeljawad [11] defined the right and left conformable fractional derivatives, conformable fractional integrals and the fractional Laplace transform. this conformable differential operators become the subject of many contribution in several reserchers. Moreover, the nonlocal cauchy problem, associated with the Fractional derivative, have been investigated by many authors [2,3,4,7,8,14].
In this paper, we are studying the following nonlocal-delay cauchy problem with conformable fractional derivative.,

$$\begin{cases} \mathrm{D}_t^{\alpha}\left[u(t) - \psi((t, u_t)\right] + Au(t) = \xi((t, u_t) & t \in (0, a] \\ u_0(\tau) + \phi((u_{t_1}, u_{t_2}, \ldots, u_{t_n})(\tau) = \varphi(\tau), & \tau \in [-r, 0] \end{cases} \tag{1.1}$$

where :
• D_t^{α} is the Conformable fractional derivative of order $0 < \alpha < 1$,
•(-A) is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ of operators on E witch E is a Banach space with the norm $|.|$,
• $\psi, \xi : [0, \infty) \times \mathcal{C} \to E$ and $\phi : \mathcal{C}^n \to \mathcal{C}$ are given functions, where $\mathcal{C} := C(I, E)$ the continuous functions from $I \subset \mathbb{R}$ into$E$ with the norm $||u|| = Sup_{t\in I}|u(t)|$
• $\varphi \in C([-r, 0], E)$ and define u_t by $u_t(\tau) = u(t + \tau)$, for $\tau \in [-r, 0]$.
This paper is divided into 4 sections. The second section is divoted to some concepts and basic properties of the conformable fractional calculus, in the third Section we define the mild solution of the problem and

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prove the existence and uniqueness under some conditions, and in the last section we give an example of application of the main results.

2. Preliminaries

We start, in this section, by recaling some basic concepts and theorems of the conformable fractional calculus.

Definition 2.1 *([10]). Given a function $f : [0, \infty) \longrightarrow \mathcal{R}$. Then the "conformable fractional derivative" of f of order α is defined by:*

$$\mathrm{D}_t^\alpha(f)(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} \tag{2.1}$$

for all $t > 0, \alpha \in (0, 1)$. If f is α-differentiable in some $(0, a), a > 0$, and $\lim_{t \to 0^+} D^\alpha f(t)$ exists, then define:

$$D_t^\alpha f(0) = \lim_{t \to 0^+} D_t^\alpha(f)(t)$$

The fractional integral $I_a^\alpha(.)$ associated with the conformable fractional derivative is defined by

$$I_a^\alpha(f)(t) = \int_0^t s^{\alpha-1} f(s) \mathrm{d}s \tag{2.2}$$

Theorem 2.1 *([10]). If $f(.)$ is a continuous function in the domain of $I^\alpha(.)$, then we have*

$$D^\alpha \left(I^\alpha(f)(t)\right) = f(t) \tag{2.3}$$

Definition 2.2 *. The Laplace transform of a function $f(.)$ is defined by*

$$L(f(t))(\lambda) := \int_0^{+\infty} e^{-\lambda t} f(t) \mathrm{d}t, \quad \lambda > 0. \tag{2.4}$$

The adapted transform is given by the following definition.

Definition 2.3 *([11]). The conformable fractional Laplace transform of order $\alpha \in]0, 1]$ of a function $f(.)$ is defined by*

$$L_\alpha(f(t))(\lambda) := \int_0^{+\infty} t^{\alpha-1} e^{-\lambda(t^\alpha/\alpha)} f(t) \mathrm{d}t, \quad \lambda > 0 \tag{2.5}$$

Proposition 2.1 *([11]) If $f(.)$ is α differentiable function, then we have the following results:*

$$I^\alpha \left(D^\alpha f(.)\right)(t) = f(t) - f(0) \tag{2.6}$$

$$L_\alpha \left(D^\alpha f(t)\right)(\lambda) = \lambda L_\alpha(f(t))(\lambda) - f(0) \tag{2.7}$$

Proposition 2.2 *([11]). For two functions $f(.)$ and $g(.)$, we have*

$$L_\alpha\left(f\left(\frac{t^\alpha}{\alpha}\right)\right)(\lambda) = L(f(t))(\lambda) \tag{2.8}$$

$$L_\alpha\left(\int_0^t s^{\alpha-1} f\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) g(s) \mathrm{d}s s\right)(\lambda) = L(f(t))(\lambda) L_\alpha(g(t))(\lambda) \tag{2.9}$$

