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Some results on MKKCR-type coupling on complete metric space

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ABSTRACT: The terms Cyclic Meir-Keeler Kannan-Chatterjea-Reich (MKKCR) type coupling, Cyclic Meir-Keeler Kannan-Reich (MKKR) type coupling, and Cyclic Meir-Keeler Chatterjea-Reich (MKCR) type coupling are defined in this study by fusing the concepts of coupled fixed points, cyclic contractions, and Meir-Keeler mappings. In the context of entire metric space, several results of strongly coupled fixed points are produced for these contraction mappings. We provide an illustration to back up our main finding. It has also been argued how our main finding can be applied to the question of whether a class of nonlinear integral equations exists. Our findings generalize a number of previously published findings on coupled fixed points, particularly findings related to Chatterjea and Kannan type contractions.

Key Words: Coupled Fixed point, strong Coupled Fixed point, cyclic Contraction, best proximity points, cyclic Meir-Keeler Kannan-Chatterjea-Reich (MKKCR) type contraction.

Contents

1. Introduction

The study of fixed point theory has been quite popular due to its prominent role in the study of non linear analysis. Banach Contraction Principle(BCP) was one of the landmark achievement in the field of fixed points.The contraction principle has been generalized in many directions. One of the direction is to study the theory of fixed points in different spaces. BCP was established in metric space. In 2006, Mustafa and Sims [25] defined a generalized metric space and establish fixed point results for contractive mappings in generalized metric space. Using the concept of generalized metric space, Tripathy et.al. [7,8,9] defined generalized fuzzy metric space and establish fixed point results in fuzzy setting.

Another very common direction is modification in the definition of contraction. Some popular extensions in this line are that of Kannan [17], Chatterjea [10] and Reich [21], etc. Another important generalisation of Banach Contraction Principle was given by Meir and Keeler [19] in the year 1969. With the help of weakly uniformly strict contraction, they established a result for existence and uniqueness of fixed point. A mapping $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping if for any $\alpha > 0$, $\exists\ \delta > 0$ such that $\alpha \leq t < \alpha + \delta$ implies $\mu(t) < \alpha, \forall\ t \in \mathbb{R}^+$. It is worth mentioning that in a Meir-Keeler mapping μ, $\mu(t) < t$ for all t.

Fixed point theory within the context of cyclic contraction maps is another significant extension. The concept of cyclic mapping in the field of fixed points was initiated by Kirk et al. [18], in 2003.

The study of Coupled fixed point was initiated by Guo and Laksmikantham [15], in 1987. However, it gain its popularity with the works of Bhaskar and Lakshmitham [6]. Blending it with the concept of cyclic mappings, Choudhury and Maity [12] introduced the notion of couplings and strong coupled fixed point in fixed point theory. They defined cyclic coupled Kannan type contraction and established an existence result for strong coupled fixed point. In 2017, Choudhury et al. [13] used the idea of coupling to extend Banach contraction and Chatterjea type contractions. Henceforth, many authors used the idea of coupling in different directions. Some of the works can be seen [1,2,3,4,5,20,22,26], etc.

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In the mean time, Chen and Kuo [11] established the notions of cyclic Meir-Keeler-Kannan- Chatterjea-Reich contractions and cyclic Meir-Keeler-Kannan-Chatterjea contractive pairs and established some best proximity results for these contractions. Recently, Eshi et al. [14] introduced the concept of cyclic MKKCR contractions and cyclic MKKCR contractive pairs and demonstrated certain fixed point and best proximity point results, based on the work of Chen and Kuo.

In this paper, motivated by the idea of coupling and following Chen and Kuo [11], we establish some existence results for strong coupled fixed point for such contractions in the framework of metric space.

Next, we present some of the preliminary definitions in couplings.

Definition 1.1 *[12] Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of a set $\mathcal{Y}$. Then, a mapping $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ is said to be cyclic mapping or coupling with reference to the sets $\mathcal{U}$ and $\mathcal{V}$ if*

(1) $\mathcal{H}(u,v)\in\mathcal{V}$ when $u\in\mathcal{U}$ and $v\in\mathcal{V}$.

(2) $\mathcal{H}(u,v)\in\mathcal{U}$ when $u\in\mathcal{V}$ and $v\in\mathcal{U}$.

