Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–24.
ISSN-0037-8712
doi:10.5269/bspm.69390

A Novel Approach to AG-groupoids via Soft Intersection Product Operation

Bashar Khan, Imtiaz Ahmad, Aman Ullah, Faruk Karaaslan* and Mohammad Rashad

ABSTRACT: This study investigates some general properties of soft sets on a symmetry-preserving structure AG-groupoid. Various weaknesses in the papers Karaaslan [22] and Sezgin [39] are rectified, and some generalizations and simplifications. Also, it is investigated that the family of soft sets over AG-groupoid is again an AG-groupoid when equipped with the soft intersection-product (SI-product) operation. Moreover, it is proven that the family of soft sets over an AG-groupoid possesses the properties of being medial, paramedial, Bol*, and nuclear square under the SI-product operation.The application of the soft set theory, specifically the SI-product, is extended to AG-groupoids. Some properties of left abelian distributive (LAD) AG-groupoids (LAD-AG-groupoids) and right abelian distributive (RAD) AG-groupoids (RAD-AG-groupoids) under the SI-product of soft sets are investigated. Counterexamples to show that the family of soft sets on a set S ($S(S)$) is not soft intersection-left abelian distributive (SI-LAD) and soft intersection-right abelian distributive (SI-RAD) are given. Also, the conditions under which they become SI-LAD and SI-RAD are investigated. Some of the examples in this article were obtained using modern techniques such as Mace4 and Prover9.

Key Words: Soft sets, AG-groupoids, Soft intersection product, SI-product.

Contents

1 Introduction **1**

2 Preliminaries **3**

3 Some General Properties of AG-groupoid based on SI-product **4**

4 Properties of abelian distributive AG-groupoids via the product of soft sets **7**

5 Properties of Right Abelian Distributive AG-groupoid **15**

6 Conclusion **22**

1. Introduction

Researchers strive to resolve complex problems involving uncertain data in various disciplines like engineering, medical science, and economics. The existing methodologies have severe problems in dealing effectively with the problems of uncertainty. In this connection, the researchers introduced different ideas, such as the fuzzy set theory [47] and intuitionistic fuzzy sets [8] as alternative effective tools. Molodtsov [30] revealed that both of these concepts have their own complications and limitations and consequently, he initiated a new approach in 1999 to deal these uncertainty of fuzzy subsets of Zadeh [47] and the intuitionistic fuzzy sets of Atanassov [8]. This new approach of soft set theory by Molodtsov is expected to have comparatively limited challenges that are faced by the existing theories. The theory by Molodtsov does not require any approximation of cumbersome membership function as was required in fuzzy set theory and thus is considered appropriate to deal various problems of different fields. A new era began with the effective use of soft sets to other fields, Maji et al. [26] gave an application of soft sets in decision making and they also introduced new operations in [27] that extensively attracted the researchers. As a result, various achievements on soft sets can be observed in a variety of papers like [5,12,14,16,37]. Interestingly, the theory is extended to other algebraic structures in 2007 by Aktaş and Çağman [6]. After that, an alternative study for other algebraic structures via soft sets increased rapidly and thus made a revolution in the field of mathematics to put a hand on abstract algebra. Çağman et al. [13] in 2010, launched the concept of soft-int groups while extending the notion of Rosenfeld [35].

* Corresponding author

Submitted August 24, 2023. Published July 11, 2025
2010 *Mathematics Subject Classification:* 03E20,03E72,06D72

Aman et al. [46] extended the studies of soft-int groups to AG-groupids, while Kaygısız [23], Sezgin and Atagün [38] improved some mentioned appealing cases of soft-groups of Aktaş and Çağman. Zhan et al. [48] presented the application of soft union sets to hemirings via SU-h-ideals. Sezgin et al. [40] launched the concept of soft intersection product (SI-product) operation [39] and investigated SI-product for AG-groupoids. Similarly, Ullah et al. [46] introduced the notions of soft intersection for AG-groups. Also, Amanullah et al. [45] defined the soft uni-AG groups and derived some of their properties. Sezgin et al. [41] defined soft union group and derived its basic properties and the relation with the notion of soft intersection group. Acar et al. [2] defined the soft ring structure.

Mahmood et al. [29] introduced the concept of lattice ordered soft near rings and investigated different properties of lattice ordered soft near rings by using some operations of soft sets. Çítak [15] gave the definition of soft uni-k-ideal of a semiring by using the union operation of soft sets and investigated some algebraic applications by using soft uni-k-ideal. Jana et al. [20] introduced a new kind of soft ring structure called (α, β) -soft-intersectional ring based on some results of soft sets and intersection operations on sets. Goldar and Ray [17] presented a new definition of soft rings and soft ideals using soft elements. John and Susha [21] introduced some algebraic soft operations in terms of soft elements and study their properties. They defined the generalized soft group, its subgroup and investigated its properties. Şahin and Uluçay [36] defined the concepts of the soft proper ideal, soft maximal ideal. Jan et al. [19] discussed the notion of double framed soft and derived the basic properties of double framed soft rings. Ayub et al. [10] defined the some new types of soft roughs sets in groups based on normal soft groups. Aygün and Kamacı [9] defined some new algebraic structures of soft sets based on the XOR and XNOR products of soft sets. Öztunç et al. [32] constructed the category of soft groups and soft group homomorphisms. They also show that this structure satisfies the category conditions. Zhu and Lv [50] investigated the relationship among rough sets, soft sets and rings. Atagün and Kamacı [7] studied on some essential properties of α-intersection of soft set. Abdunabi and Shliteite [1] introduced the rough prime soft ideal and maximal soft Ideal and they studied on some of the properties of these approximations. Maheswari at al. [28] proved that the association, the associated soft subset for a subset of a semigroup, is substructure preserving. Barzegar et al. [11] defined the second type nilpotent soft subgroup and nilpotent soft group.

An AG-groupoid is a non-associative generalized structure of the commutative semigroups satisfying the left invertive law (LIL) $(xy)z = (zy)x$. For practical example of AG-groupoid, consider the rotation transformations of a square [49].

A square is rotated clockwise through 0, 90, 180, and 270 degrees, denoted by $\psi e, \psi a, \psi b$, and ψc respectively. We denote the set of these rotations by $N = \{\psi e, \psi a, \psi b, \psi c\}$. Obviously, the two consecutive rotations have the following results: $\psi e \psi e = \psi e, \psi a \psi c = \psi c \psi a = \psi e, \psi b \psi b = \psi e$. Now, define operations $*$ on N as follows: $\psi x * \psi y = \psi^{-1} x \psi y$, for all x, y in $\{e, a, b, c\}$. Then, $(N, *)$ satisfies the left invertive law, and obviously, $(N, *)$ is an AG-groupoid.

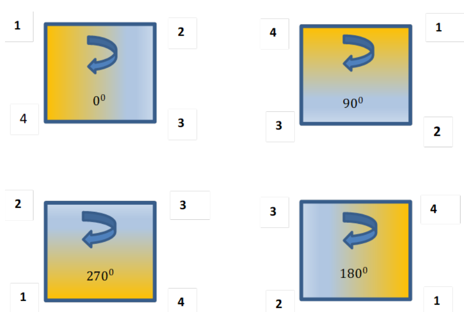

Figure 1: Rotation of a square generating an AG-groupoid

The main contributions of the paper are given below:

1. We prove certain properties of soft sets over AG-groupids and use the same to investigate other properties, simplify and rectify the already published results.

2. We investigate, AG*-groupoids, and AG-bands [31] based on these operations.

3. We mention and remove some weak results of the various published papers and investigate various general results regarding SI-product on AG-groupids

4. Various examples are constructed via modern computational techniques and produced to support the validity of produced results.

The rest of the paper is organized as follows: In section 2, some fundamental definitions and propositions related to AG-groupoid and SI-product operation are presented. Section 3 removes some weak results of the various published papers and investigates various general results regarding the SI-products on AG-groupoids . In section 4, some properties of the abelian distributive AG-groupoids under SI-product operation are investigated and some results are obtained. In section 5, properties of RAD-AG-groupoids are derived under SI-product. In section 6, conclusions related to the study are given.

2. Preliminaries

The notion of Abel Grassmann (AG) groupoid (Left almost semigroup or simply, LA-semigroup) has been introduced by Kazim and Naseeruddin [24] that generalizes the commutative semigroups and satisfy the left invertive law (LIL) $(pq)r = (rq)p$. It is easy to prove that every AG-groupoid satisfies medial law (ML), $pq \cdot rt = pr \cdot qt$. Further, left identity is unique if exists and in this case the paramedial law (PML), $pq \cdot rt = tq \cdot rp$, becomes true for such an AG-groupoid.Throughout this paper, S denotes AG-groupoid unless otherwise stated.

S is called AG*-groupoid [3] if the equivalent weak associative laws [33]: $(pq)r = q(pr), (pq)r = q(rp), \forall p, q, r \in S$, hold in S. S is known as Bol*-AG-groupoid, if $\forall p, q, r, t \in A, p(qr \cdot t) = (pq \cdot r)t$ [42,43]. S is called

1. Self-dual if the identity, $p(qr) = r(qp)$ holds in S and is called Bol* if $p((qu)v) = ((pq)u)v$.

2. S is called bi-commutative (BC) if $(pq)(rs) = (sr)(qp)$, and Moufang if $(pq)(sp) = (p(qs))p$.

3. AG** if $p(qr) = q(pr)$, S is called left [right] alternative if $p(pq) = (pp)q\, [p(qq) = (pq)q]$ and is called flexible if $p(qp) = (pq)p$.

4. Jordan if, $p((qq)r) = (qq)(pr)$.

5. Left abelian distributive (LAD)/right abelian distributive (RAD) if $p(qr) = (pq)(rp)\ /(pq)r = (rp)(qr)$.

6. Slim if $p(qr) = pr$.

7. AG-3-band if it satisfies the identity $p(pp) = (pp)p = p$.

8. Right commutative, (RC) if $p(qr) = p(rq)$.

9. Left commutative, (LC) if $p(qr) = p(rq)$.

10. Left alternative, if $(pp)q = p(pq)$.

11. Left (Right) nuclear square if $p^2(qr) = (p^2q)r(p(qr^2) = (pq)r^2)$

12. Middle nuclear square, if $(pq^2)r = p(q^2r)$.

13. Nuclear square, if it is left, right and middle nuclear square.

14. Cyclic associative (CA), if $p(qr) = r(pq)$

15. Right permutable (RP*), if $p((qr)s) = r((qp)s)$. for all $p, q, r, s \in S$ [3,18,34,42,43]

Definition 2.1 *Let U be the universal set, E the set of parameters and $S \subseteq E$. Then,*

1. A soft set α_S over U is defined by $\alpha_S : E \rightarrow \mathcal{P}(U)$ wherein, $\alpha_S(s) = \varphi$ if $s \notin S$. In this case, α_S is called the approximation function and $\alpha_S = \{(x, \alpha_S(x)) : x \in E, \alpha_S(x) \in \mathcal{P}(U)\}$. $S(U)$ is the collection of all soft sets over U or by $S(A)$ if parameter set is restricted to A [30].

