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Primary ideal bundles of Lie algebra bundles

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ABSTRACT: In this paper, we study primary ideal bundles of a Lie algebra bundle and investigate their relations with prime, semiprime and irreducible ideal bundles. We characterize primary ideal bundles in terms of sections of Lie algebra bundles. Also, the concepts of prime Lie algebra bundles and primary Lie algebra bundles are introduced and studied briefly.

Key Words: Lie algebra bundle, prime ideal bundle, semi prime ideal bundle, irreducible ideal bundle, primary ideal bundle, prime Lie algebra bundle, primary Lie algebra bundle.

Contents

1. Introduction

The concept of prime ideals and primary ideals has been studied in Lie algebras in [2],[3],[5],[4]. Lie algebra bundles were defined and studied in [6], [8], [9]. It is of interest how the notion of primary ideals of Lie algebras can be applied to Lie algebra bundles. We define primary ideal bundles of a Lie algebra bundle and study the relations between prime and primary ideal bundles. A characterisation of primary ideal bundles in terms of sections of Lie algebra bundles is done. We have deduced that if $\xi \in Max - \triangleleft$, then the notions of prime ideal bundles, primary ideal bundles and irreducible ideal bundles are equivalent. At the end we introduce the concept of prime Lie algebra bundles and primary Lie algebra bundles.

We begin with some definitions required in the context of this paper.

A **Lie algebra bundle** is a vector bundle $\xi = (E, p, X)$ in which each fibre ξ_x is a Lie algebra and for each x in X, there is an open neighbourhood U of x, a Lie algebra L and a homeomorphism $\phi : U \times L \to p^{-1}(U)$ such that for each y in U, $\phi_y : L \to p^{-1}(y)$ is a Lie algebra isomorphism. A Lie algebra bundle with a semisimple Lie algebra structure on each of its fibre is called a **semisimple Lie algebra bundle**. A **section** S of a Lie algebra bundle ξ is a continuous map $s : X \to E$ such that $p \circ s = Id$. An ideal bundle P of a Lie algebra bundle ξ is said to be a **prime ideal bundle** if for each x in X, P_x is a prime ideal of ξ_x. An ideal subbundle Q of a Lie algebra bundle ξ is said to be **semiprime** if for any ideal bundle H of ξ, $[H, H] \subseteq Q$ implies $H \subseteq Q$. An ideal bundle N of a Lie algebra bundle ξ is said to be **irreducible** if $N = H \cap K$ for ideal bundles H and K of ξ gives $N = H$ or $N = K$. If η be an ideal subbundle of ξ then $r(\eta) = \cup_{x \in X} r(\eta_x)$ is an ideal subbundle of ξ called the **prime radical** of η. The union of all prime radicals of the fibres of ξ is denoted by $r(0)$ and we call $r(0)$ as the **prime radical** of ξ and denote it as $r(\xi)$.

We assume that the base space X of any Lie algebra bundle is compact Hausdorff and all underlying vector spaces are real and finite dimensional.

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2. Primary ideal bundles

It is proved in [7] that the intersection of any two ideal bundles of a semisimple Lie algebra bundle ξ is an ideal bundle of ξ. We extend this result to a finite intersection of ideal bundles.

Theorem 2.1 *Any finite intersection of ideal bundles of a semisimple Lie algebra bundle ξ is an ideal bundle of ξ.*

Proof: Let $\xi_1, \xi_2, \ldots$ be ideal bundles of ξ. We prove the result by induction. For $n = 2$, $\xi_1 \cap \xi_2$ is an ideal bundle of ξ. Assume the result for all $n-1$, $n \geq 3$. Then $\xi' = \bigcap_{i=1}^{n-1} \xi_i$ is an ideal bundle of ξ. Let the local triviality of ξ' be $\phi : U \times I' \to p^{-1}(U)$ for an open set U containing $x \in X$. If the local triviality of ξ_n is $\phi : U \times I_n \to p^{-1}(U)$ then by definition of intersection of ideal bundles, $\xi' \cap \xi_n$ is an ideal bundle of ξ with the local triviality condition, $\phi : U \times (I' \cap I_n) \to p^{-1}(U)$. Thus we can conclude that finite intersection of ideal bundles is an ideal bundle of ξ. □

From Theorem 2.1 for any semisimple Lie algebra bundle ξ, we have the following corollaries.