Lemma 2.1 *(Krasnoselskii FPT, [13]).Let E be a Banach space, let B be a bounded closed and convex subset of E and let $P1; P2$ be maps of B into E such that $P_1 x + P_2 y \in B$ for every pair $x; y \in B$. If P_1 is a contraction and P_2 is completely continuous, then the equation$P_1 x + P_2 x = x$ has a solution on B.*

In this paper,let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. For $0 < \beta \leq 1$, we define the fractional power A^{β} as a closed linear operator on its domain $D\left(A^{\beta}\right)$.
For analytic semigroup $\{T(t)\}_{t\geq 0}$, the following properties will be used [1].
(a) There is a $M \geq 1$ such that

$$M := \sup_{t\in[0,+\infty)} |T(t)| < \infty$$

(b) for any $\beta \in (0,1]$, there exists a positive constant C_{β} such that

$$\left|A^{\beta}T(t)\right| \leq \frac{C_{\beta}}{t^{\beta}}, \quad 0 < t \leq a$$

3. Main Results

First, we give the definition of mild solutions for the Cauchy problem (1.1). so, applying the fractional Laplace transform in equation, we obtain ;

$$\begin{aligned}\lambda L_{\alpha}(u(t)-\psi(t,u_t)(\lambda) =&(u_0-\psi(0,u_0)-AL_{\alpha}[u(t)-\psi(t,u_t)](\lambda)\\ &-AL_{\alpha}(\psi(t,u_t)(\lambda)+L_{\alpha}(\xi(t,u_t)(\lambda)\end{aligned}$$

Then:

$$\begin{aligned}L_{\alpha}(u(t)-\psi(t,u(t))(\lambda) =&(\lambda+A)^{-1}[(\varphi(0)-\phi\left(u_{t_1},u_{t_2},\ldots,u_{t_n}\right)(\tau)-\psi(0,u_0))]\\ &-A(\lambda+A)^{-1}L_{\alpha}(\psi(t,u_t))(\lambda)\\ &+(\lambda+A)^{-1}L_{\alpha}(\xi(t,u_t))(\lambda)\end{aligned}$$

Using the inverse fractional Laplace transform combined with (2.8) and (2.9), we obtain

$$\begin{cases} u(t) =T\left(\dfrac{t^{\alpha}}{\alpha}\right)[(\varphi(0)-\phi\left(u_{t_1},u_{t_2},\ldots,u_{t_n}\right)(0)-\psi(0,u_0))] \\ \qquad +\psi(t,u_t)-\displaystyle\int_0^t s^{\alpha-1}AT\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\psi(s,u(s))\mathrm{d}s \\ \qquad +\displaystyle\int_0^t s^{\alpha-1}T\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\xi(s,u(s))\mathrm{d}s; \qquad t\in[0,a] \\ u_0(\tau)+\phi\left(u_{t_1},u_{t_2},\ldots,u_{t_n}\right)(\tau)=\varphi(\tau); \qquad \tau\in[-r,0] \end{cases} \tag{3.1}$$

Now, we can introduce the following definition.

Definition 3.1 *A function $u \in \mathcal{C} := C([-r;a];E)$ is called a mild solution of the Cauchy problem (1.1) if :*

$$\begin{cases} u(t) =T\left(\dfrac{t^{\alpha}}{\alpha}\right)[\varphi(0)-\phi\left(u_{t_1},u_{t_2},\ldots,u_{t_n}\right)(0)-\psi(0,u_0)] \\ \qquad +\psi(t,u_t)-\displaystyle\int_0^t s^{\alpha-1}AT\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\psi(s,u(s))\mathrm{d}s \\ \qquad +\displaystyle\int_0^t s^{\alpha-1}T\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\xi(s,u(s))\mathrm{d}s; \qquad t\in[0,a] \\ \\ u_0(\tau)+\phi\left(u_{t_1},u_{t_2},\ldots,u_{t_n}\right)(\tau)=\varphi(\tau); \qquad \tau\in[-r,0] \end{cases} \tag{3.2}$$

3.1. Existence

To obtain the existence of mild Solutions, we will need the foollowing assumptions:

