Bol. Soc. Paran. Mat. (3s.) **v. 2025 (43)** : 1–11.
©SPM – E-ISSN-2175-1188 ISSN-0037-8712
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.69521

Inner higher derivations on Hilbert C*-modules

Sayed Khalil Ekrami

ABSTRACT: Let $\mathcal{M}$ be a Hilbert C*-module. We define a product $\pi_{\mathbf{e}}$ on $\mathcal{M}$, making $\mathcal{M}$ into a Banach algebra. Then we study inner derivations and inner higher derivations of Banach algebra $(\mathcal{M}, \pi_{\mathbf{e}})$. We find the general form of the family of inner derivations corresponding to an inner higher derivation on unital Banach algebra $(\mathcal{M}, \pi_{\mathbf{e}})$, with the identity element $1_{\mathcal{M}}$. We show that if $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are sequences in $\mathcal{M}$ such that $v_0 = w_0 = 1_{\mathcal{M}}$ and $(\mathbf{v} * \mathbf{w})_n = (\mathbf{w} * \mathbf{v})_n = 0$ for all $n \in \mathbb{N}$ and $\{\varphi_n\}_{n=0}^{\infty}$ is the inner higher derivation on $(\mathcal{M}, \pi_{\mathbf{e}})$ defined by

$$\varphi_n(x) = \sum_{i=0}^{n} \langle v_i, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle w_{n-i},$$

for all $x \in \mathcal{M}$ and each non-negative integer n, then the corresponding sequence of inner derivations $\{\psi_n\}_{n=1}^{\infty}$, defined on $(\mathcal{M}, \pi_{\mathbf{e}})$ by

$$\psi_n(x) = \Big\langle \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1}\ r_1 \langle v_{r_1}, \mathbf{e}\rangle\langle v_{r_2}, \mathbf{e}\rangle \ldots \langle v_{r_{i-1}}, \mathbf{e}\rangle v_{r_i} \Big), \mathbf{e} \Big\rangle x$$

$$+ \langle x, \mathbf{e}\rangle \sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1}\ r_i \langle w_{r_1}, \mathbf{e}\rangle\langle w_{r_2}, \mathbf{e}\rangle \ldots \langle w_{r_{i-1}}, \mathbf{e}\rangle w_{r_i} \Big),$$

for all $x \in \mathcal{M}$ and all $n \in \mathbb{N}$.

Key Words: Derivation, inner derivation, higher derivation, Hilbert C*-module.

Contents

1. Introduction

A derivation of an algebra A is a linear mapping δ from A into itself such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$. For a fixed $a \in A$, the mapping $\delta_a(x) = ax - xa$ is clearly a derivation, which is called an inner derivation implemented by a. A sequence of linear mappings $\{d_n : A \to A\}_{n=0}^{\infty}$ with $d_0 = I$ (the identity mapping on A), is called a higher derivation on A, if it satisfies the equation $d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$ for all $a, b \in A$ and each non-negative integer n. Mirzavaziri defined the concept of inner higher derivations on algebras in [9] as a sequence of linear mappings $\{d_n : A \to A\}_{n=0}^{\infty}$ satisfying $(n+1)d_n(x) = \sum_{i=0}^{n} a_{i+1}d_{n-i}(x) - d_{n-i}(x)a_{i+1}$ for each $n = 0, 1, 2, \ldots$ and $x \in A$ in which $\{a_n\}_{n=1}^{\infty}$ is a sequence in A and then gave a characterization for inner higher derivations on a torsion free algebra A. He showed that each higher derivation on A is inner provided that each derivation on A is inner. For more information about derivations and higher derivations on algebras, the reader refer to [1,2,4,5,6,7,10,11,12,13].

In this paper, we discuss about inner higher derivations on Hilbert C*-modules. The notion of a Hilbert C*-module is a generalization of a Hilbert space in which the inner product takes its values in a C*-algebra (see [8]). Let $\mathcal{A}$ be a C*-algebra and $\mathcal{M}$ be a complex linear space which is a left $\mathcal{A}$-module with a compatible scalar multiplication. The space $\mathcal{M}$ is called a left inner product $\mathcal{A}$-module, if there is an $\mathcal{A}$-valued inner product $(x, y) \mapsto \langle x, y\rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ such that for each $x, y, z \in \mathcal{M}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$,

(i) $\langle x, x\rangle \geq 0$ and the equality holds if and only if $x = 0$,

Submitted September 06, 2023. Published March 18, 2025
2010 *Mathematics Subject Classification:* 16W25, 46L57, 47B47, 46H40.

(ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$,

(iii) $\langle ax, y\rangle = a\langle x, y\rangle$,

(iv) $\langle x, y\rangle^* = \langle y, x\rangle$.

It follows from the conditions (i)-(iv) that, $\langle x, x\rangle$ is a positive element in C*-algebra $\mathcal{A}$, the inner product is conjugate-linear in its second variable and $\langle x, ay\rangle = \langle x, y\rangle a^*$ for all $x, y \in \mathcal{M}, a \in \mathcal{A}$. An inner product $\mathcal{A}$-module $\mathcal{M}$ which is complete with respect to the norm $\|x\|_{\mathcal{M}} = \|\langle x, x\rangle\|_{\mathcal{A}}^{\frac{1}{2}}$ is called a Hilbert $\mathcal{A}$-module or a Hilbert C*-module over the C*-algebra $\mathcal{A}$. For example, every C*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module under the $\mathcal{A}$-valued inner product $\langle a, b\rangle = ab^*$ $(a, b \in \mathcal{A})$. Every complex Hilbert space is a left Hilbert $\mathbb{C}$-module. The notion of a right Hilbert $\mathcal{A}$-module can be defined similarly.

Let $\mathcal{M}$ be a (left) Hilbert C*-module over a C*-algebra $\mathcal{A}$ and I be the identity mapping on $\mathcal{M}$. Ekrami [3] showed that for any Hilbert C*-module higher derivation $\{\varphi_n\}_{n=0}^{\infty}$ on a Hilbert C*-module $\mathcal{M}$, with $\varphi_0 = I$, there exists a unique sequence of Hilbert C*-module derivations $\{\psi_n\}_{n=1}^{\infty}$ on $\mathcal{M}$ such that

$$\psi_n = \sum_{k=1}^{n}\Big(\sum_{r_1+r_2+\cdots+r_k=n} (-1)^{k-1}\ r_1\varphi_{r_1}\varphi_{r_2}\ldots\varphi_{r_k}\Big),$$

for each positive integers n, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{k} r_j = n$.

Every Hilbert C*-module $\mathcal{M}$ is a Banach space but there is no algebraic product on it. In this paper, using the $\mathcal{A}$-valued inner product defined on $\mathcal{M}\times\mathcal{M}$, we define a product $\pi_{\mathbf{e}}$ on $\mathcal{M}$, making $\mathcal{M}$ into a Banach algebra. Then we study inner derivations and inner higher derivations of Banach algebra $(\mathcal{M}, \pi_{\mathbf{e}})$. We find the general form of the family of inner derivations corresponding to an inner higher derivation. We show that if $(\mathcal{M}, \pi_{\mathbf{e}})$ is unital, with the identity element $1_{\mathcal{M}}$ and $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are sequences in $\mathcal{M}$ such that $v_0 = w_0 = 1_{\mathcal{M}}$ and $(\mathbf{v}*\mathbf{w})_n = (\mathbf{w}*\mathbf{v})_n = 0$ for all $n \in \mathbb{N}$ and $\{\varphi_n\}_{n=0}^{\infty}$ is the inner higher derivation on $(\mathcal{M}, \pi_{\mathbf{e}})$ defined by

$$\varphi_n(x) = \sum_{i=0}^{n}\langle v_i, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle w_{n-i},$$

for all $x \in \mathcal{M}$ and each non-negative integer n, then the corresponding sequence of inner derivations $\{\psi_n\}_{n=1}^{\infty}$, defined on $(\mathcal{M}, \pi_{\mathbf{e}})$ by

$$\begin{aligned}\psi_n(x) = &\Big\langle \sum_{i=1}^{n}\Big(\sum_{\sum_{j=1}^{i} r_j=n} (-1)^{i-1}\ r_1\langle v_{r_1}, \mathbf{e}\rangle\langle v_{r_2}, \mathbf{e}\rangle\ldots\langle v_{r_{i-1}}, \mathbf{e}\rangle v_{r_i}\Big), \mathbf{e}\Big\rangle x\\ &+ \langle x, \mathbf{e}\rangle\sum_{i=1}^{n}\Big(\sum_{\sum_{j=1}^{i} r_j=n} (-1)^{i-1}\ r_i\langle w_{r_1}, \mathbf{e}\rangle\langle w_{r_2}, \mathbf{e}\rangle\ldots\langle w_{r_{i-1}}, \mathbf{e}\rangle w_{r_i}\Big),\end{aligned}$$

for all $x \in \mathcal{M}$ and all $n \in \mathbb{N}$.