2. Let β_S and γ_S be soft sets over the common universe U. Then, the SI-product $(\beta_S * \gamma_S)$ is defined in [39] as follows:

$$(\beta_S * \gamma_S)(x) = \begin{cases} \cup_{x=yz} \{\beta_S(y) \cap \gamma_S(z)\} & \exists y, z \in S, \text{such that } x = yz \\ \emptyset, & \text{otherwise.} \end{cases}$$

In the rest of the paper, U shall have the meaning of an initial universe, E the set of parameters, $P(U)$ the power set of U. Also, $A \subseteq E$ and S AG-groupoid. Sezer [39] proved that under the operation as defined in the Definition 2.1, the set $((S(S), *)$ satisfies the left invertive law, $(j_S * k_S) * l_S = (l_S * k_S) * j_S$ and hence:

Proposition 2.1 *Let S be an AG-groupoid. Then, so is $((S(S), *)$.*

We list now some of the properties of the SI-product and soft set operations that arise in [39] as follow:

Theorem 2.2 *[39] Let $\alpha_S, \beta_S, \gamma_S$ be any soft sets over S. Then, the following are true.*

1. $(\alpha_S * \beta_S) * \gamma_S = \alpha_S * (\beta_S * \gamma_S)$, the associative law for SI-product

2. $\alpha_S * \beta_S \neq \beta_S * \alpha_S$, in general

3. $\alpha_S * (\beta_S \tilde{\bigcup} \gamma_S) = (\alpha_S * \beta_S) \tilde{\bigcup} (\alpha_S * \gamma_S)$

4. $(\alpha_S \tilde{\bigcup} \beta_S) * \gamma_S = (\alpha_S * \gamma_S) \tilde{\bigcup} (\beta_S * \gamma_S)$

5. $\alpha_S * (\beta_S \tilde{\bigcap} \gamma_S) = (\alpha_S * \beta_S) \tilde{\bigcap} (\alpha_S * \gamma_S)$

6. $(\alpha_S \tilde{\bigcap} \beta_S) * \gamma_S = (\alpha_S * \gamma_S) \tilde{\bigcap} (\beta_S * \gamma_S)$

7. If $\lambda_A, \mu_A \in S(U)$ such that $\lambda_A \subseteq \alpha_A$ and $\mu_A \subseteq \beta_S$, then $\lambda_A * \mu_A \subseteq \alpha_S * \beta_S$.

Proposition 2.2 *[39] For an AG-groupoid S . The following hold:*

1. $((S(S), *)$ satisfies the left invertive law, and hence is medial AG-groupoid.

2. Let S be an AG-monoid and $\alpha_S, \beta_S, \gamma_S$ and δ_S be any elements of $S(S)$. Then, the following properties hold in $(S(S), *)$:

   (a) $\alpha_S * (\beta_S * \gamma_S) = \beta_S * (\alpha_{S^*} \gamma_S)$,

   (b) $(\alpha_S * \beta_S) * (\gamma_S * \delta_S) = (\delta_S * \gamma_S) * (\beta_S * \alpha_S)$.

3. Some General Properties of AG-groupoid based on SI-product

In this section, we extend Proposition 2.2 and investigate various results for the AG-groupoid $(S(S), *)$.

In AG-groupoids the paramedial law does not hold without the left identity. Sezer [39] proved that parmedial law holds in $S(S)$ when identity is allowed in it though it is not the case. Similarly, for $S(S)$ the following is investigated by [22] under requirement of the AG*:

Proposition 3.1 *[22, Proposition 3.4] $S(A)$ is parmedial for any AG*-groupoid A.*

However, here the situation is different and we investigate that the result holds in general in $(S(S), *)$ without the requirement of the left identity. To this end we first list the following lemmas:

Lemma 3.1 *[22] Let $\alpha_s, \beta_S, \gamma_S \in S(S)$. Then, the following holds,*

$$\alpha_S * (\beta_S * \gamma_S) = (\gamma_S * \beta_S) * \alpha_S. \tag{3.1}$$

Lemma 3.2 *Let $\alpha_S, \beta_S, \gamma_S, \delta_S \in S(S)$. Then, the following holds:*

$$\alpha_S * \{\beta_S * (\gamma_S * \delta_S)\} = \alpha_S * \{\gamma_S * (\beta_S * \delta_S)\}.$$

Lemma 3.3 *Let $\alpha_S, \beta_S, \gamma_S, \delta_S$ be arbitrary elements of $S(S)$. Then,*

$$\alpha_S * \{\beta_S * (\gamma_S * \delta_S)\} = \delta_S * \{\gamma_S * (\alpha_S * \beta_S)\}.$$

We prove the following important property. AG-groupoid is not paramedial in general [4] but here in this study, the situation is different and thus improves various weakness in the available published research so far done in this area.

Proposition 3.2 *Let S be an AG-groupoid. Then, $S(S)$ is paramedial AG-groupoid.*

Proof. For any elements $\alpha_S, \beta_S, \gamma_S$ and δ_S of $S(S)$. Then,

$$\begin{aligned}
(\alpha_S * \beta_S) * (\gamma_S * \delta_S) &= (\delta_S * \gamma_S) * (\alpha_S * \beta_S) \ \text{(by Lem.3.1)} \\
&= (\delta_S * \alpha_S) * (\gamma_S * \beta_S) \ \text{(by ML)} \\
&= \delta_S * \{\alpha_S * (\gamma_S * \beta_S)\} \ \text{(by assoc.)} \\
&= \delta_S * \{\gamma_S * (\alpha_S * \beta_S)\} \ \text{(by Lem. 3.2)} \\
&= \delta_S * \{(\beta_S * \alpha_S) * \gamma_S\} \ \text{(by Lem. 3.1)} \\
&= \delta_S * \{\beta_S * (\alpha_S * \gamma_S)\} \ \text{(by assoc.)} \\
&= \gamma_S * \{\beta_S * (\delta_S * \alpha_S)\} \ \text{(by Lem. 3.3)} \\
&= \gamma_S * \{(\alpha_S * \delta_S) * \beta_S)\} \ \text{(by Lem. 3.1)} \\
&= \gamma_S * \{(\beta_S * \delta_S) * \alpha_S\} \ \text{(by LIL)} \\
&= \{\alpha_S * (\beta_S * \delta_S)\} * \gamma_S\} \ \text{(by Lem. 3.1)} \\
&= \{(\alpha_S * \beta_S) * \delta_S)\} * \gamma_S \ \text{(by assoc.)} \\
&= (\gamma_S * \delta_S) * (\alpha_S * \beta_S) \ \text{(by LIL)} \\
&= (\beta_S * \alpha_S) * (\gamma_s * \delta_S) \ \text{(by Lem. 3.1)} \\
&= (\delta_S * \gamma S) * (\beta_S * \alpha_S) \ \text{(by Lem.3.1)} \\
&= (\delta_S * \beta_S) * (\gamma_S * \alpha_S) \ \text{(by ML)}
\end{aligned}$$

The following results are proved by Sezer [39] and that of Karaaslan [22]. We prove it as a direct consequence of the above facts, without the requirement of the left identity or the requirement of the AG* and other restrictions or conditions.

Theorem 3.4 *[39, Theorem 3.6] Let S be an AG-groupoid with left identity and $\alpha_S, \beta_S, \delta_S$ and γ_S be any elements of $S(S)$. Then, $(\alpha_S * \beta_S) * (\gamma_S * \delta_S) = (\delta_S * \gamma_S) * (\beta_S * \alpha_S)$.*

Theorem 3.5 *[22, Theorem 3.5] Let A be an AG*-groupoid and $\alpha_A, \beta_A, \delta_A$ and $\gamma_A \in S(A)$. Then,*

$$\alpha_A * \{(\beta_A * \delta_A) * \gamma_A\} = \{(\alpha_A * \beta_A) * \delta_A)\} * \gamma_A.$$

We further draft the two requirements for the following Theorem 3.7 to prove that $S(S)$ is Bol*.

Theorem 3.6 *[22, Theorem 3.8] Let A be an AG-groupoid and* $\alpha_A, \beta_A, \delta_A, \gamma_A \in S(A)$ *such that* $\alpha_A*(\beta_A*\delta_A)*\gamma_A = (\delta_A*\alpha_A)*(\beta_A*\gamma_A)$ *and* $\alpha_A*(\beta_A*\delta_A)*\gamma_A = (\alpha_A*\gamma_A)*(\delta_A*\beta_A)$*. Then,* $\alpha_A*(\beta_A*\delta_A)*\gamma_A = (\alpha_A*\beta_A)*\delta_A*\gamma_A$*.*

The following result reveals the unique and distinct non-classical algebraic properties of $S(S)$ that deviate from those observed in the general structure of AG-groupoids. It is important to note that not all AG-groupoids exhibit self-duality, Bol* properties, or bi-commutativity being subclasses. However, these properties are upheld within the structure under the application of SI-product over its soft sets.

Theorem 3.7 *Let S be an AG -groupoid. Then, the following hold in* $S(S)$*.*

1. $S(S)$ is self-dual,
2. $S(S)$ is Bol*,
3. $S(S)$ is bi-commutative.

Proof. Let $\alpha_S, \beta_S, \delta_S$ and γ_S be any elements of $S(S)$.

1.

$$\begin{aligned} \alpha_S*(\beta_S*\gamma_S) &= (\alpha_S*\beta_S)*\gamma_S (\text{by assoc.}) \\ &= (\gamma_S*\beta_S)*\alpha_S (\text{by LIL}) \\ &= \gamma_S*(\beta_S*\alpha_S) \ (\text{by assoc.}) \\ \Rightarrow \alpha_S*(\beta_S*\gamma_S) &= \gamma_S*(\beta_S*\alpha_S). \end{aligned}$$

Therefore, $S(S)$ is self-dual.

2.

$$\begin{aligned} \alpha_S*\{(\beta_S*\delta_S)*\gamma_S\} &= \alpha_{S*}\{\beta_S*(\delta_S*\gamma_S)\} \ (\text{by assoc.}) \\ &= (\alpha_S*\beta_S)*(\delta_S*\gamma_S) \ (\text{by assoc.}) \\ &= \{(\alpha_S*\beta_S)*\delta_S)\}*\gamma_S \ (\text{by assoc.}) \\ \Rightarrow \alpha_S*\{(\beta_S*\delta_S)*\gamma_S\} &= \{(\alpha_S*\beta_S)*\delta_S)\}*\gamma_S. \end{aligned}$$

Thus, $S(S)$ is Bol*.

3.

$$\begin{aligned} (\alpha_S*\beta_S)*(\gamma_S*\delta_S) &= (\alpha_{S*}\gamma_S)*(\beta_{S*}\delta_S) \ (\text{by ML}) \\ &= (\delta_S*\gamma_S)*(\beta_S*\alpha_S) \ (\text{by PML}) \\ \Rightarrow (\alpha_S*\beta_S)*(\gamma_S*\delta_S) &= (\delta_S*\gamma_S)\circ(\beta_S*\alpha_S). \end{aligned}$$

Hence, $S(S)$ is bi-commutative

The following result appears in [22] with a corollary. We generalize and improve it as in the next result, without the requirement of AG*.