Corollary 2.1 *The prime radical bundle of any ideal bundle of ξ is a semiprime ideal bundle of ξ.*

Proof: The prime radical bundle of an ideal bundle η of ξ is given by $r(\eta) = \bigcup_{x \in X} r(\eta_x)$, $r(\eta_x)$ is the prime radical of ξ_x. If I is any ideal of ξ_x and $I^2 \subseteq r(\eta_x)$ then by definition of $r(\eta_x)$, $I \subseteq r(\eta_x)$. Thus each fibre of $r(\eta)$ is semiprime and the corollary is proved. □

Corollary 2.2 *Let η be an ideal bundle of ξ. If $A^2 \subseteq \eta$, A an ideal bundle of ξ, then $A \subseteq r(\eta)$.*

Proof: By definition, $r(\eta)$ is the prime radical $r(\eta_x)$ on each of its fibre η_x. If $A^2 \subseteq \eta$ then $A_x^2 \subseteq r(\eta_x)$ and hence $A \subseteq r(\eta)$. □

As defined for Lie algebras in [4], we proceed to define primary ideal bundles of Lie algebra bundles.

Definition 2.1 *An ideal bundle P of a Lie algebra bundle ξ is said to be a primary ideal bundle of ξ if the ideal P_x is a primary ideal in ξ_x.*

It follows that, if A, B are ideal bundles of ξ and $[A, B] \subseteq P$, then $A \subseteq r(P)$ or $B \subseteq r(P)$.

Theorem 2.2 *If $r(P)$ is a prime ideal bundle of ξ, then P is a primary ideal bundle of ξ.*

Proof: Let A, B be ideal bundles of ξ and $[A, B] \subseteq P$. $P \subseteq r(P)$ and hence $[A, B] \subseteq r(P)$ from which we get $A \subseteq r(P)$ or $B \subseteq r(P)$. Therefore P is a primary ideal bundle of ξ. □

Corollary 2.3 *Every prime ideal bundle of ξ is primary.*

Proof: A, B be ideal bundles of ξ and let $[A, B] \subseteq P$, P is a prime ideal bundle of ξ. Then $A \subseteq P \subseteq r(P)$ or $B \subseteq P \subseteq r(P)$ so that P is primary. □

Definition 2.2 *A prime ideal bundle of ξ is called a* ***prime radical ideal bundle*** *of ξ if its prime radical bundle is itself that is, $r(P) = P$.*

Theorem 2.3 *Let P be a primary ideal bundle of ξ. If P is a prime radical ideal bundle of ξ then P is a prime ideal bundle of ξ.*

Proof: Since P is primary for any two ideal bundles A, B of ξ such that $[A, B] \subseteq P$, $A \subseteq r(P)$ or $B \subseteq r(P)$. Also $r(P) = P$ and hence the theorem follows. □

Lemma 2.1 *If M is an ideal bundle of ξ with $r(M) = \xi$, then M is a primary ideal bundle.*

Proof: If A, B are ideal bundles of ξ, $[A,B] \subseteq M$, then since $r(M) = \xi$, $A \subseteq r(M)$ or $B \subseteq r(M)$. Therefore M is a primary ideal bundle of ξ. □

Lemma 2.2 *If M is a maximal prime radical ideal bundle of ξ then M is a prime ideal bundle of ξ.*

Proof: If M is not a prime ideal bundle of ξ then ξ is the only prime ideal bundle containing M. Hence we have $r(M) = \xi$ which is not possible. □

Theorem 2.4 *Every maximal ideal bundle M of ξ is a primary ideal bundle of ξ.*

Proof: Suppose M is a maximal ideal bundle of ξ. Then we have two cases.
Case(i): When $r(M) = M$.
By Lemma 2.1, M is a prime ideal bundle and hence M is a primary ideal bundle.
Case(ii): When $r(M) = \xi$.
This case follows from Lemma 2.2. □

We now proceed to obtain a characterization of primary ideal bundles using sections of Lie algebra bundle.