For more information about derivations and higher derivations on Hilbert C*-modules, the reader refer to [3,12].

2. The Results

Let $\mathcal{M}$ be a (left) Hilbert $\mathcal{A}$-module and let $\mathbf{e}$ be an arbitrary element in $\mathcal{M}$ with $\|\mathbf{e}\| = 1$. The mapping $\pi_{\mathbf{e}} : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ defind by $\pi_{\mathbf{e}}(x, y) = \langle x, \mathbf{e}\rangle y$ for all $x, y \in \mathcal{M}$, is a product on $\mathcal{M}$. Validity of equation

$$\pi_{\mathbf{e}}\big(\pi_{\mathbf{e}}(x, y)z\big) = \langle x, \mathbf{e}\rangle\langle y, \mathbf{e}\rangle z = \pi_{\mathbf{e}}\big(x, \pi_{\mathbf{e}}(y, z)\big),$$

shows that $(\mathcal{M}, \pi_{\mathbf{e}})$ is an associative algebra. With a simple calculation we can show that $(\mathcal{M}, \pi_{\mathbf{e}})$ is a Banach algebra. $(\mathcal{M}, \pi_{\mathbf{e}})$ is called unital, if there exists an element $1_{\mathcal{M}}$ in $\mathcal{M}$ such that $\langle x, \mathbf{e}\rangle 1_{\mathcal{M}} = \langle 1_{\mathcal{M}}, \mathbf{e}\rangle x = x$ for all $x \in \mathcal{M}$.

A linear mapping $\psi : (\mathcal{M}, \pi_{\mathbf{e}}) \to (\mathcal{M}, \pi_{\mathbf{e}})$ is called a derivation on $(\mathcal{M}, \pi_{\mathbf{e}})$, if it satisfies the equation

$$\psi(\langle x, \mathbf{e}\rangle y) = \langle \psi(x), \mathbf{e}\rangle y + \langle x, \mathbf{e}\rangle \psi(y),$$

for all $x, y \in \mathcal{M}$.

Example 2.1 *Let $M_2(\mathbb{C})$ be the C^*-algebra of 2×2 complex matrices and* $E = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix}$. *The mapping $\psi : (M_2(\mathbb{C}), \pi_{\boldsymbol{E}}) \to (M_2(\mathbb{C}), \pi_{\boldsymbol{E}})$ defined by*

$$\psi(X) = \begin{bmatrix} 0 & 0 \\ x_{11}+x_{21} & x_{12}+x_{22} \end{bmatrix},$$

for all $X \in M_2(\mathbb{C})$, is a derivation on $(M_2(\mathbb{C}), \pi_{\boldsymbol{E}})$.

For a fixed element $w \in \mathcal{M}$, the linear mapping $\psi_w : (\mathcal{M}, \pi_{\mathbf{e}}) \to (\mathcal{M}, \pi_{\mathbf{e}})$ defined by

$$\psi_w(x) = \langle w, \mathbf{e}\rangle x - \langle x, \mathbf{e}\rangle w,$$

for all $x \in \mathcal{M}$, is a derivation, which is called an *inner derivation* on $(\mathcal{M}, \pi_{\mathbf{e}})$.

A sequence of linear mappings $\{\varphi_n : (\mathcal{M}, \pi_{\mathbf{e}}) \to (\mathcal{M}, \pi_{\mathbf{e}})\}_{n=0}^{\infty}$ with $\varphi_0 = I$, is called a *higher derivation* on $(\mathcal{M}, \pi_{\mathbf{e}})$, if it satisfies the equation

$$\varphi_n(\langle x, \mathbf{e}\rangle y) = \sum_{i=0}^{n} \langle \varphi_i(x), \mathbf{e}\rangle \varphi_{n-i}(y), \tag{2.1}$$

for all $x, y \in \mathcal{M}$ and each non-negative integer n.