Definition 1.2 *[12] An element $(w,w)\in\mathcal{Y}\times\mathcal{Y}$ is called a strong coupled fixed point of the mapping $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ if $\mathcal{H}(w,w)=w$.*

Definition 1.3 *[12] Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of a metric space $(\mathcal{Y},\mathfrak{d})$. Then, a mapping $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{X}$ is said to be cyclic coupled Kannan type contraction with reference to the sets $\mathcal{U}$ and $\mathcal{V}$ if $\mathcal{H}$ is cyclic with reference to the sets $\mathcal{U}$ and $\mathcal{V}$ such that for some $k\in\left(0,\frac{1}{2}\right)$, the inequality $\mathfrak{d}(\mathcal{H}(a,b),\mathcal{H}(u,v))\le k\left(\mathfrak{d}(a,\mathcal{H}(a,b))+\mathfrak{d}(u,\mathcal{H}(u,v))\right)$ holds, for $b,u\in\mathcal{U}$ and $a,v\in\mathcal{V}$.*

Definition 1.4 *[13] Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of a metric space $(\mathcal{Y},\mathfrak{d})$. Then, a mapping $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ is said to be Banach type coupling with reference to the sets $\mathcal{U}$ and $\mathcal{V}$ if $\mathcal{H}$ is cyclic with reference to the sets $\mathcal{U}$ and $\mathcal{V}$ such that for some $t\in(0,1)$, the inequality $\mathfrak{d}(\mathcal{H}(a,b),\mathcal{H}(u,v))\le \frac{t}{2}\left(\mathfrak{d}(a,u)+\mathfrak{d}(b,v)\right)$ holds, for $b,u\in\mathcal{U}$ and $a,v\in\mathcal{V}$.*

Definition 1.5 *[13] Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of a metric space $(\mathcal{Y},\mathfrak{d})$. Then, a mapping $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ is said to be Chatterjea type coupling with reference to the sets $\mathcal{U}$ and $\mathcal{V}$ if $\mathcal{H}$ is cyclic with reference to the sets $\mathcal{U}$ and $\mathcal{V}$ such that for some $t\in(0,1/2)$, the inequality $\mathfrak{d}(\mathcal{H}(a,b),\mathcal{H}(u,v))\le t\left(\mathfrak{d}(a,\mathcal{H}(u,v))+\mathfrak{d}(u,\mathcal{H}(a,b))\right)$ holds, for $b,u\in\mathcal{U}$ and $a,v\in\mathcal{V}$,*

2. Main Result

Throughout the paper, we use the usual notation $\mathbb{N}_n=\{1,2,...,n\}$. Using the notions of coupled fixed points, cyclic contractions and Meir-Keeler mapping, MKKCR type coupling can be defined in the following manner.

Definition 2.1 *Let $\mathcal{U}$ and $\mathcal{V}$ are two non-empty subsets of a metric space $(\mathcal{Y},\mathfrak{d})$, and $\mu:\mathbb{R}^+\to\mathbb{R}^+$ is a linear Meir-Keeler mapping. Then, the mapping $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ is said to be MKKCR type coupling, if*

(1) $\mathcal{H}$ is cyclic with reference to the sets $\mathcal{U}$ and $\mathcal{V}$.

(2) for $b,u\in\mathcal{U},a,v\in\mathcal{V}$ and $l\in\mathbb{N}_n\cup\{0\}$,
$$\mathfrak{d}(\mathcal{H}(a,b),\mathcal{H}(u,v))\le\mu\left(\frac{\mathfrak{d}(a,\mathcal{H}(a,b))+\mathfrak{d}(u,\mathcal{H}(u,v))+\mathfrak{d}(a,\mathcal{H}(u,v))+\mathfrak{d}(u,\mathcal{H}(a,b))+4l\mathfrak{d}(a,u)}{4(l+1)}\right).$$

Lemma 2.1 *Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty closed subsets of a complete metric space $(\mathcal{Y},\mathfrak{d})$. Let $\mu:\mathbb{R}^+\to\mathbb{R}^+$ be an increasing linear Meir-Keeler mapping, and $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ be a MKKCR type coupling. Then, for any $u_0\in\mathcal{U}$ and $v_0\in\mathcal{V}$ such that $u_{m+1}=\mathcal{H}(v_m,u_m)$ and $v_{m+1}=\mathcal{H}(u_m,v_m)$, $(\mathfrak{d}(u_{m+1},v_{m+2})+\mathfrak{d}(v_{m+1},u_{m+2}))\to 0$.*

Proof: Let $u_0 \in \mathcal{U}$ and $v_0 \in \mathcal{V}$ such that $u_{m+1} = \mathcal{H}(v_m, u_m)$ and $v_{m+1} = \mathcal{H}(u_m, v_m)$. Then, $u_m \in \mathcal{U}$ and $v_m \in \mathcal{V}$ for all $m \geq 0$.
Now,