Proposition 3.3 *[22, Proposition 3.6] Let A be an AG*-groupoid. Then, for all* α_A, β_A *and* $\gamma_A \in S(A)$

1. $\alpha_A^2*(\beta_A*\gamma_A) = (\alpha_A^2*\beta_A)*\gamma_A$ where $(\alpha_A^2 = \alpha_A*\alpha_A)$
2. $(\alpha_A*\beta_A)*\gamma_A^2 = \alpha_A^2*(\beta_A*\gamma_A^2)$ where $(\gamma_A^2 = \gamma_A*\gamma_A)$
3. $(\alpha_A*\beta_A^2)*\gamma_A = (\alpha_A*\beta_A^2)*\gamma_A)$ where $(\beta_A^2 = \beta_A*\beta_A)$.

Corollary 3.8 *[22, Corollary 3.7.] Let A be an AG*-groupoid. Then,* $S(A)$ *is nuclear square.*

Theorem 3.9 *For any AG-groupoid S any of the following holds in $S(S)$.*

1. $S(S)$ is left/right alternative,
2. $S(S)$ is nuclear square,
3. $S(S)$ is locally associative,
4. $S(S)$ is Moufang,
5. $S(S)$ is Jordan, and
6. $S(S)$ is flexible.

Proof The proof follows simply by the associativity in $S(S)$ as in Theorem 2.2, and other above mentioned relevant properties in $S(S)$.

4. Properties of abelian distributive AG-groupoids via the product of soft sets

Here, now we describe and investigate some other properties of an abelian distributive AG-groupoid under the SI-product for its soft sets. We first give a counterexample to show that $S(S)$ is not SI-LAD and thus we proceed to seek under what conditions it becomes LAD.

Example 4.1 *Let $A = \{x, y, w, z\}$ be an AG-groupoid an in the following table and α_A, β_A and $\delta_A \in S(A)$ be soft sets over $U = \{1,2,3,4,5\}$. Then, $S(A)$ is not an LAD-AG-groupoid.*

$\cdot$	x	y	z
x	x	x	x
y	z	z	z
z	x	x	x

Let the three soft sets over A be as follows:

$$\begin{aligned}\alpha_A &= \{(x,\{1,2,3,4,5\}),(y,\{3,4,5\}),(z,\{1,2,3,4,5\})\}\\ \beta_A &= \{(x,\{1,2,3,4\}),(y,\{3,4\}),(z,\{1,2,3,4\})\} \text{ and}\\ \delta_A &= \{(x,\{1,2,3,5\}),(y,\{1,3,5\}),(z,\{1,2,3,5\})\}\end{aligned}$$

over $U = \{1,2,3,4,5\}$. Then,

$$\begin{aligned}(\beta_A * \delta_A)(x) = &\{\beta_A(x)\cap\delta_A(x)\}\cup\{\beta_A(x)\cap\delta_A(y)\}\cup\\ &\{\beta_A(x)\cap\delta_A(z)\}\cup\{\beta_A(z)\cap\delta_A(x)\}\cup\\ &\{\beta_A(z)\cap\delta_A(y)\}\cup\{\beta_A(z)\cap\delta_A(z)\}\\ &=\{1,2,3\},\end{aligned}$$

$$(\beta_A * h_A)(y) = \emptyset, \text{ and}$$

$$\begin{aligned}(\beta_A * \delta_A)(z) = &\{\beta_A(y)\cap\delta_A(x)\}\cup\{\beta_A(y)\cap\delta_A(y)\}\\ &\cup\{\beta_A(y)\cap\delta_A(z)\}\\ &=\{3\}.\end{aligned}$$

Claim: We claim that $(S(A), *)$ is not LAD, i.e. it does not satisfy the condition $\alpha_A * (\beta_A * \delta_A) = (\alpha_A * \beta_A) * (\delta_A * \alpha_A)$. Since

$(\alpha_A * (\beta_A * \delta_A))(z) = \bigcup_{z=pq} \{\alpha_A(p) \cap (\beta_A * \delta_A)(q)\}$. Now, z appears in the given table as, $z = yx = yy = yz$, with $p = y$, and $q = x, y, z$. Now, $(\beta_A * \delta_A)(q) = \bigcup_{q=rs} \{\beta_A(r) \cap \delta_A(s)\} = \{1, 2, 3\}$ and $\alpha_A(y) = \{3, 4, 5\}$. Thus,

$$\begin{aligned}\alpha_A * (\beta_A * \delta_A)(z) &= \bigcup_{z=pq} \{\alpha_A(p) \cap (\beta_A * \delta_A)(q)\} \\ &= \{\alpha_A(y) \cap (\beta_A * \delta_A)(x)\} \cup \{\alpha_A(y) \cap (\beta_A * \delta_A)(y)\}\} \\ &\quad \cup \{\alpha_A(y) \cap (\beta_A * \delta_A)(z)\} = \{3\} \cup \emptyset \cup \{3\} = \{3\}.\end{aligned}$$

Similarly, $((\alpha_A * \beta_A) * (\delta_A * \alpha_A))(z) = \cup_{z=pq} \{(\alpha_A * \beta_A)(p) \cap (\delta_A * \alpha_A)(q)\}$. Now, $z = yx = yy = yz$, with $p = y$, and $q = x, y, z$. Since for $p = y$, $(\alpha_A * \beta_A)(p) = \emptyset$. So is, $(\alpha_A * \beta_A) * (\delta_A * \alpha_A)(z) = \emptyset$.

Thus, $\alpha_A * (\beta_A * \delta_A) \neq (\alpha_A * \beta_A) * (\delta_A * \alpha_A)$.

As by Example 4.1, $S(S)$ is not LAD under the SI-product operation. However, it does hold under various conditions as discussed in the following result.

Theorem 4.2 *Let $S(S)$ satisfy at least one of the following properties over an AG-groupid S. Then, $S(S)$ is necessarily a LAD-AG-groupid.*

1. $S(S)$ is RAD
2. $S(S)$ is left distributive (LD)
3. $S(S)$ is slim
4. $S(S)$ is idempotent
5. AG-3-band.

Proof. Let $\alpha_s, \beta_S, \gamma_S$and δ_S be arbitrary elements of $S(S)$. Then in each case we prove that $S(S)$ is LAD-AG-groupoid or is a commutative AG-groupoid that becomes an LAD.

1. Let $S(S)$ is RAD. Then, by Lemma 2.1,

$$\begin{aligned}\alpha_S * (\beta_S * \gamma_S) &= (\gamma_S * \beta_S) * \alpha_S \quad \text{(by Lem. 2.1)} \\ &= (\alpha_S * \gamma_S) * (\beta_S * \alpha_S) \quad \text{(by RAD)} \\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \quad \text{(by ML)} \\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S).\end{aligned}$$

2. Let $S(S)$ is LD. Then,

$$\begin{aligned}\alpha_S * (\beta_S * \gamma_S) &= (\alpha_S * \beta_S) * (\alpha_S * \gamma_S) \quad \text{(by LD)} \\ &= (\alpha_S * \alpha_S) * (\beta_S * \gamma_S) \quad \text{(by ML)} \\ &= (\gamma_S * \alpha_S) * (\beta_S * \alpha_S) \quad \text{(by PML)} \\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \quad \text{(by Lem. 2.1)} \\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S).\end{aligned}$$

3. Let $S(S)$ is Slim. Then,

$$\begin{aligned}\alpha_S * (\beta_S * \gamma_S) &= \alpha_S * \{\alpha_S * (\beta_S * \gamma_S)\} \quad \text{(by assum. 3)} \\ &= \{(\beta_S * \gamma_S) * \alpha_S\} * \alpha_S \quad \text{(by Lemma 2.1)} \\ &= (\alpha_S * \alpha_S) * (\beta_S * \gamma_S) \quad \text{(by LIL)} \\ &= (\alpha_S * \beta_S) * (\alpha_S * \gamma_S) \quad \text{(by ML)} \\ &= (\gamma_S * \alpha_S) * (\beta_S * \alpha_S) \quad \text{(by LIL)} \\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \quad \text{(by Lem. 2.1)} \\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S).\end{aligned}$$

Furthermore, $S(S)$ is commutative as;

$$\begin{aligned}\alpha_S*\beta_S &= (\alpha_S*\gamma_S)*\beta_S \quad \text{(by assump.)}\\ &= (\beta_S*\gamma_S)*\alpha_S \quad \text{(by LIL)}\\ &= \beta_S*\alpha_S \qquad \text{(by assump.)}\\ &= \beta_S*\alpha_S.\end{aligned}$$

4. Let $S(S)$ is idempotent. Then,

$$\begin{aligned}\alpha_S*\beta_S &= (\alpha_S*\alpha_S)*\beta_S \quad \text{(by idempotency)}\\ &= (\beta_S*\alpha_S)*\alpha_S \quad \text{(by LIL)}\\ &= \beta_S*(\alpha_S*\alpha_S) \quad \text{(by assoc.)}\\ &= \beta_S*\alpha_S \qquad \text{(by idempotency)}\\ &= \beta_S*\alpha_S.\end{aligned}$$

i.e. $S(S)$ is commutative AG-groupoid. Hence,

$$\begin{aligned}\alpha_S*(\beta_S*\gamma_S) &= (\alpha_S*\alpha_S)*(\beta_S*\gamma_S) \quad \text{(by assump.)}\\ &= (\alpha_S*\beta_S)*(\alpha_S*\gamma_S) \quad \text{(by ML)}\\ &= (\alpha_S*\beta_S)*(\gamma_S*\alpha_S) \quad \text{(by commutativity)}\\ &= (\alpha_S*\beta_S)*(\gamma_S*\alpha_S).\end{aligned}$$

5. Let $S(S)$ is AG-3-band. Then,

$$\begin{aligned}\alpha_S*\beta_S &= \{\alpha_S*(\alpha_S*\alpha_S)\}*\beta_S \quad \text{(by assump.)}\\ &= \{(\alpha_S*\alpha_S)*\alpha_S\}*\beta_S \quad \text{(by associativity)}\\ &= (\beta_S*\alpha_S)*(\alpha_S*\alpha_S) \quad \text{(by LIL)}\\ &= \beta_S*\{\alpha_S*(\alpha_S*\alpha_S)\} \quad \text{(by assoc. law)}\\ &= \beta_S*\alpha_S \qquad \text{(by assump.)}\\ &= \beta_S*\alpha_S.\end{aligned}$$

Hence, $S(S)$ is a commutative AG-groupoid, therefore satisfying the LAD property.

Remark. By Theorem 2.2 $S(S)$ is not commutative in general. However, in above theorem, we see in the last three cases that $S(S)$ is commutative AG-groupoid and thus is an LAD. Further, we study the sub-classes of LAD-AG-grouoids under SI-products of soft sets.