Proposition 2.1 *Let $\{\varphi_n\}_{n=0}^{\infty}$ be a sequence of linear mappings on $(\mathcal{M}, \pi_{e})$ with $\varphi_0 = I$, for which there exists a sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{M}$ such that*

$$\varphi_{n+1}(x) = \frac{1}{n+1} \sum_{k=0}^{n} \Big(\langle v_{k+1}, \boldsymbol{e}\rangle \varphi_{n-k}(x) - \langle \varphi_{n-k}(x), \boldsymbol{e}\rangle v_{k+1} \Big), \tag{2.2}$$

for all $x \in \mathcal{M}$ and each non-negative integer n. Then $\{\varphi_n\}_{n=0}^{\infty}$ is a higher derivation on $(\mathcal{M}, \pi_{e})$.

Proof: Assume that for sequence of linear mappings $\{\varphi_n\}_{n=0}^{\infty}$ on $(\mathcal{M}, \pi_{\mathbf{e}})$ with $\varphi_0 = I$ and sequence $\{v_n\}_{n=1}^{\infty}$ of elements of $\mathcal{M}$, the equation (2.2) holds. Using induction on n, we show that $\{\varphi_n\}_{n=0}^{\infty}$ satifies the equation (2.1).

Putting $n = 0$ in (2.2), we have $\varphi_1(x) = \langle v_1, \mathbf{e}\rangle x - \langle x, \mathbf{e}\rangle v_1 = \varphi_{v_1}(x)$ for all $x \in \mathcal{M}$. Thus φ_1 is an inner derivation on $(\mathcal{M}, \pi_{\mathbf{e}})$. Suppose that the equation (2.1) is true for all $k \le n$. For $n+1$ we have

$$\begin{aligned}
&(n+1)\varphi_{n+1}(\langle x, \mathbf{e}\rangle y) \\
&= \sum_{k=0}^{n} \Big(\langle v_{k+1}, \mathbf{e}\rangle \varphi_{n-k}(\langle x, \mathbf{e}\rangle y) - \langle \varphi_{n-k}(\langle x, \mathbf{e}\rangle y), \mathbf{e}\rangle v_{k+1} \Big) \\
&= \sum_{k=0}^{n} \psi_{v_{k+1}}\big(\varphi_{n-k}(\langle x, \mathbf{e}\rangle y)\big) \\
&= \sum_{k=0}^{n} \psi_{v_{k+1}}\Big(\sum_{i=0}^{n-k} \langle \varphi_i(x), \mathbf{e}\rangle \varphi_{n-k-i}(y) \Big) \\
&= \sum_{k=0}^{n} \sum_{i=0}^{n-k} \langle \psi_{v_{k+1}}(\varphi_i(x)), \mathbf{e}\rangle \varphi_{n-k-i}(y) + \sum_{k=0}^{n} \sum_{i=0}^{n-k} \langle \varphi_i(x), \mathbf{e}\rangle \psi_{v_{k+1}}(\varphi_{n-k-i}(y)) \\
&= \sum_{k=0}^{n} \sum_{i=0}^{n-k} \langle \psi_{v_{k+1}}(\varphi_i(x)), \mathbf{e}\rangle \varphi_{n-k-i}(y) + \sum_{i=0}^{n} \sum_{k=0}^{n-i} \langle \varphi_i(x), \mathbf{e}\rangle \psi_{v_{k+1}}(\varphi_{n-k-i}(y)),
\end{aligned}$$

for all $x, y \in \mathcal{M}$. Now, putting $k+i=r$ in the first summation, changing the indices in the second summation and using assumption, we have