Theorem 2.5 *Let P be an ideal bundle of a Lie algebra bundle ξ. Then the following conditions are equivalent.*

1. *P is a primary ideal bundle.*
2. *If $[s(x), H_x] \subseteq P_x$ for some section s of ξ and ideal bundle H in ξ then either $s(x) \in r(P_x)$ or $H_x \subseteq r(P_x)$.*
3. *If $[s_1(x), \langle s_2(x)\rangle] \subseteq P_x$ for some sections s_1 and s_2 of ξ then $s_1(x) \in r(P_x)$ or $s_2(x) \in r(P_x)$.*

Proof: (1) $\implies$ (3): For $x \in X$, set

$$(V_0)_x = (s_1(x))$$
$$(V_i)_x = [\dots[(s_1(x)), \underbrace{\xi_x], \dots, \xi_x]}_{i \text{ times}}, \text{ for all } i$$

If for sections s_1, s_2 of ξ and $x \in X$, $[s_1(x), \langle s_2(x)\rangle] \subseteq P_x$, we assert that $[(V_i)_x, \langle s_2(x)\rangle] \subseteq P_x$ for all $i \geq 0$. We prove this by induction. For $i = 0$, $[s_1(x), \langle s_2(x)\rangle] \subseteq P_x$ implies $[(s_1(x)), \langle s_2(x)\rangle] \subseteq P_x$ so that $[(V_0)_x, \langle s_2(x)\rangle] \subseteq P_x$. Let us assume the assertion is true for $i-1$, $i \geq 1$. Then

$$[(V_i)_x, \langle s_2(x)\rangle] = [[(V_{i-1})_x, \xi_x], \langle s_2(x)\rangle].$$

Using Jacobi identity,

$$\begin{aligned}[(V_i)_x, \langle s_2(x)\rangle] &\subseteq [[(V_{i-1})_x, \langle s_2(x)\rangle], \xi_x] + [(V_{i-1})_x, [\xi_x, \langle s_2(x)\rangle]] \\ &\subseteq [P_x, \xi_x] + [(V_{i-1})_x, \langle s_2(x)\rangle] \\ &\subseteq P_x.\end{aligned}$$

By definition of $\langle s_1(x)\rangle$, $\langle s_1(x)\rangle = \sum_{i=0}^{\infty} (V_i)_x$. Therefore,

$$\begin{aligned}[\langle s_1(x)\rangle, \langle s_2(x)\rangle] &= \left[\sum_{i=0}^{\infty} (V_i)_x, \langle s_2(x)\rangle\right] \\ &= \sum_{i=0}^{\infty} [(V_i)_x, \langle s_2(x)\rangle] \\ &\subseteq P_x.\end{aligned}$$

Since P_x is a primary ideal in ξ_x, $\langle s_1(x)\rangle \subseteq r(P_x)$ or $\langle s_2(x)\rangle \subseteq r(P_x)$ from which it follows that $s_1(x) \in r(P_x)$ or $s_2(x) \in r(P_x)$. This proves (3).

(3) $\implies$ (2): If $s_1(x) \in P_x$ for all x then nothing to prove. Otherwise we prove that $K_x \subseteq P_x$.