Proposition 4.1 *Let A be an LAD-AG-groupoid. Then, $S(A)$ is right commutative (RC).*

Proof. It is known that LAD-AG-groupoid also satisfies the property of an RC groupoid [4]. Thus, for any $\alpha_A, \beta_A, \gamma_A$ and any element x of A that is not appearing as a product of its two elements. Then,we get $(\alpha_A*(\beta_A*\gamma_A))(x) = (\alpha_A*(\gamma_A*\beta_A))(x) = \emptyset$. Now, we consider that the element x can be written as a product of p and q. Then,

$$
\begin{aligned}
(\alpha_A * (\beta_A * \gamma_A))(x) &= \bigcup_{x=pq} \{\alpha_A(p) \cap (\beta_A * \gamma_A)(q)\} \\
&= \bigcup\nolimits_{x=pq} \{\alpha_A(p) \cap (\cup_{q=rs} \{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=p(rs)} \{\alpha_A(p) \cap (\{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=p(rs)} \{\alpha_A(p) \cap \{\gamma_A(s) \cap \beta_A(r)\}\} \\
&= \bigcup\nolimits_{x=p(sr)} \{\alpha_A(p) \cap \{\cup_{q=sr} \gamma_A(s) \cap \beta_A(r)\}\} \\
&= \bigcup\nolimits_{x=pq} \{\alpha_A(p) \cap (\gamma_A * \beta_A)(q)\} \\
&= (\alpha_A * (\gamma_A * \beta_A))(x). \\
&\Rightarrow (\alpha_A * (\beta_A * \gamma_A)) \\
&= (\alpha_A * (\gamma_A * \beta_A)).
\end{aligned}
$$

Proposition 4.2 *Let A be LAD-AG-groupoid. Then $S(A)$ is self-dual AG-groupoid.*

Proof. It is known that every LAD-AG-groupoid is self-dual [4]. Thus, for any soft sets, $\alpha_A, \beta_A, \gamma_A$ we have to show that $(\alpha_A * (\beta_A * \gamma_A))(x) = (\gamma_A (\beta_A * \alpha_A))(x)$. Since

$$
\begin{aligned}
(\alpha_A * (\beta_A * \gamma_A))(x) &= \bigcup_{x=pq} \{\alpha_A(p) \cap (\beta_A * \gamma_A)(q)\} \\
&= \bigcup\nolimits_{x=pq} \{\alpha_A(p) \cap (\cup_{q=rs} \{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=p(rs)} \{\alpha_A(p) \cap (\{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=s(rp)} \{\gamma_A(s) \cap (\{\beta_A(r) \cap \alpha_A(p)\})\} \\
&= \bigcup\nolimits_{x=st} \{\gamma_A(s) \cap \{\cup_{t=rp} (\beta_A(r) \cap \alpha_A(p))\}\} \\
&= \bigcup\nolimits_{x=st} \{\gamma_A(s) \cap (\beta_A * \alpha_A)(t)\} \\
&= (\gamma_A * (\beta_A * \alpha_A))(x).
\end{aligned}
$$

Proposition 4.3 *If A is LAD-AG-groupoid, then $S(A)$ is AG**.*

Proof. From [4], it is known that LAD-AG-groupoid is AG**. Thus, for any $\alpha_A, \beta_A, \gamma_A$ we have to prove $(\alpha_A * (\beta_A * \gamma_A))(x) = (\beta_A * (\alpha_A * \gamma_A))(x)$ for any x written as the product of its two elements, otherwise $(\alpha_A * (\beta_A * \gamma_A))(x) = (\beta_A * (\alpha_A * \gamma_A))(x) = \emptyset$. Now, for any element x that is appears as a product of p and q. Then,

$$
\begin{aligned}
(\alpha_A * (\beta_A * \gamma_A))(x) &= \bigcup_{x=pq} \{\alpha_A(p) \cap (\beta_A * \gamma_A)(q)\} \\
&= \bigcup\nolimits_{x=pq} \{\alpha_A(p) \cap (\cup_{q=rs} \{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=p(rs)} \{\alpha_A(p) \cap (\{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=r(ps)} \{\beta_A(r) \cap (\{\alpha_A(p) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=rq} \{\beta_A(r) \cap \{\cup_{q=ps} (\alpha_A(p) \cap \gamma_A(s))\}\} \\
&= \bigcup\nolimits_{x=rq} \{\beta_A(r) \cap (\alpha_A * \gamma_A)(q)\} \\
&= (\beta_A * (\alpha_A * \gamma_A))(x).
\end{aligned}
$$

Proposition 4.4 *Let A be an LAD-AG-groupoid. Then, $S(A)$ is left distributive (LD) AG-groupoid.*

Proof. From [4], we know that LAD-AG-groupoid is LD-AG-groupoid. Thus, for any $\alpha_A, \beta_A, \gamma_A$ we have to show that $(\alpha_A * (\beta_A * \gamma_A))(x) = ((\alpha_A * \beta_A) * (\alpha_A * \gamma_A))(x)$. Thus,

$$\begin{aligned}
(\alpha_A * (\beta_A * \gamma_A))(x) &= \bigcup_{x=pq} \{\alpha_A(p) \cap (\beta_A * \gamma_A)(q)\} \\
&= \bigcup\nolimits_{x=pq} \{\alpha_A(p) \cap (\cup_{q=rs} \{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=p(rs)} \{\alpha_A(p) \cap (\{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=p(rs)} \{(\alpha_A(p) \cap \alpha_A(p)) \cap (\{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=(pr)(sp)} \{\alpha_A(p) \cap \beta_A(r) \cap (\{\gamma_A(s) \cap \alpha_A(p)\})\} \\
&= \bigcup\nolimits_{x=(pr)(ps)} \{\alpha_A(p) \cap \beta_A(r) \cap (\{\alpha_A(p) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=qv} \{\cup_{q=pr} (\alpha_A * \beta_A)(q) \cap \{\cup_{v=ps} (\alpha_A * \gamma_A)(v)\}\} \\
&= \bigcup\nolimits_{x=qv} ((\alpha_A * \beta_A) * (\alpha_A * \gamma_A))(x) \\
&= ((\alpha_A * \beta_A) * (\alpha_A * \gamma_A))(x).
\end{aligned}$$

Proposition 4.5 *Let A be an LAD-AG-groupoid. Then, $S(A)$ is CA-AG-groupoid.*

Proof. From [4], we know that LAD-AG-groupoid is CA-AG-groupoid . To provide that $S(A)$ is CA-AG-groupoid, we should provide that $(\alpha_A * (\beta_A * \gamma_A))(x) = (\gamma_A * (\alpha_A * \beta_A))(x)$, for any $\alpha_A, \beta_A, \gamma_A \in S(A)$. Since

$$\begin{aligned}
(\alpha_A * (\beta_A * \gamma_A))(x) &= \bigcup_{x=pq} \{\alpha_A(p) \cap (\beta_A * \gamma_A)(q)\} \\
&= \bigcup\nolimits_{x=pq} \{\alpha_A(p) \cap (\cup_{q=rs} \{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=p(rs)} \{\alpha_A(p) \cap (\{\beta_A(r) \cap \gamma_A(s)\})\} \\
&= \bigcup\nolimits_{x=s(pr)} \{\gamma_A(s) \cap (\{\alpha_A(p) \cap \beta_A(r)\})\} \\
&= \bigcup\nolimits_{x=sq} \{\gamma_A(s) \cap \{\cup_{q=pr} (\alpha_A(p) \cap \beta_A(r))\}\} \\
&= \bigcup\nolimits_{x=sq} \{\gamma_A(s) \cap (\alpha_A * \beta_A)(q)\} \\
&= (\gamma_A * (\alpha_A * \beta_A))(x).
\end{aligned}$$

Thus, $(\alpha_A * (\beta_A * \gamma_A)) = (\gamma_A * (\alpha_A * \beta_A))$.

Theorem 4.3 *Let e be the right identity element of an AG-groupoid S and α_S, β_S are arbitrary elements of $S(S)$. Then, $S(S)$ is commutative semigroup. i.e. $\alpha_S * \beta_S = \beta_S * \alpha_S$.*

Proof. For any $s \in S$, if s does not appear as a product of two elements in S, then $(\alpha_S * \beta_S)(s) = (\beta_S * \alpha_S)(s) = \emptyset$. Otherwise, $s = yz$, for $y, z \in S$ and since $yz = ye \cdot ze = (ze \cdot e)y = (ze \cdot y) = zy$. Thus,

$$\begin{aligned}
(\alpha_S * \beta_S)(s) &= \bigcup\nolimits_{s=yz} \{\alpha_S(y) \cap \beta_S(z)\} \\
&= \bigcup\nolimits_{s=zy} \{\beta_S(z) \cap \alpha_S(y)\} \\
&= (\beta_S * \alpha_S)(s) \\
\Rightarrow \alpha_S * \beta_S &= \beta_S * \alpha_S.
\end{aligned}$$

In the subsequent analysis, we delve into an intriguing characterization of the AG-groupoid when subjected to the SI-product. Typically, in an AG-groupoid with a right identity, it transforms into an AG-monoid and exhibits commutativity, leading to the acquisition of various other properties, including LAD. However, it should be noted that for any AG-groupoid S with a right identity, $S(S)$ does not qualify as an LAD structure under the SI-product operation, as illustrated by the example.

Example 4.4 *Let $A = \{x, y, z\}$ be an AG-groupoid with right identity element x defined by the following table and α_A, β_A and γ_A be soft sets over $U = \{1,2,3,4,5,6,7,8\}$. Then, $S(A)$ is not an LAD-AG-groupoid.*

$*$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

Consider three soft sets on A over $U = \{1,2,3,4,5,6,7,8\}$ given as follows:

$$\begin{aligned}\alpha_A &= \{(x,\{1,2,5\}),(y,\{2,3,7\}),(z,\{1,7,8\})\}\\ \beta_A &= \{(x,\{4,5,6\}),(y,\{3,7,8\}),(z,\{2,3,7,8\})\}\\ \gamma_A &= \{(x,\{1,2,3\}),(y,\{3,5,8\}),(z,\{1,3,8\})\}\end{aligned}$$

Then, $(\beta_A*\gamma_A)(x) = \{3,8\}$, $(\beta_A*\gamma_A)(y) = \{3,5,8\}$ $(\beta_A*\gamma_A)(z) = \{2,3,8\}$, $(\gamma_A*\alpha_A)(x) = \{2,3,8\}$,$(\alpha_A*\beta_A)(x) = \{2,3,5,7,8\}$.