$$\begin{aligned}
&(n+1)\varphi_{n+1}(\langle x, \mathbf{e}\rangle y)\\
&= \sum_{r=0}^{n}\Big\langle \sum_{k=0}^{r}\psi_{v_{k+1}}(\varphi_{r-k}(x)), \mathbf{e}\Big\rangle \varphi_{n-r}(y) + \sum_{r=0}^{n}\langle \varphi_r(x), \mathbf{e}\rangle \Big(\sum_{k=0}^{n-r}\psi_{v_{k+1}}(\varphi_{n-k-r}(y))\Big)\\
&= \sum_{r=0}^{n}\langle (r+1)\varphi_{r+1}(x), \mathbf{e}\rangle \varphi_{n-r}(y) + \sum_{r=0}^{n}\langle \varphi_r(x), \mathbf{e}\rangle \Big((n-r+1)\varphi_{n-r+1}(y)\Big)\\
&= \sum_{r=1}^{n+1} r\langle \varphi_r(x), \mathbf{e}\rangle \varphi_{n-r+1}(y) + \sum_{r=0}^{n}(n-r+1)\langle \varphi_r(x), \mathbf{e}\rangle \varphi_{n-r+1}(y)\\
&= (n+1)\langle \varphi_{n+1}(x), \mathbf{e}\rangle \varphi_0(y) + \sum_{r=1}^{n} r\langle \varphi_r(x), \mathbf{e}\rangle \varphi_{n-r+1}(y)\\
&\quad + \sum_{r=1}^{n}(n-r+1)\langle \varphi_r(x), \mathbf{e}\rangle \varphi_{n-r+1}(y) + (n+1)\langle \varphi_0(x), \mathbf{e}\rangle \varphi_{n+1}(y)\\
&= (n+1)\langle \varphi_{n+1}(x), \mathbf{e}\rangle \varphi_0(y) + (n+1)\sum_{r=1}^{n}\langle \varphi_r(x), \mathbf{e}\rangle \varphi_{n-r+1}(y) + (n+1)\langle \varphi_0(x), \mathbf{e}\rangle \varphi_{n+1}(y)\\
&= (n+1)\sum_{r=0}^{n+1}\langle \varphi_r(x), \mathbf{e}\rangle \varphi_{n+1-r}(y),
\end{aligned}$$

for all $x, y \in \mathcal{M}$. This shows that $\{\varphi_n\}_{n=0}^{\infty}$ is a higher derivation on $(\mathcal{M}, \pi_{\mathbf{e}})$ and completes the proof. □

Definition 2.1 *A higher derivation satisfying in Proposition 2.1, is called an inner higher derivation on $(\mathcal{M}, \pi_e)$.*

The next corollary follows from Proposition 2.1.

Corollary 2.1 *A sequence of linear mappings $\{\varphi_n\}_{n=0}^{\infty}$ on $(\mathcal{M}, \pi_e)$ with $\varphi_0 = I$, is an inner higher derivation on $(\mathcal{M}, \pi_e)$, if and only if there is a sequence of inner derivations $\{\psi_{v_n}\}_{n=1}^{\infty}$ on $(\mathcal{M}, \pi_e)$ defined by $\psi_{v_n}(x) = \langle v_n, \mathbf{e}\rangle x - \langle x, \mathbf{e}\rangle v_n$, such that*

$$(n+1)\varphi_{n+1} = \sum_{k=0}^{n}\psi_{v_{k+1}}\varphi_{n-k}, \tag{2.3}$$

for each non-negative integer n.

Lemma 2.1 *Let $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ be two sequences in $\mathcal{M}$, and $\{\varphi_n\}_{n=0}^{\infty}$ be a sequence of mappings defined on $(\mathcal{M}, \pi_e)$ by*

$$\varphi_n(x) = \sum_{i=0}^{n}\langle v_i, \boldsymbol{e}\rangle \langle x, \boldsymbol{e}\rangle w_{n-i},$$

for all $x \in \mathcal{M}$ and each non-negative integer n. Then we have

$$\sum_{i=0}^{n}\langle \varphi_i(x), \boldsymbol{e}\rangle \varphi_{n-i}(y) = \sum_{i=0}^{n}\varphi_i\Big(\langle x, \boldsymbol{e}\rangle \langle (\boldsymbol{w} * \boldsymbol{v})_{n-i}, \boldsymbol{e}\rangle y\Big),$$

for all $x, y \in \mathcal{M}$ and each non-negative integer n, in which $(\boldsymbol{w} * \boldsymbol{v})_k = \sum_{i=0}^{k}\langle w_i, \boldsymbol{e}\rangle v_{k-i}$, $k = 0, 1, 2, \cdots$.