(H_1) : The function $\xi(t,\cdot):\mathcal{C}\to E$ is continuous for almost all $t\in[0,a]$,and for every each $u\in\mathcal{C}$, the function $\xi(\cdot,u):[0,a]\to E$ is strongly measurable,

(H_2) : for every each $m>0$ there exists a function $\varepsilon_m \in L^{\infty}([0,a],\mathbb{R}^+)$ such that: $\sup_{\|u\|\leq m}\|\xi(t,u)\| \leq \varepsilon_m(t)$, for almost all $t\in[0,a]$

(H_3) : there exists a constant $R_0>0$ such that: $\|\phi(u_{t_1},u_{t_2},\ldots,u_{t_n})-\phi(v_{t_1},v_{t_2},\ldots,v_{t_n})\| \leq R_0\|u-v\|$, for $u,v\in C([-r,a],E)$,

(H_4) : $\psi:[0,a]\times\mathcal{C}\to E$ is a continuous function and there is $\eta\in(0,1)$; $K,K_1>0$ such that $\psi\in D(A^{\eta})$ and for any $u,v\in\mathcal{C}, t\in[0,a]$, the function $A^{\eta}\psi(\cdot,u)$ is strongly measurable and $A^{\eta}\psi(t,\cdot)$ satisfies the Lipschitz condition

$$|A^{\eta}\psi(t,u)-A^{\eta}\psi(t,v)| \leq K\|u-v\|$$

and Inequality:

$$|A^{\eta}\psi(t,u)| \leq K_1(\|u\|+1)$$

Theorem 3.1 *If $T(t)_{t>0}$ is compact and $(H_1)-(H_4)$ are satisfied, then the nonlocal Cauchy problem with delay (1.1) has a mild solution provided that:*

$$R_1 := MR_0 + (M+1)\left|A^{-\eta}\right|K + \frac{C_{1-\eta}K}{\eta\alpha^{\eta}}a^{\alpha\eta} < 1 \tag{3.3}$$

Proof: We prove the existence of mild solution for the nonlocal Cauchy problem (1.1) by applying Krasnoselskii's fixed point theorem.
First, we define the function $w\in C([-r,a],E)$ such that $|w(t)|\equiv 0, t\in[-r,a]$.for any positive constant m and $u\in B_m=\{u\in C([-r,a],E): \|u\| |\leq m\}$, and according to the hypothesis (H$_3$), for $t\in[0,a]$, we have:

$$\begin{aligned}\left|T(\frac{t^{\alpha}}{\alpha})\left[\varphi(0)-\phi(u_{t_1},u_{t_2},\ldots,u_{t_n})(0)-\psi(0,u_0)\right]\right| &\leq M(|\varphi(0)|+R_0\|u-w\|)\\ &+M|\phi(w_{t_1},w_{t_2},\ldots,w_{t_n})|\\ &+M\left|A^{-\eta}A^{\eta}\psi(0,u_0)\right|\\ &\leq M[\|\varphi\|+R_0m+\|\phi(w_{t_1},w_{t_2},\ldots,w_{t_n})\|]\\ &+M\left|A^{-\eta}\right|K_1(m+1)\end{aligned} \tag{3.4}$$

$\{T(t)\}_{t\geq 0}$ is a uniformly continuous analytic semigroup, then $s^{\alpha-1}AT\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\psi(s,u_s)$ is continuous on $[0,t]$.
For every $u\in B_m; t\in[0,t]$, using the properties (a), (b) and the hypothesis (H_4) we have :

$$\begin{aligned}\int_0^t\left|s^{\alpha-1}AT\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\psi(s,u_s)\right|\mathrm{ds} &= \int_0^t\left|s^{\alpha-1}A^{1-\eta}T\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)A^{\eta}\psi(s,u_s)\right|ds\\ &\leq \int_0^t s^{\alpha-1}\frac{\alpha^{1-\eta}C_{1-\eta}}{(t^{\alpha}-s^{\alpha})^{(1-\eta)}}K_1(m+1)ds\\ &= K_1(m+1)C_{1-\eta}\int_0^t\frac{\alpha^{1-\eta}s^{\alpha-1}}{(t^{\alpha}-s^{\alpha})^{(1-\eta)}}ds\end{aligned}$$