$$\begin{aligned}
&\mathfrak{d}(u_{m+1}, v_{m+2})\\
&= \mathfrak{d}(\mathcal{H}(v_m, u_m), \mathcal{H}(u_{m+1}, v_{m+1})\\
&\leq \mu\Big(\frac{1}{4(l+1)}\big(\mathfrak{d}(v_m, \mathcal{H}(v_m, u_m)) + \mathfrak{d}(u_{m+1}, \mathcal{H}(u_{m+1}, v_{m+1})) + \mathfrak{d}(v_m, \mathcal{H}(u_{m+1}, v_{m+1}))\\
&+ \mathfrak{d}(u_{m+1}, \mathcal{H}(v_m, u_m)) + 4l\mathfrak{d}(v_m, u_{m+1})\big)\Big)\\
&\leq \mu\left(\frac{\mathfrak{d}(v_m, u_{m+1}) + \mathfrak{d}(u_{m+1}, v_{m+2}) + \mathfrak{d}(v_m, v_{m+2}) + \mathfrak{d}(u_{m+1}, u_{m+1}) + 4l\mathfrak{d}(v_m, u_{m+1})}{4(l+1)}\right)\\
&\leq \mu\left(\frac{2\mathfrak{d}(u_{m+1}, v_{m+2}) + (4l+2)\mathfrak{d}(v_m, u_{m+1})}{4(l+1)}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathfrak{d}(u_{m+1}, v_{m+2}) &< \frac{2\mathfrak{d}(u_{m+1}, v_{m+2}) + (4l+2)\mathfrak{d}(v_m, u_{m+1})}{4(l+1)}\\
\mathfrak{d}(u_{m+1}, v_{m+2}) &< \mathfrak{d}(v_m, u_{m+1}).
\end{aligned}$$

Also,

$$\begin{aligned}
&\mathfrak{d}(v_{m+1}, u_{m+2})\\
&= \mathfrak{d}(\mathcal{H}(u_m, v_m), \mathcal{H}(v_{m+1}, u_{m+1})\\
&\leq \mu\Big(\frac{1}{4(l+1)}\big(\mathfrak{d}(u_m, \mathcal{H}(u_m, v_m)) + \mathfrak{d}(v_{m+1}, \mathcal{H}(v_{m+1}, u_{m+1})) + \mathfrak{d}(u_m, \mathcal{H}(v_{m+1}, u_{m+1}))\\
&+ \mathfrak{d}(v_{m+1}, \mathcal{H}(u_m, v_m)) + 4l\mathfrak{d}(u_m, v_{m+1})\big)\Big)\\
&\leq \mu\left(\frac{\mathfrak{d}(u_m, v_{m+1}) + \mathfrak{d}(v_{m+1}, u_{m+2}) + \mathfrak{d}(u_m, u_{m+2}) + \mathfrak{d}(v_{m+1}, v_{m+1}) + 4l\mathfrak{d}(u_m, v_{m+1})}{4(l+1)}\right)\\
&\leq \mu\left(\frac{2\mathfrak{d}(v_{m+1}, u_{m+2}) + (4l+2)\mathfrak{d}(u_m, v_{m+1})}{4(l+1)}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathfrak{d}(v_{m+1}, u_{m+2}) &< \frac{2\mathfrak{d}(v_{m+1}, u_{m+2}) + (4l+2)\mathfrak{d}(u_m, v_{m+1})}{4(l+1)}\\
\mathfrak{d}(v_{m+1}, u_{m+2}) &< \mathfrak{d}(u_m, v_{m+1}).
\end{aligned}$$

Hence, $\mathfrak{d}(u_{m+1}, v_{m+2}) + \mathfrak{d}(v_{m+1}, u_{m+2}) < \mathfrak{d}(v_m, u_{m+1}) + \mathfrak{d}(u_m, v_{m+1})$. So, $(\mathfrak{d}(u_{m+1}, v_{m+2}) + \mathfrak{d}(v_{m+1}, u_{m+2}))$ is a decreasing sequence which is bounded below. This implies that there exists an $\alpha \geq 0$ such that $\lim_{m\to\infty} (\mathfrak{d}(u_{m+1}, v_{m+2}) + \mathfrak{d}(v_{m+1}, u_{m+2})) = \alpha$. Here, $\alpha = \inf\{\mathfrak{d}(u_{m+1}, v_{m+2}) + \mathfrak{d}(v_{m+1}, u_{m+2}) : m \in \mathbb{N} \cup \{0\}\}$.
We claim: $\alpha = 0$. On the contrary, let $\alpha > 0$. Then, there exists a $\delta > 0$ and a natural number n_0 such that

$$\begin{aligned}
&\alpha \leq \mathfrak{d}(u_n, v_{n+1}) + \mathfrak{d}(v_n, u_{n+1}) \leq \alpha + \delta, \ \forall n \geq n_0\\
\Longrightarrow &\mu(\mathfrak{d}(u_n, v_{n+1}) + \mathfrak{d}(v_n, u_{n+1})) < \alpha.
\end{aligned}$$