Proof: Trivially each φ_n is linear. For all $x, y \in \mathcal{M}$ and each non-negative integer n, we have

$$\begin{aligned}\sum_{i=0}^{n} \varphi_i\Big(\langle x, \mathbf{e}\rangle\langle(\mathbf{w} * \mathbf{v})_{n-i}, \mathbf{e}\rangle y\Big) &= \sum_{i=0}^{n} \varphi_i\Big(\langle x, \mathbf{e}\rangle\Big\langle \sum_{j=0}^{n-i}\langle w_j, \mathbf{e}\rangle v_{n-i-j}, \mathbf{e}\Big\rangle y\Big)\\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} \varphi_i\Big(\langle x, \mathbf{e}\rangle\langle w_j, \mathbf{e}\rangle\langle v_{n-i-j}, \mathbf{e}\rangle y\Big)\\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{k=0}^{i}\langle v_k, \mathbf{e}\rangle\Big\langle\langle x, \mathbf{e}\rangle\langle w_j, \mathbf{e}\rangle\langle v_{n-i-j}, \mathbf{e}\rangle y, \mathbf{e}\Big\rangle w_{i-k}\\ &= \sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{j=0}^{n-i}\langle v_k, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle\langle w_j, \mathbf{e}\rangle\langle v_{n-i-j}, \mathbf{e}\rangle\langle y, \mathbf{e}\rangle w_{i-k}.\end{aligned}$$

We can write $\sum_{i=0}^{n}\sum_{k=0}^{i} x_{i,k} = \sum_{i=0}^{n}\sum_{k=i}^{n} x_{k,i}$, in which

$$x_{i,k} = \sum_{j=0}^{n-i}\langle v_k, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle\langle w_j, \mathbf{e}\rangle\langle v_{n-i-j}, \mathbf{e}\rangle\langle y, \mathbf{e}\rangle w_{i-k}.$$

Thus we conclude that

$$\begin{aligned}\sum_{i=0}^{n} \varphi_i\Big(\langle x, \mathbf{e}\rangle\langle(\mathbf{w} * \mathbf{v})_{n-i}, \mathbf{e}\rangle y\Big) &= \sum_{i=0}^{n}\sum_{k=i}^{n}\sum_{j=0}^{n-k}\langle v_i, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle\langle w_j, \mathbf{e}\rangle\langle v_{n-k-j}, \mathbf{e}\rangle\langle y, \mathbf{e}\rangle w_{k-i}\\ &= \sum_{i=0}^{n}\sum_{k=0}^{n-i}\sum_{j=0}^{n-i-k}\langle v_i, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle\langle w_j, \mathbf{e}\rangle\langle v_{n-k-i-j}, \mathbf{e}\rangle\langle y, \mathbf{e}\rangle w_{k}.\end{aligned}$$

We can write $\sum_{k=0}^{n-i}\sum_{j=0}^{n-i-k} x_{k,j} = \sum_{j=0}^{n-i}\sum_{k=0}^{n-i-j} x_{k,j}$, in which

$$x_{k,j} = \langle v_i, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle\langle w_j, \mathbf{e}\rangle\langle v_{n-k-i-j}, \mathbf{e}\rangle\langle y, \mathbf{e}\rangle w_{k}.$$

Thus

$$\sum_{i=0}^{n} \varphi_i\Big(\langle x, \mathbf{e}\rangle\langle(\mathbf{w} * \mathbf{v})_{n-i}, \mathbf{e}\rangle y\Big) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{k=0}^{n-i-j}\langle v_i, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle\langle w_j, \mathbf{e}\rangle\langle v_{n-k-i-j}, \mathbf{e}\rangle\langle y, \mathbf{e}\rangle w_{k}.$$

In the summation $\sum_{i=0}^{n}\sum_{j=0}^{n-i}$, we have $0 \le i + j \le n$. Thus if we put $i + j = r$, then we can write it as the form $\sum_{r=0}^{n}\sum_{i+j=r}$. Putting $j = r - i$, we indeed have

$$\begin{aligned}\sum_{i=0}^{n} \varphi_i\Big(\langle x, \mathbf{e}\rangle\langle(\mathbf{w} * \mathbf{v})_{n-i}, \mathbf{e}\rangle y\Big) &= \sum_{r=0}^{n}\sum_{i=0}^{r}\sum_{k=0}^{n-r}\langle v_i, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle\langle w_{r-i}, \mathbf{e}\rangle\langle v_{n-k-r}, \mathbf{e}\rangle\langle y, \mathbf{e}\rangle w_{k}\\ &= \sum_{r=0}^{n}\Big\langle\sum_{i=0}^{r}\langle v_i, \mathbf{e}\rangle\langle x, \mathbf{e}\rangle w_{r-i}, \mathbf{e}\Big\rangle\sum_{k=0}^{n-r}\langle v_{n-r-k}, \mathbf{e}\rangle\langle y, \mathbf{e}\rangle w_{k}\\ &= \sum_{r=0}^{n}\langle\varphi_r(x), \mathbf{e}\rangle\varphi_{n-r}(y).\end{aligned}$$