Then

$$\int_0^t\left|s^{\alpha-1}AT\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\psi(s,u(s))\right|\mathrm{ds} \leq \frac{C_{1-\eta}K_1(m+1)}{\eta\alpha^{\eta}}a^{\alpha\eta} \tag{3.5}$$

therefore : $\left|s^{\alpha-1}AT\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\psi(s,u_s)\right|$ is integrable in the Lebesgue sense for all $s\in[0,t]$ and $t\in[0,a]$
For every $m>0$, we define two operators (P_1) et (P_2) on B_m as follows:

$$\begin{cases}(P_1u)(t) = T(\frac{t^{\alpha}}{\alpha})\left[\varphi(0)-\phi(u_{t_1},u_{t_2},\ldots,u_{t_n})(0)-\psi(0,u_0)\right]\\ \qquad +\psi(t,u_t)-\displaystyle\int_0^t s^{\alpha-1}AT\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\psi(s,u_s)ds; \quad t\in[0,a]\\ (P_1u)(\tau) = \varphi(\tau)-\phi(u_{t_1},u_{t_2},\ldots,u_{t_n})(\tau; \quad \tau\in[-r,0]\end{cases} \tag{3.6}$$

and

$$\begin{cases} (P_2u)(t) = \displaystyle\int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \xi(s, u(s)) \mathrm{d}s; & t \in [0, a] \\ (P_2u)(\tau) = 0; & \tau \in [-r, 0] \end{cases} \tag{3.7}$$

Then, u is a mild solution of (1.1) if and only if $u = P_1u + P_2u$ has a solution $u \in B_m$. So we determine m_0 such that $P_1 + P_2$ has a fixed point in B_{m_0}.
We choose m_0 such that

$$\begin{aligned} m_0 &\geq M\left[\|\varphi\| + R_0 m_0 + \|\phi(w_{t_1}, w_{t_2}, \ldots, w_{t_n})\| + \left|A^{-\eta}\right| K_1 (m_0 + 1)\right] \\ &+ \left|A^{-\eta}\right| K_1 (m_0 + 1) + \frac{M}{\alpha} \left|\epsilon_{m_0}\right|_{L^\infty([0,a];\mathbb{R}^+)} a^\alpha + \frac{C_{1-\eta} K_1 m_0}{\eta \alpha^\eta} a^{\alpha\eta} \end{aligned} \tag{3.8}$$

We show that $P_1 + P_2$ has a fixed point, by using Krasnoselskii's Fixed Point Theorem, in three steps
Step 1: $P_1u + P_2v \in B_{m_0}$, for any $u, v \in B_{m_0}$.
Clearly, for every pair $u; v \in B_{m_0}, (P_1u)(t)$ and $(P_2v)(t)$ are continuous in $t \in [-r, a]$.
Let $u, v \in B_{m_0}$ and $t \in [0, a]$,according to $(H_1), (H_2)$ and (3.4),(3.5) we get:

$$\begin{aligned} |(P_1u)(t) + (P_2v)(t)| &\leq \left|T(\frac{t^\alpha}{\alpha})\right| |[\varphi(0) - \phi(u_{t_1}, u_{t_2}, \ldots, u_{t_n})(0) - \psi(0, u_0)]| \\ &+ \left|\int_0^t s^{\alpha-1} AT\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \psi(s, u(s)) \mathrm{d}s\right| \\ &+ |\psi(t, u_t)| + \left|\int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \xi(s, u(s)) \mathrm{d}s\right| \\ &\leq M\left[\|\varphi\| + R_0 m_0 + \|\phi(w_{t_1}, w_{t_2}, \ldots, w_{t_n})\|\right] \\ &+ M\left|A^{-\eta}\right| K_1 (m_0 + 1) + \left|A^{-\eta}\right| K_1 (m_0 + 1) \\ &+ \frac{M}{\alpha} \left|\epsilon_{m_0}\right|_{L^\infty([0,a];\mathbb{R}^+)} a^\alpha + \frac{C_{1-\eta} K_1 m_0}{\eta \alpha^\eta} a^{\alpha\eta} \\ &\leq m_0. \end{aligned}$$

For $\tau \in [-r, 0]$

$$\begin{aligned} |(P_1x)(\tau) + (P_2y)(\tau)| &\leq M\left[\|\varphi\| + R_0 m_0 + \|\phi(w_{t_1}, w_{t_2}, \ldots, w_{t_n})\|\right] \\ &\leq m_0 \end{aligned}$$