We show that $(S(A),*)$ is not LAD, i.e. it does not satisfy the condition $\alpha_A*(\beta_A*\gamma_A) = (\alpha_A*\beta_A)*(\gamma_A*\alpha_A)$. Since

$$\begin{aligned}\alpha_A*(\beta_A*\gamma_A)(z) &= \bigcup_{z=pq}\{\alpha_A(p)\cap(\beta_A*\gamma_A)(q)\}\\ &= \{\alpha_A(x)\cap(\beta_A*\gamma_A)(z)\}\cup\{\alpha_A(y)\cap(\beta_A*\gamma_A)(y)\}\\ &\cup \{\alpha_A(z)\cap(\beta_A*\gamma_A)(x)\}\\ &= \{\{1,2,5\}\cap\{2,3,8\}\}\cup\{\{2,3,7\}\cap\{3,5,8\}\}\\ &= \cup\{\{1,7,8\}\cap\{3,8\}\}\\ &= \{2\}\cup\{3\}\cup\{8\} = \{2,3,8\}.\end{aligned}$$

Now, $(\alpha_A*\beta_A)(x) = \{2,3,5,7,8\}, (\gamma_A*\alpha_A)(x) = \{2,3,8\}, (\alpha_A*\beta_A)(y) = \{3,7,8\}, (\gamma_A*\alpha_A)(z) = \{1,3\}$, $(\alpha_A*\beta_A)(z) = \{3\}, (\gamma_A*\alpha_A)(y) = \{1,2,3,5,8\}$
Thus, $((\alpha_A*\beta_A)*(\gamma_A*\alpha_A))(z) = \cup_{z=pq}\{(\alpha_A*\beta_A)(p)\cap(\gamma_A*\alpha_A)(q)\}$.

$$\begin{aligned}((\alpha_A*\beta_A)*(\gamma_A*\alpha_A))(z) &= \bigcup_{y=pq}\{(\alpha_A*\beta_A)(p)\cap(\beta_A*\gamma_A)(q)\}\\ &= \{(\alpha_A*\beta_A)(x)\cap(\gamma_A*\alpha_A)(z)\}\\ &\cup \{(\alpha_A*\beta_A)(y)\cap(\gamma_A*\alpha_A)(y)\}\\ &\cup \{(\alpha_A*\beta_A)(z)\cap(\gamma_A*\alpha_A)(x)\}\\ &= \{\{2,3,5,7,8\}\cap\{1,3\}\}\cup\{\{3,7,8\}\cap\{1,2,3,5,8\}\}\\ &\cup \{\{3\}\cap\{2,3,8\}\}\\ &= \{3\}\cup\{3,8\}\cup\{3\} = \{3,8\}.\end{aligned}$$

Thus, $\alpha_A*(\beta_A*\gamma_A) \neq (\alpha_A*\beta_A)*(\gamma_A*\alpha_A)$.

$S(S)$ is a non-commutative structure but it is an associative AG-groupoid, so left invertive law holds in it. Further, it is paramedial, bi-commutative, and nuclear square. Using these properties we proceed to prove the following characterization under some restriction of the left abelian distributive property.

Theorem 4.5 *Let S be an AG-groupoid and $S(S)$ be the AG groupoid. Then,*

1. *$S(S)$ is AG*,*
2. *$S(S)$ is AG**,*
3. *$S(S)$ is LD-AG-groupoid,*
4. *S(S) is RD-AG-groupoid,*
5. *$S(S)$ is cyclic associative (CA),*
6. *$S(S)$ is right abelian distributive (RAD),*
7. *S(S) is left commutative (LC) AG-gropupoid (LC-AG-groupoid) and right commutative (RC) AG-groupoid (RC-AG-groupoid),*
8. *$S(S)$ is Stein AG-groupoid,*
9. *$S(S)$ is right permuatable (RP*) AG-groupoid (RP*-AG-groupoid).*

Proof. Let $\alpha_S, \beta_S, \gamma_S$ and δ_S be arbitrary elements of $S(S)$. Then,

1.

$$\begin{aligned}
(\alpha_S * \beta_S) * \gamma_S &= \alpha_S * (\beta_S * \gamma_S) \text{ (by assoc.)}\\
&= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by LAD)}\\
&= (\alpha_S * \gamma_S) * (\beta_S * \alpha_S) \text{ (by Prop. 2.2)}\\
&= \alpha_S * (\gamma_S * \beta_S) \text{ (by LAD)}\\
&= (\beta_S * \gamma_S) * \alpha_S \text{(by Lem. 3.1)}\\
&= \beta_S * (\gamma_S * \alpha_S) \text{ (by assoc.)}\\
&= (\beta_S * \gamma_S) * (\alpha_S * \beta_S) \text{ (by LAD)}\\
&= (\beta_S * \alpha_S) * (\gamma_S * \beta_S) \text{ (by BC law)}\\
&= \beta_S * (\alpha_S * \gamma_S) \text{ (by LAD)}\\
\Rightarrow\ & (\alpha_S * \beta_S) * \gamma_S = \beta_S * (\alpha_S * \gamma_S).
\end{aligned}$$

Hence, $S(S)$ is AG^*-groupoid.

2.

$$\begin{aligned}
\alpha_S * (\beta_S * \gamma_S) &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by LAD)}\\
&= (\alpha_S * \gamma_S) * (\beta_S * \alpha_S) \text{ (by ML)}\\
&= \alpha_S * (\gamma_S * \beta_S) \text{ (by LAD)}\\
&= (\beta_S * \gamma_S) * \alpha_S \text{ (by Lem. 3.1)}\\
&= \beta_S * (\gamma_S * \alpha_S) \text{ (by assoc.)}\\
&= (\beta_S * \gamma_S) * (\alpha_S * \beta_S) \text{ (by LAD)}\\
&= (\beta_S * \alpha_S) * (\gamma_S * \beta_S) \text{ (by ML)}\\
&= (\beta_S * \alpha_S) * (\gamma_S * \beta_S) \text{ (by PML)}\\
&= \beta_S * (\alpha_S * \gamma_S) \text{ (by LAD)}\\
\Rightarrow\ & \alpha_S * (\beta_S * \gamma_S) = \beta_S * (\alpha_S * \gamma_S).
\end{aligned}$$

Thus, $S(S)$ is AG^{**}-groupoid

3.

$$\begin{aligned}\alpha_S*(\beta_S*\gamma_S) &= (\alpha_S*\beta_S)*(\gamma_S*\alpha_S) \text{ (by LAD)}\\ &= (\alpha_S*\gamma_S)*(\beta_S*\alpha_S) \text{ (by ML)}\\ &= (\alpha_S*\gamma_S)*(\beta_S*\alpha_S) \text{ (by PML)}\\ &= (\alpha_S*\beta_S)*(\alpha_S*\gamma_S) \text{ (by Prop. 2.2)}\end{aligned}$$

Thus, $\alpha_S*(\beta_S*\gamma_S) = (\alpha_S*\beta_S)*(\alpha_S*\gamma_S)$ and so $S(S)$ is LD-AG-groupoid.

4.

$$\begin{aligned}(\alpha_S*\beta_S)*\gamma_S &= (\gamma_S*\beta_S)*\alpha_S \text{ (by LIL)}\\ &= \gamma_S*(\beta_S*\alpha_S) \text{ (by assoc.)}\\ &= (\gamma_S*\beta_S)*(\alpha_S*\gamma_S) \text{ (by LAD)}\\ &= \{(\alpha_S*\gamma_S)*\beta_S\}*\gamma_S \text{ (by LIL)}\\ &= (\alpha_S*\gamma_S)*(\beta_S*\gamma_S) \text{ (by assoc.)}\end{aligned}$$

It is concluded that $(\alpha_S*\beta_S)*\gamma_S = (\alpha_S*\gamma_S)*(\beta_S*\gamma_S)$. This shows that $S(S)$ is RD-AG-groupoid.

5.

$$\begin{aligned}\alpha_S*(\beta_S*\gamma_S) &= (\gamma_S*\beta_S)*\alpha_S \text{ (by Lem. 3.1)}\\ &= \gamma_S*(\beta_S*\alpha_S) \text{ (by assoc.)}\\ &= (\gamma_S*\beta_S)*(\alpha_S*\gamma_S) \text{ (by LAD)}\\ &= (\alpha_S*\gamma_S)*(\beta_S*\gamma_S) \text{ (by ML)}\\ &= (\alpha_S*\beta_S)*(\alpha_S*\gamma_S) \text{ (by LIL)}\\ &= (\gamma_S*\alpha_S)*(\beta_S*\gamma_S) \text{ (by ML)}\\ &= \gamma_S*(\alpha_S*\beta_S) \text{ (by LAD)}\end{aligned}$$

Then, $\alpha_S*(\beta_S*\gamma_S) = \gamma_S*(\alpha_S*\beta_S)$ and so $S(S)$ is CA-AG-groupoid.

6.

$$\begin{aligned}(\alpha_S*\beta_S)*\gamma_S &= \alpha_S*(\beta_S*\gamma_S) \text{ (by assoc.)}\\ &= (\gamma_S*\beta_S)*\alpha_S \text{ (by Lem. 3.1)}\\ &= \gamma_S*(\beta_S*\alpha_S) \text{ (by assoc.)}\\ &= (\gamma_S*\beta_S)*(\alpha_S*\gamma_S) \text{ (by LAD)}\\ &= (\gamma_S*\alpha_S)*(\beta_S*\gamma_S) \text{ (by ML)}\end{aligned}$$

Here, we see that $(\alpha_S*\beta_S)*\gamma_S = (\gamma_S*\alpha_S)*(\beta_S*\gamma_S)$. Therefore, $S(S)$ with LAD-AG-groupoid is RAD-AG-groupoid.

7.

$$\begin{aligned}\alpha_S*(\beta_S*\gamma_S) &= (\alpha_S*\beta_S)*(\gamma_S*\alpha_S) \text{ (by LAD)}\\ &= (\alpha_S*\gamma_S)*(\alpha_S*\beta_S) \text{ (by Prop. 2.2)}\\ &= (\alpha_S*\alpha_S)*(\gamma_S*\beta_S) \text{ (by ML)}\\ &= (\beta_S*\alpha_S)*(\gamma_S*\alpha_S) \text{ (by PML)}\\ &= (\alpha_S*\gamma_S)*(\beta_S*\alpha_S) \text{ (by Prop.2.2)}\\ &= \alpha_S*(\gamma_S*\beta_S), \text{ (by LAD)}\\ &= \alpha_S*(\gamma_S*\beta_S).\end{aligned}$$

Hence, when $S(S)$ is LAD, then it is RC.

Now, let us show that if $S(S)$ is LAD, then it is LC.

$$\begin{aligned}
(\alpha_S * \beta_S) * \gamma_S &= (\alpha_S * \beta_S) * \gamma_S, \text{ (by assoc.)} \\
&= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by LAD)} \\
&= (\alpha_S * \gamma_S) * (\beta_S * \alpha_S) \text{ (by BC law)} \\
&= \alpha_S * (\gamma_S * \beta_S) \text{ (by LAD)} \\
&= (\beta_S * \gamma_S) * \alpha_S \text{ (by Lem. 3.1)} \\
&= \beta_S * (\gamma_S * \alpha_S) \text{ (by assoc.)} \\
&= (\beta_S * \gamma_S) * (\alpha_S * \beta_S) \text{ (by LAD)} \\
&= (\beta_S * \alpha_S) * (\gamma_S * \beta_S) \text{ (by Prop. 2.2)} \\
&= \beta_S * (\alpha_S * \gamma_S) \text{ (by LAD)} \\
&= (\beta_S * \alpha_S) * \gamma_S \text{ (by assoc.)} \\
&= (\beta_S * \alpha_S) * \gamma_S.
\end{aligned}$$

This shows that If $S(S)$ is LAD-AG-groupoid, the it is LC-AG-groupoid.