This completes the proof. □

Corollary 2.2 *Let $(\mathcal{M},\pi_{\boldsymbol{e}})$ be unital, with the identity element $1_{\mathcal{M}}$ and $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ be sequences in $\mathcal{M}$ such that $v_0=w_0=1_{\mathcal{M}}$ and $(\boldsymbol{v}*\boldsymbol{w})_n=(\boldsymbol{w}*\boldsymbol{v})_n=0$ for all $n\in\mathbb{N}$. Then the sequence of mappings $\{\varphi_n\}_{n=0}^{\infty}$ on $(\mathcal{M},\pi_{\boldsymbol{e}})$ defined by*

$$\varphi_n(x)=\sum_{i=0}^{n}\langle v_i,\boldsymbol{e}\rangle\langle x,\boldsymbol{e}\rangle w_{n-i},$$

for all $x\in\mathcal{M}$ and each non-negative integer n, is a higher derivation on $(\mathcal{M},\pi_{\boldsymbol{e}})$.

Proof: For $n=0$, we hane $\varphi_0(x)=\langle v_0,\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_0=x=I(x)$ for all $x\in\mathcal{M}$. Thus $\varphi_0=I$. Let $n\in\mathbb{N}$. By Lemma 2.1, we have

$$\begin{aligned}\sum_{i=0}^{n}\langle\varphi_i(x),\mathbf{e}\rangle\varphi_{n-i}(y)&=\sum_{i=0}^{n}\varphi_i\Big(\langle x,\mathbf{e}\rangle\langle(\mathbf{w}*\mathbf{v})_{n-i},\mathbf{e}\rangle y\Big)\\&=\varphi_n\Big(\langle x,\mathbf{e}\rangle\langle(\mathbf{w}*\mathbf{v})_{0},\mathbf{e}\rangle y\Big)+\sum_{i=0}^{n-1}\varphi_i\Big(\langle x,\mathbf{e}\rangle\langle(\mathbf{w}*\mathbf{v})_{n-i},\mathbf{e}\rangle y\Big)\\&=\varphi_n\Big(\langle x,\mathbf{e}\rangle y\Big),\end{aligned}$$

for all $x,y\in\mathcal{M}$. □

In the next proposition, we find the general form of the family of inner derivations corresponding to an inner higher derivation.

Proposition 2.2 *Let $(\mathcal{M},\pi_{\boldsymbol{e}})$ be unital, with the identity element $1_{\mathcal{M}}$ and $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ be sequences in $\mathcal{M}$ such that $v_0=w_0=1_{\mathcal{M}}$ and $(\boldsymbol{v}*\boldsymbol{w})_n=(\boldsymbol{w}*\boldsymbol{v})_n=0$ for all $n\in\mathbb{N}$. Let $\{\varphi_n\}_{n=0}^{\infty}$ be the (inner) higher derivation on $(\mathcal{M},\pi_{\boldsymbol{e}})$ defined by*

$$\varphi_n(x)=\sum_{i=0}^{n}\langle v_i,\boldsymbol{e}\rangle\langle x,\boldsymbol{e}\rangle w_{n-i},$$

for all $x\in\mathcal{M}$ and each non-negative integer n. Then the corresponding sequence of inner derivations $\{\psi_n\}_{n=1}^{\infty}$, defined on $(\mathcal{M},\pi_{\boldsymbol{e}})$ by