Therfore, $P_1u + P_2v \in B_{m_0}$, for every $u, v \in B_{m_0}$
Step 2: P_1 is contraction on B_{m_0} .
Let $u, v \in B_{m_0}$ and $t \in [0, a]$,

$$\begin{aligned} |(P_1u)(t) - (P_1v)(t)| &\leq \left|T(\frac{t^\alpha}{\alpha})\left[\phi(u_{t_1}, u_{t_2}, \ldots, u_{t_n})(0) - \phi(v_{t_1}, v_{t_2}, \ldots, v_{t_n})(0)\right]\right| \\ &+ \left|T(\frac{t^\alpha}{\alpha})\left[\psi(0, u_0) - \psi(0, v_0)\right]\right| + |\psi(t, u_t) - \psi(t, v_t)| \\ &+ \int_0^t s^{\alpha-1} \left|A^{1-\eta} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\left[A^\eta \psi(s, u_s) - A^\eta \psi(s, v_s)\right]\right| ds \\ &\leq M R_0 \|u - v\| + (M + 1)\left|A^{-\eta}\right| K \|u - v\| + \frac{C_{1-\eta} K}{\eta \alpha^\eta} a^{\alpha\eta} \|u - v\| \\ &= \left(M R_0 + (M + 1)\left|A^{-\beta}\right| K + \frac{C_{1-\eta} K}{\eta \alpha^\eta} a^{\alpha\eta}\right) \|u - v\| \end{aligned}$$

For $\tau \in [-r, 0]$ we have :

$$|(P_1u)(\tau) - (P_1v)(\tau)| \leq M R_0 \|u - v\|, \quad \tau \in [-r, 0]$$

Then ; $\|P_1u - P_1v\| \leq \left(MR_0 + (M+1)\left|A^{-\beta}\right|K + \frac{C_{1-\eta}K}{\eta\alpha^{\eta}}a^{\alpha\eta}\right)\|u-v\|$. Therefore according to (3.3),P_1 is a contraction.

Step 3 : (P_2) is completely continuous .
First, we show that P_2 is continuous on B_{m_0}.
Let $\{u^n\} \subseteq B_{m_0}$ where $u^n \to u$ on B_{m_0}. then: $u_t^n \to u_t$ for $t \in [0,a]$ according to (H$_1$),we have:

$$\xi\left(s, u_s^n\right) \to \xi\left(s, u_s\right), s \in [0,a], \qquad \text{as } n \to \infty.$$

Since $s^{\alpha-1}\left|\xi\left(s,u_s^n\right) - \xi\left(s,u_s\right)\right| \leq 2s^{\alpha-1}\epsilon_{m_0}(s)$, according to the dominated convergence theorem, we have:

$$\begin{aligned}\left|\left(P_2u^n\right)(t) - \left(P_2u\right)(t)\right| &= \left|\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\left[\xi\left(s,u_s^n\right) - \xi\left(s,u_s\right)\right]\mathrm{d}s\right| \\ &\leq M\int_0^t s^{\alpha-1}\left|\xi\left(s,u_s^n\right) - \xi\left(s,u_s\right)\right|\mathrm{d}s \to 0, \qquad \text{as } n \to \infty\end{aligned}$$

which implies P_2 is continuous.

Next, we will show that $\{P_2u, u \in B_{m_0}\}$ is relatively compact,
For $u \in B_{m_0}$ and $t_1, t_2 \in [0,\tau]$ such that $t_1 < t_2$. we have :

$$\begin{aligned}\left|\left(P_2u\right)(t_2) - \left(P_2u\right)(t_1)\right| = &\int_{t_1}^{t_2} s^{\alpha-1}T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right)\xi(s,u(s))\mathrm{d}s. \\ &+ \left[T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I\right]\int_0^{t_1} s^{\alpha-1}T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)\xi(s,u(s))\mathrm{d}s\end{aligned}$$

Using the assumption (H_3), we obtain:

$$\begin{aligned}\left|P_2(u)(t_2) - P_2(u)(t_1)\right| \leq &\left(\frac{a^\alpha}{\alpha}\left|\epsilon_{m_0}\right|_{L^\infty([0,a],\mathbb{R}^+)}\right)\sup_{t\in[0,a]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \cdot \left[\left|T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I\right|\right] \\ &+ \sup_{t\in[0,a]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left|\epsilon_{m_0}\right|_{L^\infty([0,a],\mathbb{R}^+)}\left[\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right]\end{aligned}$$