8.

$$\begin{aligned}
\alpha_S * (\beta_S * \gamma_S) &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by LAD)} \\
&= (\alpha_S * \gamma_S) * (\beta_S * \alpha_S) \text{ (by Prop. 2.2)} \\
&= \alpha_S * (\gamma_S * \beta_S), \text{ (by LAD)} \\
&= (\beta_S * \gamma_S) * \alpha_S, \text{ (by Lem. 3.1)} \\
&= \beta_S * (\gamma_S * \alpha_S), \text{ (by assoc.)} \\
&= (\beta_S * \gamma_S) * \alpha_S, \text{ (by assoc.)} \\
&= (\beta_S * \gamma_S) * \alpha_S.
\end{aligned}$$

Thus, $S(S)$ with LAD property is Stein AG-groupoid.

9.

$$\begin{aligned}
(\alpha_S * \beta_S) * \gamma_S &= \alpha_S * (\beta_S * \gamma_S) \text{ (by assoc.)} \\
&= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by LAD)} \\
&= (\alpha_S * \gamma_S) * (\beta_S * \alpha_S) \text{ (by BC law)} \\
&= \alpha_S * (\gamma_S * \beta_S) \text{ (by LAD)} \\
&= (\alpha_S * \gamma_S) * \beta_S \text{ (by assoc.)} \\
&= (\alpha_S * \gamma_S) * \beta_S.
\end{aligned}$$

Hence, $S(S)$ with LAD property is RP^*-AG-groupoid.

5. Properties of Right Abelian Distributive AG-groupoid

For an AG-groupoid S it has been depicted in Example 4.4 that $(S(A), *)$ is not LAD over the SI-product operation. We also provide an example to show that $(S(A), *)$ is not SI-RAD

Example 5.1 *Let $A = \{x, y, z\}$ be an AG-groupoid defined by the following table and α_A, β_A and γ_A be soft sets over $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then, $S(A)$ is not an SI-RAD AG-groupoid.*

$*$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

Consider three soft sets on A over $U = \{1,2,3,4,5,6,7,8\}$ *given as follows:*

$$\begin{aligned}\alpha_A &= \{(x,\{1,2,5\}),(y,\{2,3,7\}),(z,\{1,7,8\})\}\\ \beta_A &= \{(x,\{4,5,6\}),(y,\{3,7,8\}),(z,\{2,3,7,8\})\}\\ \gamma_A &= \{(x,\{1,2,3\}),(y,\{3,5,8\}),(z,\{1,3,8\})\}\end{aligned}$$

Then, $(\beta_A*\gamma_A)(x) = \{3,8\}$, $(\gamma_A*\alpha_A)(x) = \{2,3,8\}$, $(\alpha_A*\beta_A)(x) = \{2,3,5,7,8\}$, $(\alpha_A*\beta_A)(y) = \{3,7,8\}$, $(\alpha_A*\beta_A)(z) = \{3\}$ $(\beta_A*\gamma_A)(x) = \{3,8\}$, $(\beta_A*\gamma_A)(y) = \{3,5,8\}$, $(\beta_A*\gamma_A)(z) = \{2,3,8\}$.

We show that $(S(A),*)$ *is not RAD, i.e. it does not satisfy the condition* $(\alpha_A*\beta_A)*\gamma_A = (\gamma_A*\alpha_A)*(\beta_A*\gamma_A)$. *Since* $((\alpha_A*\gamma_A)*\gamma_A)(z) = \bigcup_{z=pq}\{(\alpha_A*\beta_A)(p)\cap\gamma_A(q)\}$.

Wherein, z appears in the table as,

$$\begin{aligned}(\alpha_A*\beta_A)*\gamma_A(z) &= \bigcup_{z=pq}\{(\alpha_A*\beta_A)(p)\cap\gamma_A(q)\}\\ &= \{(\alpha_A*\beta_A)(x)\cap\gamma_A(z)\}\\ &\cup \{(\alpha_A*\beta_A)(y)\cap\gamma_A(y)\}\\ &\cup \{(\alpha_A*\beta_A)(z)\cap\gamma_A(x)\}\\ &= \{\{2,3,5,7,8\}\cap\{1,3,8\}\}\\ &\cup \{\{3,7,8\}\cap\{3,5,8\}\}\cup\{\{3\}\cap\{1,2,3\}\}\\ &= \{3,8\}\cup\{3,8\}\cup\{3\} = \{3,8\}.\end{aligned}$$

Further since,
$(\alpha_A*\beta_A)(x) = \{2,3,5,7,8\}$, $(\gamma_A*\alpha_A)(x) = \{2,3,8\}$, $(\alpha_A*\beta_A)(y) = \{3,7,8\}$, $(\gamma_A*\alpha_A)(z) = \{1,3\}$, $(\alpha_A*\beta_A)(z) = \{3\}$, $(\gamma_A*\alpha_A)(y) = \{1,2,3,5,8\}$. *Thus,*

$$\begin{aligned}&((\gamma_A*\alpha_A)*(\beta_A*\gamma_A))(z)\\ &= \bigcup_{z=pq}\{(\gamma_A*\alpha_A)(p)\cap(\beta_A*\gamma_A)(q)\}\\ &= \{(\gamma_A*\alpha_A)(x)\cap(\beta_A*\gamma_A)(z)\}\\ &\cup \{(\gamma_A*\alpha_A)(y)\cap(\beta_A*\gamma_A)(y)\}\\ &\cup \{(\gamma_A*\alpha_A)(z)\cap(\beta_A*\gamma_A)(x)\}\\ &= \{\{2,3,8\}\cap\{2,3,8\}\}\cup\{\{1,2,3,5,8\}\cap\{3,5,8\}\}\\ &\cup \{\{1,3\}\cap\{3,8\}\}\\ &= \{2,3,8\}\cup\{3,5,8\}\cup\{3\} = \{2,3,5,8\}.\end{aligned}$$

Thus, $(\alpha_A*\beta_A)*\gamma_A \neq (\gamma_A*\alpha_A)*(\beta_A*\gamma_A)$.

Proposition 5.1 *Let A be LC-AG-groupoid. Then, S(A) is LC-AG-groupoid.*

Proof. From [4], we know that RAD-AG-groupoid is LC-AG-groupoid. To show that $S(A)$ is a LC-AG-groupoid, we should prove that for all $\alpha_A,\beta_A,\gamma_A\in S(A)$ and $x\in A$ $((\alpha_A*\beta_A)*\gamma_A)(x) = ((\beta_A*\alpha_A)*\gamma_A)$. We consider that if x does not appears as a product of two elements in A, then $(\alpha_A*\beta_A)(x) = (\beta_A*\alpha_A)(x) = \emptyset$. Let $x = pq$, for $p,q\in A$. Then,

$$\begin{aligned}
((\alpha_A * \beta_A) * \gamma_A)(x) &= \bigcup_{x=pq} \{(\alpha_A * \beta_A)(p) \cap (\gamma_A)(q)\} \\
&= \bigcup\nolimits_{x=pq} \{\cup_{p=rs} \{\alpha_A(r) \cap \beta_A(s)\} \cap (\gamma_A(q))\} \\
&= \bigcup\nolimits_{x=(rs)q} \{(\alpha_A(r) \cap \beta_A(s)) \cap \gamma_A(q)\} \\
&= \bigcup\nolimits_{x=(sr)q} \{(\beta_A(s) \cap \alpha_A(r)) \cap \gamma_A(q)\} \\
&= \bigcup\nolimits_{x=pq} \{\cup_{p=sr} (\beta_A * \alpha_A)(p) \cap \gamma_A(q)\} \\
&= \bigcup\nolimits_{x=pq} \{(\beta_A * \alpha_A)(p) \cap \gamma_A(q)\} \\
&= \bigcup\nolimits_{x=pq} \{((\beta_A * \alpha_A) * \gamma_A)(x)\} \\
&= ((\beta_A * \alpha_A) * \gamma_A)(x)
\end{aligned}$$

Proposition 5.2 *Let A be an RAD-AG-groupoid. Then, $S(A)$ is RD-AG-groupoid.*

Proof. From [4] we know that if A is RAD, then it is RD. Thus, for any $\alpha_A, \beta_A, \gamma_A$ we have to show that $((\alpha_A * \beta_A) * \gamma_A)(x) = ((\alpha_A * \gamma_A) * (\beta_A * \gamma_A))(x)$. Then,

$$\begin{aligned}
((\alpha_A * \beta_A) * \gamma_A)(x) &= \bigcup_{x=pq} \{(\alpha_A * \beta_A)(p) \cap \gamma_A(q)\} \\
&= \bigcup\nolimits_{x=pq} \{(\cup_{p=rs} \{\alpha_A(r) \cap \beta_A(s)\}) \cap \gamma_A(q)\} \\
&= \bigcup\nolimits_{x=(rs)q} \{(\{\alpha_A(r) \cap \beta_A(s)\}) \cap \gamma_A(q)\} \\
&= \bigcup\nolimits_{x=(rs)q} \{(\alpha_A(r) \cap \beta_A(s)) \cap (\{\gamma_A(q) \cap \gamma_A(q)\})\} \\
&= \bigcup\nolimits_{x=(rq)(sq)} \{\alpha_A(r) \cap \gamma_A(q) \cap (\{\beta_A(s) \cap \gamma_A(q)\})\} \\
&= \bigcup\nolimits_{x=(rq)(sq)} \{\{\alpha_A(r) \cap \gamma_A(q)\} \cap (\{\beta_A(s) \cap \gamma_A(q)\})\} \\
&= \bigcup\nolimits_{x=uv} \{\{\cup_{u=rq} (\alpha_A * \gamma_A)(q)\} \cap \{\cup_{v=sq} (\beta_A * \gamma_A)(v)\}\} \\
&= \bigcup\nolimits_{x=uv} ((\alpha_A * \gamma_A) * (\beta_A * \gamma_A))(x) \\
&= ((\alpha_A * \gamma_A) * (\beta_A * \gamma_A))(x).
\end{aligned}$$

Hence, $(\alpha_A * \beta_A) * \gamma_A = (\alpha_A * \gamma_A) * (\beta_A * \gamma_A)$.

As by Example 5.1, $S(A)$ is not RAD under the SI-product operation. However, it does hold under the following conditions as mentioned below.