Let $y \in K_x$. Then $Y = \{x\}$ is a closed subspace of X. Set $s(x) = y$ on Y. Then $s : Y \to \xi|Y$ is a section of $\xi|Y$. Since X is compact Hausdorff, X has a finite open subcover $\{U_i\}$. Let t_i be sections of $E|U_i$ and $\{p_i\}$ be a partition of unity with $supp(p_i) \subset U_i$. Define,

$$s_i(x) = \begin{cases} p_i(x)t_i(x) & x \in U_i \\ 0 & \text{otherwise} \end{cases}$$

Then $s_2 = \sum_i s_i(x)$ is a section of ξ and $s_2|Y = s$.

Therefore for $y \in K_x$ we can find a section s_2 of ξ such that $y = s_2(x)$.

Now, we have, $[s_1(x), \langle s_2(x)\rangle] \subseteq P_x$ and from (3), $s_1(x) \in r(P_x)$ or $s_2(x) \in r(P_x)$. But $s_1(x) \notin r(P_x)$. It follows that $K_x \subseteq r(P_x)$. Therefore (2) is proved.

(2) $\implies$ (1): Let H, K be ideal bundles of ξ with $[H_x, K_x] \subseteq P_x$ and $H_x \nsubseteq r(P_x)$. If s is a section of ξ, $s(x) \in H_x \setminus r(P_x)$ we have $[s(x), K_x] \subseteq P_x$. From *(ii)* $K_x \subseteq r(P_x)$ which proves (1). □

Analogous to Lie algebras, a Lie algebra bundle ξ is said to satisfy the **maximal condition for ideal bundles** if for every chain of ideal bundles $\eta_1 \subseteq \eta_2 \subseteq \cdots$ there exists an index m such that $\eta_i = \eta_k$ for every i, $k > m$. This is denoted as $\xi \in Max - \triangleleft$.

Theorem 2.6 *Let $\xi \in Max - \triangleleft$. An ideal bundle M is semiprime and primary if and only if M is a prime ideal bundle.*

Proof: Any semiprime ideal bundle is a prime radical ideal bundle. If M is primary then by Theorem 2.3, M is prime. Conversely, every prime ideal bundle is semiprime and primary. □

Theorem 2.7 *Let $\xi \in Max - \triangleleft$. M is semiprime and strongly irreducible if and only if M is a prime ideal.*

Proof: If M is semiprime then $r(M) = M$. Now, $M = r(M) = \bigcup_{x\in X} r(M_x)$ and $r(M_x) = \bigcap_{\alpha\in\Lambda}\{P_\alpha :$ P_α is a prime ideal of ξ_x containing $M_x\}$. Since M is strongly irreducible, $N_\beta \subseteq M_x$ for some $\beta \in \Lambda$. Also $M_x \subseteq N_\beta$. Therefore $M_x = N_\beta$ whence, M is prime. Converse is true since every prime ideal bundle is semiprime and strongly irreducible. □

As an immediate consequence of Theorem 2.6 and Theorem 2.7, we have the following corollary.

Corollary 2.4 *Let $\xi \in Max - \triangleleft$ and M be a semiprime ideal bundle of ξ. Then the following conditions are equivalent:*

1. *M is prime.*
2. *M is strongly irreducible in ξ.*
3. *M is primary.*

3. Prime and primary Lie algebra bundle

Prime rings play an important role in the theory of radicals of rings. Analogously, in the theory of Lie algebras, prime Lie algebras [2] and primary Lie algebras [4] are discussed. In this section, we define and discuss prime Lie algebra bundles and primary Lie algebra bundles.

Definition 3.1 *A Lie algebra bundle ξ is called a **prime Lie algebra bundle** if for nonzero ideal bundles A, B of ξ, $[A, B] \neq 0$.*

Definition 3.2 *A Lie algebra bundle ξ is called a **semiprime Lie algebra bundle** if for nonzero ideal bundle A of ξ, $A^2 \neq 0$.*

We observe that an ideal bundle of ξ is prime (semiprime) if and only if the zero bundle is a prime(semiprime) ideal bundle of ξ.