Then $\{P_2u, u \in B_{m_0}\}$ is equicontinuous , and uniformly bounded.
Next we show that for every $t \in [-r,a], V(t) = \{(P_2u)(t), u \in B_{m_0}\}$ is relatively compact on E.
For $t \in [-r,0], V(t)$ is relatively compact on E.
For $0 < t \leq a$ fixe, and for every $\varepsilon \in (0,t)$, we define $P_{2,\varepsilon}$ on B_{m_0}:

$$\begin{aligned}(P_{2,\varepsilon}u)(t) &= \int_0^{t-\varepsilon} s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\xi(s,u_s)\mathrm{d}s \\ &= T\left(\frac{\varepsilon^\alpha}{\alpha}\right)\int_0^{t-\varepsilon} s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha - \varepsilon^\alpha}{\alpha}\right)\xi(s,u_s)\mathrm{d}s\end{aligned}$$

Since $T\left(\frac{\varepsilon^\alpha}{\alpha}\right)$ is compact, then $V_\varepsilon(t) = \{(P_{2,\varepsilon}u)(t), u \in B_{m_0}\}$ is relatively compact.
For every $\varepsilon > 0$, we have

$$\begin{aligned}\left|\left(P_2u\right)(t) - \left(P_{2,\varepsilon}u\right)\right| &= \left|\int_{t-\varepsilon}^t s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)f(s,u_s)\mathrm{d}s\right| \\ &\leq M\left|\epsilon_{m_0}\right|_{L^\infty([0,a],\mathbb{R}^+)}\int_{t-\varepsilon}^t s^{\alpha-1}\mathrm{d}s \\ &\leq M\left|\epsilon_{m_0}\right|_{L^\infty([0,a],\mathbb{R}^+)}\varepsilon^\alpha\end{aligned}$$

So there are relatively compact sets close to the set $V(t), t > 0$. Hence the set $V(t), t > 0$ is also relatively compact.

According to the theorem of Ascoli-Arzela; $\{P_2 u, u \in B_{m_0}\}$ is relatively compact. Therefore, P_2 is completely continuous .
Finally, the conditions of the Krasnoselskii theorem are verified, then $P_1 u + P_2 u$ has a fixed point, and the nonlocal neutral evolution problem(1.1) has a mild solution. □

3.2. Existence and Uniqueness of mild solution

To show the existence and uniqueness result for the non-local Cauchy problem (1.1), we apply the Banach contraction principle and we add the following assumptions:

(H_5) : $\xi(t, u_t)$ is strongly measurable for every $u \in C([-r, a], B_m)$ and almost all $t \in [0, a]$.

(H_6) : there exists a constant $\alpha_1 \in (0, \alpha)$ and $\rho \in L^{\frac{1}{\alpha_1}}([0, a], \mathbb{R}^+)$ such that; for every $u,v \in C([-r, a], B_m)$, where m is a positive constant.we have:

$$|\xi(t, u_t) - \xi(t, v_t)| \leq \rho(t)\|u - v\|, t \in [0, a]$$

,

Theorem 3.2 *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_6)$ *are satisfied , Then the nonlocal Cauchy problem with delay (1.1) has a unique mild solution provided that:*

$$\begin{array}{l} R_2 := MR_0 + (M+1)\left|A^{-\eta}\right| K + \frac{C_{1-\eta} K a^{\alpha\eta}}{\eta\alpha^\eta} + \frac{M_1 M a^{(1+\mu)(1-\alpha_1)}}{(1+\mu)^{1-\alpha_1}} < 1 \\ \textit{where; } \mu = \frac{\alpha-1}{1-\alpha_1} \in (-1, 0), M_1 = \|\rho\|_{L^{\frac{1}{\alpha_1}}([0,a],\mathbb{R}^+)}. \end{array} \tag{3.9}$$