Now,

$$
\begin{aligned}
&\mathfrak{d}(u_{n+1}, v_{n+2}) + \mathfrak{d}(v_{n+1}, u_{n+2}) \\
&= \mathfrak{d}(\mathcal{H}(v_n, u_n), \mathcal{H}(u_{n+1}, v_{n+1}) + \mathfrak{d}(\mathcal{H}(u_n, v_n), \mathcal{H}(v_{n+1}, u_{n+1}) \\
&\leq \mu\left(\frac{\mathfrak{d}(v_n, \mathcal{H}(v_n, u_n)) + \mathfrak{d}(u_{n+1}, \mathcal{H}(u_{n+1}, v_{n+1})) + \mathfrak{d}(v_n, \mathcal{H}(u_{n+1}, v_{n+1})) + \mathfrak{d}(u_{n+1}, \mathcal{H}(v_n, u_n)) + 4l\mathfrak{d}(v_n, u_{n+1})}{4(l+1)}\right) \\
&+ \mu\left(\frac{\mathfrak{d}(u_n, \mathcal{H}(u_n, v_n)) + \mathfrak{d}(v_{n+1}, \mathcal{H}(v_{n+1}, u_{n+1})) + \mathfrak{d}(u_n, \mathcal{H}(v_{n+1}, u_{n+1})) + \mathfrak{d}(v_{n+1}, \mathcal{H}(u_n, v_n)) + 4l\mathfrak{d}(u_n, v_{n+1})}{4(l+1)}\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(v_n, u_{n+1}) + \mathfrak{d}(u_{n+1}, v_{n+2}) + \mathfrak{d}(v_n, v_{n+2}) + \mathfrak{d}(u_{n+1}, u_{n+1}) + 4l\mathfrak{d}(v_n, u_{n+1})}{4(l+1)}\right) \\
&+ \mu\left(\frac{\mathfrak{d}(u_n, v_{n+1}) + \mathfrak{d}(v_{n+1}, u_{n+2}) + \mathfrak{d}(u_n, u_{n+2}) + \mathfrak{d}(v_{n+1}, v_{n+1}) + 4l\mathfrak{d}(u_n, v_{n+1})}{4(l+1)}\right) \\
&\leq \mu\left(\frac{4(l+1)\mathfrak{d}(v_n, u_{n+1})}{4(l+1)}\right) + \mu\left(\frac{4(l+1)\mathfrak{d}(u_n, v_{n+1})}{4(l+1)}\right) \\
&\leq \mu(\mathfrak{d}(v_n, u_{n+1})) + \mu(\mathfrak{d}(u_n, v_{n+1})) \\
&\leq \mu(\mathfrak{d}(v_n, u_{n+1}) + \mathfrak{d}(u_n, v_{n+1})) \\
&\leq \alpha,
\end{aligned}
$$

which leads to a contradiction. Hence, $(\mathfrak{d}(u_{m+1}, v_{m+2}) + \mathfrak{d}(v_{m+1}, u_{m+2})) \to 0$, as $m \to \infty$. □

Lemma 2.2 *Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty closed subsets of a complete metric space $(\mathcal{Y}, \mathfrak{d})$. Let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing linear Meir-Keeler mapping, and $\mathcal{H} : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ be a MKKCR type coupling and $\mathcal{U} \cap \mathcal{V} \neq \emptyset$. Then, for any $u_0 \in \mathcal{U}$ and $v_0 \in \mathcal{V}$ such that $u_{m+1} = \mathcal{H}(v_m, u_m)$ and $v_{m+1} = \mathcal{H}(u_m, v_m)$, the sequences (u_m) and (v_m) are cauchy and converges to the same limit.*

Proof: Taking $u_0 \in \mathcal{U}$ and $v_0 \in \mathcal{V}$ such that $u_{m+1} = \mathcal{H}(v_m, u_m)$ and $v_{m+1} = \mathcal{H}(u_m, v_m)$, we have, $u_m \in \mathcal{U}$ and $v_m \in \mathcal{V}$ for all $m \geq 0$