Theorem 5.2 *Let $S(S)$ satisfy at least one of the following properties over an AG-groupid S. Then $S(S)$ is necessarily an RAD-AG-groupid.*

1. *$S(S)$ is LAD*
2. *$S(S)$ is LD*
3. *$S(S)$ is RD*
4. *$S(S)$ is slim*

5. *$S(S)$ is idempotent*

Proof. Let $\alpha_S, \beta_S, \gamma_S$ and δ_S be any elements of $S(S)$. Then,

1. Let $S(S)$ be LAD. Then,

$$\begin{aligned}(\alpha_S * \beta_S) * \gamma_S &= \alpha_S * (\beta_S * \gamma_S) \text{ (by assoc.)}\\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by LAD)}\\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by PML)}\\ &= \{(\gamma_S * \alpha_S) * \beta_S\} * \alpha_S \text{ (by LIL)}\\ &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S) \text{ (by assoc.)}\\ &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S).\end{aligned}$$

2. Let $S(S)$ be LD. Then,

$$\begin{aligned}(\alpha_S * \beta_S) * \gamma_S &= \alpha_S * (\beta_S * \gamma_S) \text{ (by assoc.)}\\ &= (\gamma_S * \beta_S) * \alpha_S \quad \text{(by Lem. 2.1)}\\ &= \gamma_S * (\beta_S * \alpha_S) \quad \text{(by assoc.)}\\ &= (\gamma_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by LD)}\\ &= \{(\gamma_S * \alpha_S) * \beta_S\} * \gamma_S \text{(by LIL)}\\ &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S) \text{ (by assoc.)}\\ (\alpha_S * \beta_S) * \gamma_S &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S).\end{aligned}$$

3. Let $S(S)$ be RD. Then,

$$\begin{aligned}(\alpha_S * \beta_S) * \gamma_S &= \alpha_S * (\beta_S * \gamma_S) \text{ (by assoc.)}\\ &= (\gamma_S * \beta_S) * \alpha_S \text{ (by Lem. 2.1)}\\ &= (\gamma_S * \alpha_S) * (\beta_S * \alpha_S) \text{ (by RD)}\\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by Lem. 3.1)}\\ &= (\alpha_S * \beta_S) * (\gamma_S * \alpha_S).\end{aligned}$$

4. Let $S(S)$ be Slim. Then,

$$\begin{aligned}(\alpha_S * \beta_S) * \gamma_S &= \alpha_S * (\beta_S * \gamma_S) \text{ (by assoc.)}\\ &= \alpha_S * \{\gamma_S * (\beta_S * \gamma_S)\} \text{ (by assump.)}\\ &= \{(\beta_S * \gamma_S) * \gamma_S\} * \alpha_S \text{ (by Lem. 2.1)}\\ &= (\alpha_S * \gamma_S) * (\beta_S * \gamma_S) \text{ (by LIL)}\\ &= (\alpha_S * \beta_S) * (\gamma_S * \gamma_S) \text{ (by ML)}\\ &= (\gamma_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by PML)}\\ &= \{(\gamma_S * \alpha_S) * \beta_A\} * \gamma_S \text{ (by LIL)}\\ &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S) \text{ (by assoc.)}\\ &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S).\end{aligned}$$

Furthermore, $S(S)$ is commutative as;

$$\begin{aligned}\alpha_S * \beta_S &= (\alpha_S * \gamma_S) * \beta_S \text{ (by assump.)}\\ &= (\beta_S * \gamma_S) * \alpha_S \text{ (by LIL)}\\ &= \beta_S * \alpha_S \text{ (by assump.)}\\ &= \beta_S * \alpha_S.\end{aligned}$$

5. Let $S(S)$ be an idempotent. Then,

$$\begin{aligned}
\alpha_S * \beta_S &= (\alpha_S * \alpha_S) * \beta_S \text{ (by assump.)} \\
&= (\beta_S * \alpha_S) * \alpha_S \text{ (by LIL)} \\
&= \beta_S * (\alpha_S * \alpha_S) \text{ (by assoc.)} \\
&= \beta_S * \alpha_S \text{ (by assump.)} \\
&= \beta_S * \alpha_S.
\end{aligned}$$

Now, we prove that $S(S)$ is RAD;

$$\begin{aligned}
(\alpha_S * \beta_S) * \gamma_S &= \alpha_S * (\beta_S * \gamma_S) \text{ (by assoc.)} \\
&= (\gamma_S * \beta_S) * \alpha_S \text{ (by Lem. 2.1)} \\
&= \gamma_S * (\beta_S * \alpha_S) \text{ (by assoc.)} \\
&= (\gamma_S * \gamma_S) * (\beta_S * \alpha_S) \text{ (by assump.)} \\
&= (\gamma_S * \beta_S) * (\gamma_S * \alpha_S) \text{ (by ML)} \\
&= \{(\gamma_S * \alpha_S) * \beta_S\} * \gamma_S \text{ (by LIL)} \\
&= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S) \text{ (by assoc.)} \\
&= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S).
\end{aligned}$$

Next, we give an alternative characterization of $S(S)$ that subclasses emerge, when it posses the RAD property. Note that this does not happen in the usual case of AG-groupoids but it only holds in $S(S)$ under the SI-product of soft sets. It means the application of SI-product to the structure of AG-groupoids has a unique impact on its characteristics. We provide some counterexamples to depict the fact that not every AG-groupoid equipped with the property of RAD is (i). AG*; (ii), AG** and (iii). CA-AG-groupoid. However, this rises in the case of $S(S)$, the unique subclass of the AG-groupoids under the SI-products of soft sets as characterized in the following theorem.

(i)

$\cdot$	1	2	3
1	2	2	2
2	3	3	3
3	3	3	3

(ii)

$*$	1	2	3
1	1	1	1
2	3	1	1
3	1	1	1

(iii)

$\circ$	1	2	3
1	1	1	1
2	3	1	1
3	1	1	1

Theorem 5.3 *For an AG-groupoid S, let $S(S)$ satisfy the RAD property. Then, any of the following hold.*

1. $S(S)$ is an AG*,
2. $S(S)$ is an AG**,
3. $S(S)$ is Left distributive (LD),
4. $S(S)$ is right distributive (RD),
5. $S(S)$ is cyclic associative (CA),
6. $S(S)$ is an LAD,
7. $S(S)$ is an LC and RC,
8. $S(S)$ is Stein,

9. $S(S)$ is an RP*.

Proof. Let $\alpha_S, \beta_S, \gamma_S$ be any elements of $S(S)$ and the RAD property, $(\alpha_S * \beta_S)*\gamma_S = (\gamma_S * \alpha_S)*(\beta_S * \gamma_S)$ holds in $S(S)$. Then,

1.

$$\begin{aligned}(\alpha_S * \beta_S) * \gamma_S &= (\gamma_S * \alpha_S) \circ (\beta_S * \gamma_S) \ \text{(by RAD)}\\ &= (\gamma_S * \beta_S) * (\alpha_S * \gamma_S) \ \text{(by ML)}\\ &= (\beta_S * \alpha_S) * \gamma_S \ \text{(by RAD)}\\ &= \beta_S * (\alpha_S * \gamma_S) \ \text{(by assoc.)}\end{aligned}$$

Hence, $S(S)$ is AG^*-groupoid.

2.

$$\begin{aligned}\alpha_S * (\beta_S * \gamma_S) &= (\alpha_S * \beta_S) * \gamma_S \ \text{(by assoc.)}\\ &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S) \ \text{(by RAD)}\\ &= (\gamma_S * \beta_S) * (\alpha_S * \gamma_S) \ \text{(by Prop. 2.2)}\\ &= (\beta_S * \alpha_S) * \gamma_S \ \text{(by RAD)}\\ &= \beta_S * (\alpha_S * \gamma_S) \ \text{(by assoc.)}\end{aligned}$$

Thus, $S(S)$ is AG^{**}-groupoid

3.

$$\begin{aligned}\alpha_S * (\beta_S * \gamma_S) &= (\gamma_S * \beta_S) * \alpha_S \ \text{(by Lem. 3.1)}\\ &= (\alpha_S * \gamma_S) * (\beta_S * \alpha_S) \ \text{(by RAD)}\\ &= (\alpha_S * \beta_S) * (\alpha_S * \gamma_S) \ \text{(by Prop. 2.2)}\end{aligned}$$

It is shown that if $S(S)$ is RAD-AG-groupoid, then it is LD-AG-groupoid.

4.

$$\begin{aligned}(\alpha_S * \beta_S) * \gamma_S &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S) \ \text{(by RAD)}\\ &= (\gamma_S * \beta_S) * (\alpha_S * \gamma_S) \ \text{(by ML)}\\ &= \{(\alpha_S * \gamma_S) * \beta_S\} * \gamma_S \ \text{(by LIL)}\\ &= (\alpha_S * \gamma_S) * (\beta_S * \gamma_S) \ \text{(by assoc.)}\end{aligned}$$

Thus, $S(S)$ is RD-AG-groupoid.

5.

$$\begin{aligned}\alpha_S * (\beta_S * \gamma_S) &= (\alpha_S * \beta_S) * \gamma_S \ \text{(by assoc.)}\\ &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S) \ \text{(by RAD)}\\ &= (\gamma_S * \beta_S) * (\gamma_S * \alpha_S) \ \text{(by Prop. 2.2)}\\ &= (\gamma_S * \beta_S) * (\alpha_S * \gamma_S) \ \text{(by RC)}\\ &= (\beta_S * \alpha_S) * \gamma_S \ \text{(by RAD)}\\ &= (\gamma_S * \alpha_S) * \beta_S \ \text{(by LIL)}\\ &= \gamma_S * (\alpha_S * \beta_S) \ \text{(by assoc.)}\end{aligned}$$

$S(S)$ is CA-AG-groupoid.

6.

$$\begin{aligned}\alpha_S*(\beta_S*\gamma_S) &= (\gamma_S*\beta_S)*\alpha_S \text{ (by Lem. 3.1)}\\ &= (\alpha_S*\gamma_S)*(\beta_S*\alpha_S) \text{ (by RAD)}\\ &= (\alpha_S*\beta_S)*(\gamma_S*\alpha_S) \text{ (by Prop. 2.2)}\end{aligned}$$

Hence, it is seen that if $S(S)$ is RAD-AG-groupoid, it is LAD-AG-groupoid.

7.

$$\begin{aligned}(\alpha_S*\beta_S)*\gamma_S &= (\gamma_S*\alpha_S)*(\beta_S*\gamma_S) \text{ (by RAD)}\\ &= (\gamma_S*\beta_S)*(\alpha_S*\gamma_S) \text{ (by ML)}\\ &= \{(\alpha_S*\gamma_S)*\beta_S\}*\gamma_S \text{ (by LIL)}\\ &= \{(\beta_S\circ\gamma_S)*\alpha_S\}*\gamma_S \text{ (by LIL)}\\ &= (\beta_S\circ\gamma_S)*(\alpha_S*\gamma_S) \text{ (by assoc.)}\\ &= (\beta_S*\alpha_S)*(\gamma_S*\gamma_S) \text{ (by ML)}\\ &= (\gamma_S*\alpha_S)*(\gamma_S*\beta_S) \text{ (by PML)}\\ &= \{(\gamma_S*\beta_S)*\alpha_S\}*\gamma_S \text{ (by LIL)}\\ &= (\gamma_S*\beta_S)*(\alpha_S*\gamma_S) \text{ (by assoc.)}\\ &= (\beta_S*\alpha_S)*\gamma_S \text{ (by RAD.)}\end{aligned}$$

It is cocluded that If $S(S)$ is RAD-AG-groupoid, then it is LC-AG-groupoid. Further,

$$\begin{aligned}\alpha_S*(\beta_S*\gamma_S) &= (\alpha_S*\beta_S)*\gamma_S \text{ (by assoc.)}\\ &= (\beta_S*\alpha_S)*\gamma_S \text{ (by LC)}\\ &= \alpha_S*(\beta_S*\gamma_S) \text{ (by AG*)}\\ &= \alpha_S*(\gamma_S*\beta_S) \text{ (by LC)}\end{aligned}$$

It is cocluded that If $S(S)$ is RAD-AG-groupoid, then it is RC-AG-groupoid.