Proof. We define the operator P on B_m by:

$$\begin{cases} (Pu)(t) & = T\left(\frac{t^\alpha}{\alpha}\right)\left[\varphi(0) - \phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)(0) - \psi(0, u_0)\right] \\ & +\psi(t, u_t) - \int_0^t s^{\alpha-1} A T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\psi(s, u(s))\mathrm{d}s \\ & + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\xi(s, u(s))\mathrm{d}s, \quad t \in [0, a] \\ (Pu)(\tau) = & -\left(\phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)(\tau) + \varphi(\tau), \quad \tau \in [-r, 0]\right. \end{cases} \tag{3.10}$$

it suffices to prove that P has a unique fixed point on B_{m_0}, where m_0 is defined by (3.8).
For every $u, v \in B_{m_0}$ and $t \in [0; a]$, according to H_3, H_4 and H_6, and applying the Holder Inequality, we have:

$$\begin{aligned} |(Pu)(t) - (Pv)(t)| \leq & \left|T(\frac{t^\alpha}{\alpha})\left[\left(\phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)\right)(0) - \left(\phi\left(v_{t_1}, v_{t_2}, \ldots, v_{t_n}\right)\right)(0)\right]\right| \\ & + \left|T(\frac{t^\alpha}{\alpha})\left[\psi(0, u_0) - \psi(0, v_0)\right]\right| + |\psi(t, u_t) - \psi(t, v_t)| \\ & + \int_0^t s^{\alpha-1}\left|A^{1-\eta} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\left[A^\eta \psi(s, u_s) - A^\eta \psi(s, v_s)\right]\right| ds \\ & + \int_0^t \left|s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\left[\xi(s, v_s) - \xi(s, u_s)\right]\right| \mathrm{d}s \\ & \leq MR_0\|u - v\| + (M+1)\left|A^{-\eta}\right| K\|u - v\| + \frac{C_{1-\eta} K}{\eta\alpha^\eta} a^{\alpha\eta}\|u - v\| \\ & + M\left(\int_0^t s^{\frac{\alpha-1}{1-\alpha_1}}\, \mathrm{d}s\right)^{1-\alpha_1} \|\rho\|_{L^{\frac{1}{\alpha_1}}[0,t]}\|u - v\| \end{aligned}$$

which implies that :

$$|(Pu)(t) - (Pv)(t)| \leq \left(MR_0 + (M+1)\left|A^{-\eta}\right| K + \frac{C_{1-\eta} K}{\eta\alpha^\eta} a^{\alpha\eta} + \frac{M_1 M a^{(1+\mu)(1-\alpha_1)}}{(1+\mu)^{1-\alpha_1}}\right)\|u - v\|$$

which means that P is a contraction according to (3.9). By applying the Banach contraction mapping principle, P has a unique fixed point on B_{m_0}. the proof is complete.

4. Application

Let $E = L^2([0,\pi], R)$. Consider the following fractional partial differential equations.

$$\begin{cases} \partial_t^\alpha \left(u(t,x) - \int_0^\pi N(x,y)u_t(\tau,y)\mathrm{d}y\right) = \partial_x^2 u(t,x) + \partial_x G\left(t, u_t(\tau,x)\right), & t \in (0,a] \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,a] \\ u(\tau,x) + \sum_{i=0}^n \int_0^\pi b(x,y)u_{t_i}(\tau,y)\mathrm{d}y = (\varphi(\tau))(x), \quad \tau \in [-r,0], \end{cases} \tag{4.1}$$

where: ∂_t^α is a conformable fractional partial derivative of order α; such $0<\alpha<1, a>0, t\in[0,a]$.
We define : $E = L^2([0,\pi], R)$,
• $G$is a given function,
•$0 < t_0 < t_1 < \cdots < t_n < a, \varphi \in C([-r,0],E)$, that is $\varphi(\tau) \in E$.
•$b(x,y) \in L^2([0,\pi]\times[0,\pi], R)$ and $u_t(\tau,z) = u(t+\tau,x), t\in[0,a], \tau\in[-r,0]$.
We take: $A = -\partial_x^2(.)$ with the domain:

$$D(A) = \{v(\cdot) \in E : v, v' \text{absolutly continuous}, v'' \in E, v(0) = v(\pi) = 0\}$$

Then $-A$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ which is compact, analytic and self-adjoint
The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}} v = \sum_{n=1}^{\infty} n \langle v, u_n\rangle u_n$$

on the space $D\left(A^{\frac{1}{2}}\right) = \{v(\cdot) \in E, \sum_{n=1}^\infty n\langle v,u_n\rangle u_n \in E\}$. The problem (4.1) can take the form in E

$$\begin{cases} D_t^\alpha \left(U(t) - \psi\left(t, U_t\right)\right) + AU(t) = \xi\left(t, U_t\right) & t \in (0,a] \\ U_0(\tau) + \left(\phi\left(U_{t_1}, \ldots, U_{t_n}\right)\right)(\tau) = \varphi(\tau) & \tau \in [-r,0], \end{cases} \tag{4.2}$$

where $U_t = u_t(\tau,\cdot)$, that is $(U(t+\tau))(x) = u(t+\tau,x), t\in[0,a], x\in[0,\pi]$.
The function $\psi : [0,a]\times\mathcal{C} \to E$ is given by

$$\left(\psi\left(t, U_t\right)\right)(x) = \int_0^\pi N(x,y)u_t(\tau,y)\mathrm{d}y.$$