$$
\begin{aligned}
&\mathfrak{d}(v_{m+1}, u_{m+1}) \\
&= \mathfrak{d}(\mathcal{H}(u_m, v_m), \mathcal{H}(v_m, u_m)) \\
&\leq \mu\left(\frac{\mathfrak{d}(u_m, \mathcal{H}(u_m, v_m)) + \mathfrak{d}(v_m, \mathcal{H}(v_m, u_m)) + \mathfrak{d}(u_m, \mathcal{H}(v_m, u_m)) + \mathfrak{d}(v_m, \mathcal{H}(u_m, v_m)) + 4l\mathfrak{d}(u_m, v_m)}{4(l+1)}\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(u_m, v_{m+1}) + \mathfrak{d}(v_m, u_{m+1}) + \mathfrak{d}(u_m, u_{m+1}) + \mathfrak{d}(v_m, v_{m+1}) + 4l\mathfrak{d}(u_m, v_m)}{4(l+1)}\right) \\
&\leq \mu(\frac{1}{4(l+1)}(\mathfrak{d}(u_m, v_{m+1}) + \mathfrak{d}(v_m, u_{m+1}) + \mathfrak{d}(u_m, v_{m+1}) + \mathfrak{d}(v_{m+1}, u_{m+1}) \\
&+ \mathfrak{d}(v_m, u_{m+1}) + \mathfrak{d}(u_{m+1}, v_{m+1}) + 4l(\mathfrak{d}(u_m, v_{m+1}) + \mathfrak{d}(v_{m+1}, u_{m+1}) + \mathfrak{d}(u_{m+1}, v_m)))) \\
&\leq \mu\left(\frac{(4l+2)(\mathfrak{d}(u_m, v_{m+1}) + \mathfrak{d}(v_m, u_{m+1})) + (4l+2)\mathfrak{d}(v_{m+1}, u_{m+1})}{4(l+1)}\right) \\
&< \frac{(4l+2)(\mathfrak{d}(u_m, v_{m+1}) + \mathfrak{d}(v_m, u_{m+1})) + (4l+2)\mathfrak{d}(v_{m+1}, u_{m+1})}{4(l+1)}.
\end{aligned}
$$

So, $\mathfrak{d}(v_{m+1}, u_{m+1}) < (4l+2)(\mathfrak{d}(u_m, v_{m+1}) + \mathfrak{d}(v_m, u_{m+1}))$. Thus, in view of Lemma 2.1, the sequence $(\mathfrak{d}(v_{m+1}, u_{m+1})) \to 0$ as $m \to \infty$. Clearly, for any $p, q \in \mathbb{N}$ such that $p < q$,
$d(u_p, u_q) \to 0$ and $d(v_p, v_q) \to 0$ as $p, q \to \infty$.
Hence, (u_m) and (v_m) are cauchy sequences in closed sets $\mathcal{U}$ and $\mathcal{V}$, respectively. Consequently, (u_m) converges to some $u \in \mathcal{U}$ and (v_m) converges to some $v \in \mathcal{V}$. Further, as $\lim\limits_{m\to\infty} \mathfrak{d}(u_m, v_m) \to 0$, we get $u = v$. Thus, (u_m) and (v_m) converges to the same limit $u \in \mathcal{U} \cap \mathcal{V}$. □

Theorem 2.1 *Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{Y}, \mathfrak{d})$ such that $\mathcal{U} \cap \mathcal{V} \neq \emptyset$. Let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing linear Meir-Keeler mapping. Then, any MKKCR type coupling $\mathcal{H} : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ has a unique strong coupled fixed point.*

Proof: Let $u_0 \in \mathcal{U}$ and $v_0 \in \mathcal{V}$ such that $u_{m+1} = \mathcal{H}(v_m, u_m)$ and $v_{m+1} = \mathcal{H}(u_m, v_m)$. Then, by Lemma 2.2, (u_m) and (v_m) are Cauchy sequences converging to the same limit, say $z \in \mathcal{U} \cap \mathcal{V}$. Since, $\mathcal{H}$ is MKKCR type coupling,

$$\begin{aligned}
&\mathfrak{d}(u_{m+1}, \mathcal{H}(z,z)) \\
&= \mathfrak{d}(\mathcal{H}(v_m, u_m), \mathcal{H}(z,z)) \\
&\leq \mu\Big(\frac{\mathfrak{d}(v_m, \mathcal{H}(v_m, u_m)) + \mathfrak{d}(z, \mathcal{H}(z,z)) + \mathfrak{d}(v_m, \mathcal{H}(z,z)) + \mathfrak{d}(z, \mathcal{H}(v_m, u_m)) + 4l\mathfrak{d}(v_m, z)}{4(l+1)}\Big) \\
&\leq \mu\Big(\frac{\mathfrak{d}(v_m, u_{m+1}) + \mathfrak{d}(z, \mathcal{H}(z,z)) + \mathfrak{d}(v_m, \mathcal{H}(z,z)) + \mathfrak{d}(z, u_{m+1}) + 4l\mathfrak{d}(v_m, z)}{4(l+1)}\Big) \\
&< \frac{\mathfrak{d}(v_m, u_{m+1}) + \mathfrak{d}(z, \mathcal{H}(z,z)) + \mathfrak{d}(v_m, \mathcal{H}(z,z)) + \mathfrak{d}(z, u_{m+1}) + 4l\mathfrak{d}(v_m, z)}{4(l+1)}.
\end{aligned}$$

As $m \to \infty$,

$$\begin{aligned}
&\mathfrak{d}(z, \mathcal{H}(z,z)) < \frac{2\mathfrak{d}(z, \mathcal{H}(z,z))}{4(l+1)} \\
\Longrightarrow &\mathfrak{d}(z, \mathcal{H}(z,z)) = 0 \\
\Longrightarrow &z = \mathcal{H}(z,z).
\end{aligned}$$