Theorem 3.1 *A Lie algebra bundle ξ having a non-zero center cannot be a semiprime Lie algebra bundle.*

Proof: Let ξ be a semiprime Lie algebra bundle. $Z(\xi)^2 \subseteq (0)$ and we have $Z(\xi) \subseteq (0)$ which is a contradiction. Therefore ξ cannot be semiprime. □

Theorem 3.2 *A semisimple Lie algebra bundle is a semiprime Lie algebra bundle.*

Proof: Every ideal bundle of a semisimple Lie algebra bundle ξ is semiprime. In particular, (0) is semiprime. Hence ξ is a semiprime Lie algebra bundle. □

Theorem 3.3 *For a proper ideal bundle P of Lie algebra bundle ξ the following conditions are equivalent:*

1. *P is a prime ideal bundle.*
2. *If I and J are ideal bundles of ξ containing P properly then $[I, J] \nsubseteq P$.*
3. *ξ/P is a prime Lie algebra bundle.*

Proof: (1) $\implies$ (2): Let I, J be ideal bundles of ξ, $I \supset P$ and $J \supset P$. If $[I, J] \subseteq P$ then $I \subseteq P$ or $J \subseteq P$ - not possible.

(2) $\implies$ (3): Let I/P, J/P be two ideal bundles of ξ/P and $[I/P, J/P] = P$. Then by (2), $[I, J] \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. Therefore $I/P = P$ or $J/P = P$. Hence ξ/P is a prime Lie algebra bundle.

(3) $\implies$ (1): Let I, J be two ideal bundles of ξ such that $[I, J] \subseteq P$. We have $(I + P)/P$ and $(J + P)/P$ are ideal bundles of ξ/P by the local triviality of ξ/P. $[(I + P)/P, (J + P)/P] = [I + P, J + P] + P$ and $[I + P, J + P] \subseteq P$. Therefore, $[(I + P)/P, (J + P)/P] = P$. Since ξ/P is a prime Lie algebra bundle, $(I + P)/P = P$ or $(J + P)/P = P$ whence $I + P \subseteq P$ or $J + P \subseteq P$. Therefore $I \subseteq P$ or $J \subseteq P$. This proves that P is prime. □

A similar result for primary Lie algebra bundles is as follows.

Theorem 3.4 *For a proper ideal bundle P of Lie algebra bundle ξ the following conditions are equivalent:*

1. *P is a primary ideal bundle.*
2. *If I and J are ideal bundles of ξ containing $r(P)$ properly then $[I, J] \nsubseteq P$.*
3. *ξ/P is a primary Lie algebra bundle.*

Proof: (1) $\implies$ (2): Let I, J be ideal bundles of ξ, $I \supset r(P)$ and $J \supset r(P)$. If $[I, J] \subseteq P$ then by assumption $I \subseteq r(P)$ or $J \subseteq r(P)$ - not possible.

(2) $\implies$ (3): Let I/P, J/P be two ideal bundles of ξ/P and $[I/P, J/P] = P$. Then by (2), $[I, J] \subseteq P$ implies $I \subseteq r(P)$ or $J \subseteq r(P)$. Therefore P is a primary ideal bundle and is the zero bundle of ξ/P. Hence ξ/P is a primary Lie algebra bundle.

(3) $\implies$ (1): Let I, J be two ideal bundles of ξ such that $[I, J] \subseteq P$. We have $(I + P)/P$ and $(J + P)/P$ are ideal bundles of ξ/P by the local triviality of ξ/P and their Lie product is P. Thus, $(I + P)/P \subseteq r(P)$ or $(J + P)/P \subseteq r(P)$. It follows that $I + P \subseteq r(P)$ or $J + P \subseteq r(P)$ or one of I, J is contained in $r(P)$. This proves that P is primary. □

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