Let $\left(N_h v\right)(x) = \int_0^\pi N(x,y)v(y)\mathrm{d}y$, for $v \in E = L^2([0,\pi],R), x\in[0,\pi]$.
The function $\xi : [0,a]\times\mathcal{C} \to E$ is given by

$$\left(\xi\left(t, U_t\right)\right)(x) = \partial_x G\left(t, u_t(\tau,x)\right),$$

and the function $\phi : \mathcal{C}^n \to \mathcal{C}$ is given by

$$\left(\phi\left(U_{t_1}, \ldots, U_{t_n}\right)\right)(\tau) = \sum_{i=0}^n b_\phi U_{t_i}(\tau)$$

where $\left(b_\phi v\right)(x) = \int_0^\pi b(x,y)v(y)\mathrm{d}y$, for $v \in E = L^2([0,\pi],R), x\in[0,\pi]$.

For $\alpha = 1/2$ and $\xi\left(t, U_t\right) = \frac{1}{t^{1/3}} \sin U_t$, then $(\mathrm{H}_1), (\mathrm{H}_2), (\mathrm{H}_5)$ and (H_6) are satisfied.
Furthermore, for $v_1, v_2 \in E$, we have

$$\begin{aligned} \left\|b_\phi v_1 - b_\phi v_2\right\|_{L^2[0,\pi]} &= \left(\int_0^\pi \left(\int_0^\pi b(x,y)\left(v_1(y) - v_2(y)\right)\mathrm{d}y\right)^2 \mathrm{d}x\right)^{1/2} \\ &\leq \left(\int_0^\pi \left(\int_0^\pi b^2(x,y)\mathrm{d}y \int_0^\pi \left[v_1(y) - v_2(y)\right]^2 \mathrm{d}y\right)\mathrm{d}x\right)^{1/2} \\ &= \left(\int_0^\pi \int_0^\pi b^2(x,y)\mathrm{d}y\ \mathrm{d}z\right)^{1/2} \left\|v_1 - v_2\right\|_{L^2[0,\pi]} \end{aligned}$$

So, $R_0 = (n+1)\left[\int_0^\pi\int_0^\pi b^2(x,y)\mathrm{d}y\ \mathrm{d}x\right]^{1/2}$, then (H_3) is satisfied.

Moreover, we assume that the following conditions hold.

(c) The function $N(x,y), x,y \in [0,\pi]$ is measurable and

$$\int_0^\pi \int_0^\pi N^2(x,y)\mathrm{d}y\ \mathrm{d}x < \infty$$

(d) the function $\partial_x N(z,y)$ is measurable, $N(0,y) = N(\pi,y) = 0$, and let

$$\bar{N} = \left(\int_0^\pi \int_0^\pi \left(\partial_x N(x,y)\right)^2\ \mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}} < \infty$$

we can take $\eta = \frac{1}{2}$ Then (H_4) is satisfied.
we conclude that Eq (4.2) has a mild solution provided that :

$$R_0 + \left(2 + 2\sqrt{2}C_{1/2}\right) \sup_{t\in[0,1]} \left(\int_0^\pi \left(\int_0^\pi \partial_x N(x,y)u_t(\tau,y)\right)^2 \mathrm{d}y\right) < 1 \tag{4.3}$$

where $R_0 = (n+1)\left[\int_0^\pi \int_0^\pi b^2(x,y)\mathrm{d}y\ \mathrm{d}x\right]^{1/2}$
Hence, by applying Theorem 3.2, the system (4.1) admits a unique mild solution

5. Conclusion

In this article, we have considered a class of conformable fractional neutral evolution equations with nonlocal-delay conditions, the existence and uniqueness results are established by using Krasnoselskii fixed point theorem,and Banach fixed point theorem. Finally, the investigation of the result has been illustrated by providing suitable examples.

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