Thus, z is a strong coupled fixed point of $\mathcal{H}$.
Next, to prove uniqueness, consider another strong coupled fixed point of $\mathcal{H}$, say ν. That is, $\mathcal{H}(\nu, \nu) = \nu$. Then,

$$\begin{aligned}
&\mathfrak{d}(z, \nu) = \mathfrak{d}(\mathcal{H}(z,z), \mathcal{H}(\nu, \nu)) \\
&\leq \mu\left(\frac{\mathfrak{d}(z, \mathcal{H}(z,z)) + \mathfrak{d}(\nu, \mathcal{H}(\nu,\nu)) + \mathfrak{d}(z, \mathcal{H}(\nu,\nu)) + \mathfrak{d}(\nu, \mathcal{H}(z,z)) + 4l\mathfrak{d}(z,\nu)}{4(l+1)}\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(z,z) + \mathfrak{d}(\nu,\nu) + \mathfrak{d}(z,\nu) + \mathfrak{d}(\nu,z) + 4l\mathfrak{d}(z,\nu)}{4(l+1)}\right) \\
&\leq \mu\left(\frac{(4l+2)\mathfrak{d}(z,\nu)}{4(l+1)}\right) \\
&< \frac{(4l+2)\mathfrak{d}(z,\nu)}{4(l+1)}.
\end{aligned}$$

This implies, $\mathfrak{d}(z, \nu) = 0$, which in turn gives $z = \nu$. Hence, z is the unique strong coupled fixed point of $\mathcal{H}$. □

Example 2.1 Consider $\mathcal{Y} = [-1, 1]$ along with the metric $\mathfrak{d}$ defined by $\mathfrak{d}(u, v) = |u - v|$. Let $\mathcal{U} = [-1, 0]$ and $\mathcal{V} = [0, 1]$ and a mapping $\mathcal{H} : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ be defined by

$$\mathcal{H}(u,v) = \begin{cases} \frac{-u}{3}, & \text{if } (u,v) \in \mathcal{U} \times \mathcal{V} \\ \frac{-v}{3}, & \text{if } (v,u) \in \mathcal{U} \times \mathcal{V}. \end{cases}$$

Also let $\mu(t) = kt$, where $k \in [1/3, 1)$. Then, μ is a linear increasing meir Keeler mapping and $\mathcal{H}$ is cyclic with reference to the sets $\mathcal{U}$ and $\mathcal{V}$. Now, let us verify if $\mathcal{H}$ satisfies the following condition:

For $b, u \in \mathcal{U}, a, v \in \mathcal{V}$ *and* $l \in \mathbb{N}_n \cup \{0\}$,

$$\mathfrak{d}(\mathcal{H}(a,b), \mathcal{H}(u,v)) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}(a,b)) + \mathfrak{d}(u, \mathcal{H}(u,v)) + \mathfrak{d}(a, \mathcal{H}(u,v)) + \mathfrak{d}(u, \mathcal{H}(a,b)) + 4l\mathfrak{d}(a,u)}{4(l+1)}\right). \quad (2.1)$$

So,

$$\mathfrak{d}(\mathcal{H}(a,b),\mathcal{H}(u,v)) = \mathfrak{d}\Big(\frac{-a}{3},\frac{-u}{3}\Big) = \Big|-\frac{a}{3}+\frac{u}{3}\Big| = \frac{1}{3}|u-a|. \tag{2.2}$$

Also,

$$\begin{aligned}
&\mu\left(\frac{\mathfrak{d}(a,\mathcal{H}(a,b))+\mathfrak{d}(u,\mathcal{H}(u,v))+\mathfrak{d}(a,\mathcal{H}(u,v))+\mathfrak{d}(u,\mathcal{H}(a,b))+4l\mathfrak{d}(a,u)}{4(l+1)}\right)\\
&= k\left(\frac{\mathfrak{d}(a,\frac{-a}{3})+\mathfrak{d}(u,\frac{-u}{3})+\mathfrak{d}(a,\frac{-u}{3})+\mathfrak{d}(u,\frac{-a}{3})+4l\mathfrak{d}(a,u)}{4(l+1)}\right)\\
&= k\left(\frac{|a+\frac{a}{3}|+|u+\frac{u}{3}|+|a+\frac{u}{3}|+|u+\frac{a}{3}|+4l|a-u|}{4(l+1)}\right)\\
&\le k\left(\frac{\frac{8}{3}\Big(|a|+|u|\Big)+(4l+2)|a-u|}{4(l+1)}\right).
\end{aligned} \tag{2.3}$$

From Equations (2.2) and (2.3), it can easily be seen that the inequality (2.1) holds for $k \in [1/3, 1)$. Thus, the necessary conditions of Theorem 2.1 are all being satisfied. Hence, $\mathcal{H}$ has a unique strong coupled fixed point. In fact, $(0,0)$ is the unique strong coupled fixed point.