8.

$$\begin{aligned}\alpha_S*(\beta_S*\gamma_S) &= (\alpha_S*\beta_S)*\gamma_S \text{ (by assoc.)}\\ &= (\gamma_S*\alpha_S)*(\beta_S*\gamma_S) \text{ (by RAD)}\\ &= (\gamma_S*\beta_S)*(\alpha_S*\gamma_S) \text{ (by ML)}\\ &= (\beta_S*\alpha_S)*\gamma_S \text{ (by RAD)}\\ &= (\gamma_S*\alpha_S)*\beta_S \text{ (by LIL)}\\ &= \gamma_S*(\alpha_S*\beta_S) \text{ (by assoc.)}\\ &= (\beta_S*\alpha_S)*\gamma_S \text{ (by Lem. 3.1)}\\ &= \alpha_S*(\beta_S*\gamma_S) \text{ (by AG*)}\\ &= (\gamma_S*\beta_S)*\alpha_S \text{ (by Lem. 3.1)}\\ &= (\beta_S*\gamma_S)*\alpha_S \text{ (by LC)}\end{aligned}$$

Hence, $S(S)$ satisfying RAD-AG-groupoid properties is Stein AG-groupoid.

9.

$$\begin{aligned}(\alpha_S * \beta_S) * \gamma_S &= (\gamma_S * \alpha_S) * (\beta_S * \gamma_S) \text{ (by RAD)}\\ &= (\gamma_S * \beta_S) * (\alpha_S * \gamma_S) \text{ (by ML)}\\ &= (\beta_S * \alpha_S) * \gamma_S \text{ (by RAD)}\\ &= (\gamma_S * \alpha_S) * \beta_S \text{ (by LIL)}\\ &= (\alpha_S * \gamma_S) * \beta_S \text{ (by LC)}\end{aligned}$$

Thus, $S(S)$ satisfying RAD-AG-groupoid property is RP*-AG-groupoid.

6. Conclusion

This paper has centered on the concept and application of the SI-product for soft sets over AG-groupoids. It has successfully rectified and improved upon various published results. Including alternative, concise, and general proofs has made understanding these results more accessible. Additionally, the construction and inclusion of examples and counterexamples have validated the obtained results and supported the corrections and improvements made to the published work. The investigation that has been conducted in this paper relies on the utilization of modern techniques such as Prover-9 and Mace-4 to generate counterexamples, which proved to be valuable tools for further exploration and investigation in this field.

Acknowledgment: The authors express their gratitude to the anonymous reviewers for their valuable insights and expert suggestions that greatly contributed to the enhancement of this paper.

References

1. F. Abdunabi and A. Shliteite, Approximations of the Prime Soft Ideal and Maximal Soft Ideal of Soft Semirings, *Lybian Journal of Basic Sciences* **14** (2022), 169-172.
2. U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, *Computers and Mathematics with Applications* **59** (2010), 3458-3463.
3. I. Ahmad, M. Rashad and M. , Some properties of AG-groupoids, *Research Journal of Recent Sciences* **2(4)** (2013), 91-93.
4. I. Ahmad, S.-Rahman M Iqbal, A Note on Left Abelian Distributive LA-semigroups *Punjab University Journal of Mathematics* **52(1)** (2020), 47-63
5. M.I. Ali, F. Feng, X. Liu, W.K. Min and M. Shabir, On some new operations in soft set theory, *Computers and Mathematics with Applications* **57** (2009), 1547–1553.
6. H. Aktaş and N. Çağman, Soft sets and soft groups, *Information Sciences* **177** (2007), 2726-2735.
7. A.O. Atagün and H. Kamacı, Set-generated soft subrings of rings, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics* **71(4)** (2022), 993-1006.
8. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986), 87-96.
9. E. Aygün and H. Kamacı, Some new algebraic structures of soft sets, *Soft Computing* **25** (2021), 8609–8626.
10. S. Ayub, M. Shabir and W. Mahmood, New types of soft rough sets in groups based on normal soft groups. *Computer Mathematics with Application* **39** (2020), 67.
11. R. Barzegar S.B. Hosseini and N. Çağman, Second type nilpotent soft subgroups, *Afrika Matematika* **34(1)** (2023), 17.
12. N. Çağman and S. Enginoğlu, Soft set theory and uni-int decision making, *European Journal of Operational Research* **207** (2010), 848–855.
13. N. Çağman, F. Çitak and H. Aktaş, Soft int-group, *Neural Computing and Applications* **21(1)** (2012), 151-158.
14. N. Çağman, Contributions to the theory of soft sets, *Journal of New Results in Science* **3(4)** (2014), 33-41.
15. F. Çítak, Soft k-uni ideals of semirings and its algebraic applications, *Journal of the Institute of Science and Technology* **8(4)** (2018), 281-294.
16. F. Feng, X. Liu, V. Leoreanu-Fotea and Y.B. Jun, Soft sets and soft rough sets. *Information Sciences* **181(6)** (2011), 1125-1137.

17. S. Golda and S.A. Ray, Classical view of Soft Ring and Soft Ideal, *Journal of Applied Science and Computation*, **6(3)** (2019), 1457-1467.

18. M. Iqbal, I. Ahmad, M. Shah, M.I Ali, On cyclic associative Abel-Grassman groupoids. (2015) arXiv preprint arXiv:1510.01316.

19. N. Jan, M.I. Siddique and T. Mahmood, On Double Framed Soft Rings, *Technical Journal* **25(02)** (2020), 126-131.

20. C. Jana, M. Pal, F. Karaaslan and A. Sezgin, (α, β)-Soft Intersectional Rings and Ideals with their Applications, *New Mathematics and Natural Computation* **15(02)** (2019), 333-350.

21. M. John and D. Susha, Generalized Soft Groups, *Journal of Computer and Mathematical Sciences* **10(6)** (2019), 1302-1313.

22. F. Karaaslan, Some properties of AG $*$-groupoids and AG-bands under SI-product operation, *Journal of Intelligent and Fuzzy Systems* **36(1)** (2019), 231-239.

23. K. Kaygisiz, On soft int-groups, *Annals of Fuzzy Mathematics and Informatics* **4(2)** (2012), 363-375.

24. M.A. Kazim and M. Naseeruddin, On almost semigroups, *The Aligarh Bulletin of Mathematics* **2** (1972), 1-7.

25. P.K. Maji, A.R. Roy and R. Biswas, An application of soft sets in a decision making problem, *Computers and Mathematics with Applications* **44** (2002), 1077-1083.

26. P.K. Maji, R. Biswas and A.R. Roy, Soft set theory. *Computers and Mathematics with Applications*, **45(4-5)** (2003), 555-562.

27. C. Maheswari, P.V. Ravi, N. V. Murthy and V. Padmavathi, From Soft Point Semigroups to Soft Semigroups, *In Journal of Physics: Conference Series* **2332(1)** (2022), 012005.

28. T. Mahmood, Z.U. Rehman and A. Sezgin, Lattice ordered soft near rings, *Korean Journal of Mathematics* **26(3)** (2018), 503-517.

29. D.A. Molodtsov, Soft set theory-first results, *Computers and Mathematics with Applications* **37** (1999), 19-31.

30. Q. Mushtaq and M. Khan, Ideals in AG-band and AG-groupoid, *Quasigroups and Related Systems* **14** (2006), 207–215.

31. S. Öztunç, S. Aslan and H. Dutta, Categorical structures of soft groups, *Soft Computing* **25** (2021), 3059-3064.

32. P.V. Protic and N. Stevanovic, AG-test and some general properties of Abel- Grassmann's groupoids, *Pure Mathematics and Applications* **4(6)** (1995), 371-383.

33. M. Rashad, I. Ahmad and M. Shah, On relation between right alternative and nuclear square AG-groupoids, *International Journal of Mathematical Forum* **8(5)** (2013), 542 237-243.

34. A. Rosenfeld, Fuzzy group, *Journal of Mathematical Analysis and Applications* **35** (1971), 512-517.

35. M. Şahin and V. Uluçay, Soft Maximal Ideals on Soft Normed Rings, *Quadruple Neutrosophic Theory And Applications* **1** (2020), 203.

36. A. Sezgin and A.O. Atagün, On operations of soft sets, *Computers and Mathematics with Applications* **61** (2011), 1457-1467.

37. A. Sezgin and A.O. Atagün, Soft groups and normalistic soft groups, *Computers and Mathematics with Applications* **62(2)** (2011), 685-698.

38. A. Sezgin Sezer, A new approach to LA-semigroup theory via the soft sets, *Journal of Intelligent and Fuzzy Systems* **26** (2014), 2483-2495.

39. A. Sezgin Sezer, N. Çağman and A.O. Atagün, Uni-soft substructures of groups, *Annals of Fuzzy Mathematics and Informatics* **9(2)** (2015), 235-246.

40. A. Sezgin, N. Çağman, Z.K. Türk and E. Muştuoğlu, Soft uni-group and its applications to group theory. *Algebra Letters*, **2018(2)** (2018), 1-20.

41. M. Shah, I. Ahmad and A. Ali, Discovery of new classes of AG-groupoids, *Research Journal of Recent Sciences* **1(11)** (2012), 47-49.

42. M. Shah, A. Ali and I. Ahmad, On introduction of new classes of AG-groupoids, *Research Journal of Recent Sciences* **2(1)** (2013), 67-70.

43. A. Ullah, F. Karaaslan and I. Ahmad, Soft Uni-Abel Grassmann's groups, *European Journal of Pure and Applied Mathematics* **11(2)** (2018), 517-536.

44. A. Ullah, I. Ahmad, F. Hayat, F. Karaaslan and M. Rashad, Soft intersection Abel-Grassmann's groups, *Journal of Hyperstructures* **7(2)** (2018), 149-173.

45. L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.

46. J., N. Çağman and A. Sezgin Sezer, Applications of soft union sets to hemirings via SU-h-ideals, *Journal of Intelligent and Fuzzy Systems* **26(3)** (2014), 1363-1370.

47. X. Zhang, M.A. Yingcang and P. Yu, On two conjectures of Abel Grassmann's groupoids, *Symmetry* **11(6)**, 816 (2019).

48. K. Zhu and Y. Lv, A novel study on soft rough rings (ideals) over rings. *Italian Journal of Pure and Applied Mathematics*, **45** (2021), 190-204.

Bashar Khan,
Department of Mathematics,
University of Malakand,
Pakistan.
E-mail address: basharkhanaboha@gmail.com

and

Imtiaz Ahmad,
Department of Mathematics,
University of Malakand,
Pakistan.
E-mail address: iahmaad@hotmail.com

and

Aman Ullah,
Department of Mathematics,
University of Malakand,
Pakistan.
E-mail address: amanswt@hotmail.com

and

Faruk Karaaslan,
Department of Mathematics,
Cankiri Karatekin University,
Turkey.
E-mail address: fkaraaslan@karatekin.edu.tr

and

Mohammed Rashad,
Department of Mathematics,
University of Malakand,
Pakistan.
E-mail address: rashad@uom.edu.pk