Definition 2.2 *If $\mathcal{U}$ and $\mathcal{V}$ are two non-empty subsets of a metric space $(\mathcal{Y},\mathfrak{d})$, and $\mu:\mathbb{R}^+\to\mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mapping $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ is said to be MKKR type coupling, if*

(1) $\mathcal{H}$ is cyclic with reference to the sets $\mathcal{U}$ and $\mathcal{V}$.

(2) for $b,u\in\mathcal{U}$, $a,v\in\mathcal{V}$ and $l\in\mathbb{N}_n\cup\{0\}$,
$$\mathfrak{d}(\mathcal{H}(a,b),\mathcal{H}(u,v)) \le \mu\left(\frac{\mathfrak{d}(a,\mathcal{H}(a,b))+\mathfrak{d}(u,\mathcal{H}(u,v))+2l\mathfrak{d}(a,u)}{2(l+1)}\right).$$

Definition 2.3 *If $\mathcal{U}$ and $\mathcal{V}$ are two non-empty subsets of a metric space $(\mathcal{Y},\mathfrak{d})$, and $\mu:\mathbb{R}^+\to\mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mapping $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ is said to be MKCR type coupling, if*

(1) $\mathcal{H}$ is cyclic with reference to the sets $\mathcal{U}$ and $\mathcal{V}$.

(2) for $b,u\in\mathcal{U}, a,v\in\mathcal{V}$ and $l\in\mathbb{N}_n\cup\{0\}$,
$$\mathfrak{d}(\mathcal{H}(a,b),\mathcal{H}(u,v)) \le \mu\left(\frac{\mathfrak{d}(a,\mathcal{H}(u,v))+\mathfrak{d}(u,\mathcal{H}(a,b))+2l\mathfrak{d}(a,u)}{2(l+1)}\right).$$

Correponding to the above two definitions, we have the following two results.

Theorem 2.2 *Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{Y},\mathfrak{d})$ such that $\mathcal{U}\cap\mathcal{V}\neq\emptyset$. Let $\mu:\mathbb{R}^+\to\mathbb{R}^+$ be an increasing Meir-Keeler mapping. Then, any MKKR type coupling $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ has a unique strong coupled fixed point.*

Theorem 2.3 *Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{Y},\mathfrak{d})$ such that $\mathcal{U}\cap\mathcal{V}\neq\emptyset$. Let $\mu:\mathbb{R}^+\to\mathbb{R}^+$ be an increasing Meir-Keeler mapping. Then, any MKCR type coupling $\mathcal{H}:\mathcal{Y}\times\mathcal{Y}\to\mathcal{Y}$ has a unique strong coupled fixed point.*

3. Application

In this section, we discuss the existence of a unique solution for a given class of non linear integral equations.
Consider the nonlinear integral equations

$$\begin{aligned} a(t) &= \int_0^r h(t, a(s), b(s))\, ds, \;\; t \in [0, r] \\ b(t) &= \int_0^r h(t, b(s), a(s))\, ds, \;\; t \in [0, r], \end{aligned} \tag{3.1}$$

where r is a real number in $(0, \infty)$ and $h : [0, r] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
Let $\mathcal{Y} = C([0, r], \mathbb{R})$ denotes the set of all continuous functions defined on $[0, r]$ and is endowed with a metric $\mathfrak{d} : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ defined by

$$\mathfrak{d}(a, b) = \sup_{t \in [0,r]} |a(t) - b(t)|, \; \forall\; a, b \in \mathcal{Y}.$$

Clearly, $(\mathcal{Y}, \mathfrak{d})$ is complete.

Theorem 3.1 *Suppose that for the given system (3.1), the following holds:*

(1) the mapping $h : [0, r] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(2) there exists closed subsets $\mathcal{U}$ and $\mathcal{V}$ with $\mathcal{U} \cap \mathcal{V} \neq \emptyset$ and for $t \in [0, r]$, $b, u \in \mathcal{U}$, $a, v \in \mathcal{V}$ and $l \in \mathbb{N}_n \cup \{0\}$,
$|h(t, a(s), b(s)) - h(t, u(s), v(s))| \leq \frac{1}{r}\mu\left(\dfrac{l(a(s) - u(s))}{(l + 1)}\right)$,
where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing linear Meir-Keeler mapping.

Then, the system (3.1) has a unique solution in $C([0, r], \mathbb{R})$.

Proof: Define a mapping $\mathcal{H} : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ by

$$\mathcal{H}(a, b)(t) = \int_0^r h(t, a(s), b(s))\, ds, \;\; t \in [0, r], a, b \in \mathcal{Y}.$$

Then, for any $b(t) \in \mathcal{U}$ and $a(t) \in \mathcal{V}$, the system (3.1) implies

$$\begin{aligned} \mathcal{H}(a, b)(t) &= \int_0^r h(t, a(s), b(s))\, ds \\ &= a(t) \in \mathcal{V} \\ \text{and } \mathcal{H}(b, a)(t) &= \int_0^r h(t, b(s), a(s))\, ds \\ &= b(t) \in \mathcal{U}. \end{aligned}$$

Thus, the mapping $\mathcal{H}$ is cyclic with reference to the sets $\mathcal{U}$ and $\mathcal{V}$.
Next, we show that $\mathcal{H}$ is MKKCR type coupling.

Consider $b, u \in \mathcal{U}$ and $a, v \in \mathcal{V}$. Then,

$$\begin{aligned}
&|\mathcal{H}(a,b)(t) - \mathcal{H}(u,v)(t)| \qquad (3.2)\\
&= |\int_0^r h(t,a(s),b(s))ds - \int_0^r h(t,u(s),v(s))ds|\\
&= |\int_0^r [h(t,a(s),b(s))ds - h(t,u(s),v(s))]ds|\\
&\leq \frac{1}{r}\int_0^r \mu(\frac{l(a(s)-u(s))}{l+1})ds\\
&= \frac{1}{r}\int_0^r \mu\left(\frac{1}{4(l+1)}(4l(a(s)-u(s)) + a(s) - \mathcal{H}(a,b)(s) + u(s) - \mathcal{H}(u,v)(s)\right.\\
&\left. + \mathcal{H}(a,b)(s) - u(s) + \mathcal{H}(u,v)(s) - a(s))\right) ds\\
&\leq \frac{1}{r}\int_0^r \mu\left(\frac{1}{4(l+1)}\left(4l \sup_{p\in[0,r]}|a(p)-u(p)| + \sup_{p\in[0,r]}|a(p)-\mathcal{H}(a,b)(p)| + \sup_{p\in[0,r]}|u(p)-\mathcal{H}(u,v)(p)|\right.\right.\\
&\left.\left. + \sup_{p\in[0,r]}|u(p)-\mathcal{H}(a,b)(p)| + \sup_{p\in[0,r]}|a(p)-\mathcal{H}(u,v)(p)|\right)\right) ds\\
&\leq \mu\left(\frac{4l\mathfrak{d}(a,u) + \mathfrak{d}(a,\mathcal{H}(a,b)) + \mathfrak{d}(u,\mathcal{H}(u,v)) + \mathfrak{d}(u,\mathcal{H}(a,b)) + \mathfrak{d}(a,\mathcal{H}(u,v))}{4(l+1)}\right).
\end{aligned}$$

This implies

$$\sup_{t\in[0,r]}|\mathcal{H}(a,b)(t) - \mathcal{H}(u,v)(t)| \leq \mu\left(\frac{4l\mathfrak{d}(a,u) + \mathfrak{d}(a,\mathcal{H}(a,b)) + \mathfrak{d}(u,\mathcal{H}(u,v)) + \mathfrak{d}(u,\mathcal{H}(a,b)) + \mathfrak{d}(a,\mathcal{H}(u,v))}{4(l+1)}\right).$$

That is,

$$\mathfrak{d}(\mathcal{H}(a,b) - \mathcal{H}(u,v)) \leq \mu\left(\frac{4l\mathfrak{d}(a,u) + \mathfrak{d}(a,\mathcal{H}(a,b)) + \mathfrak{d}(u,\mathcal{H}(u,v)) + \mathfrak{d}(u,\mathcal{H}(a,b)) + \mathfrak{d}(a,\mathcal{H}(u,v))}{4(l+1)}\right).$$

Thus, the mapping $\mathcal{H}$ is a MKKCR type coupling. As the necessary conditions of Theorem 2.1 are all being fulfilled. So, $\mathcal{H}$ has a unique strong coupled fixed point, say (z,z), in $\mathcal{U} \cap \mathcal{V}$. That is, $\mathcal{H}(z,z) = z$. Thus, (z,z) is the unique solution of the given nonlinear integral equations (3.1). □

4. Conclusion

Couplings of MKKCR type, MKKR type and MKCR type have been defined and results for existence and uniqueness of strong coupled fixed point have been established. An application of our main result with regard to the existence of solution for a class of non linear integral equations has also been discussed. Our results also generalize many existing coupled fixed point results. In particular, for $l = 0$ and $\mu(t) = kt$ with $k \in [0,1)$, Definitions 2.2 and 2.3 reduces to cyclic coupled Kannan type contraction [12] and Chatterjea type coupling [13], respectively.

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