$$\begin{aligned}\psi_n(x)=&\Big\langle\sum_{i=1}^{n}\Big(\sum_{\sum_{j=1}^{i}r_j=n}(-1)^{i-1}\ r_1\langle v_{r_1},\boldsymbol{e}\rangle\langle v_{r_2},\boldsymbol{e}\rangle\ldots\langle v_{r_{i-1}},\boldsymbol{e}\rangle v_{r_i}\Big),\boldsymbol{e}\Big\rangle x\\&+\langle x,\boldsymbol{e}\rangle\sum_{i=1}^{n}\Big(\sum_{\sum_{j=1}^{i}r_j=n}(-1)^{i-1}\ r_i\langle w_{r_1},\boldsymbol{e}\rangle\langle w_{r_2},\boldsymbol{e}\rangle\ldots\langle w_{r_{i-1}},\boldsymbol{e}\rangle w_{r_i}\Big),\end{aligned}\tag{2.4}$$

for all $x\in\mathcal{M}$ and all $n\in\mathbb{N}$.

Proof: Put

$$\begin{aligned}A_n&=\sum_{i=1}^{n}\Big(\sum_{\sum_{j=1}^{i}r_j=n}(-1)^{i-1}\ r_1\langle v_{r_1},\mathbf{e}\rangle\langle v_{r_2},\mathbf{e}\rangle\ldots\langle v_{r_{i-1}},\mathbf{e}\rangle v_{r_i}\Big),\\B_n&=\sum_{i=1}^{n}\Big(\sum_{\sum_{j=1}^{i}r_j=n}(-1)^{i-1}\ r_i\langle w_{r_1},\mathbf{e}\rangle\langle w_{r_2},\mathbf{e}\rangle\ldots\langle w_{r_{i-1}},\mathbf{e}\rangle w_{r_i}\Big).\end{aligned}$$

Then $\psi_n(x) = \langle A_n, \mathbf{e}\rangle x + \langle x, \mathbf{e}\rangle B_n$. First we prove the following equations, using induction on n.

$$\text{(i)} \quad (n+1)v_{n+1} = \sum_{k=0}^{n} \langle A_{k+1}, \mathbf{e}\rangle v_{n-k},$$
$$\text{(ii)} \quad (n+1)w_{n+1} = \sum_{k=0}^{n} \langle w_{n-k}, \mathbf{e}\rangle B_{k+1}.$$

Proof of (i): For $n = 1$ we have $A_1 = v_1$. Suppose that A_k is defined for all $k = 1, 2, \ldots, n$ as equation (i). Then for $n+1$ we have

$$\begin{aligned}
&A_{n+1}\\
&= \sum_{k=1}^{n+1} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1} (-1)^{k-1}\, r_1 \langle v_{r_1}, \mathbf{e}\rangle \langle v_{r_2}, \mathbf{e}\rangle \ldots \langle v_{r_{k-1}}, \mathbf{e}\rangle v_{r_k} \Big)\\
&= (n+1)v_{n+1} + \sum_{k=2}^{n+1} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1} (-1)^{k-1}\, r_1 \langle v_{r_1}, \mathbf{e}\rangle \langle v_{r_2}, \mathbf{e}\rangle \ldots \langle v_{r_{k-1}}, \mathbf{e}\rangle v_{r_k} \Big)\\
&= (n+1)v_{n+1} + \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k+1} r_j = n+1} (-1)^{k}\, r_1 \langle v_{r_1}, \mathbf{e}\rangle \langle v_{r_2}, \mathbf{e}\rangle \ldots \langle v_{r_k}, \mathbf{e}\rangle v_{r_{k+1}} \Big)\\
&= (n+1)v_{n+1} - \sum_{k=1}^{n} \sum_{r_{k+1}=1}^{n-(k-1)} \Big(\sum_{\substack{r_1+r_2+\ldots+r_k\\ = n+1-r_{k+1}}} (-1)^{k-1}\, r_1 \langle v_{r_1}, \mathbf{e}\rangle \langle v_{r_2}, \mathbf{e}\rangle \ldots \langle v_{r_k}, \mathbf{e}\rangle \Big) v_{r_{k+1}}\\
&= (n+1)v_{n+1} - \sum_{k=1}^{n} \sum_{i=1}^{n-(k-1)} \Big(\sum_{\substack{r_1+r_2+\ldots+r_k\\ = n+1-i}} (-1)^{k-1}\, r_1 \langle v_{r_1}, \mathbf{e}\rangle \langle v_{r_2}, \mathbf{e}\rangle \ldots \langle v_{r_k}, \mathbf{e}\rangle \Big) v_i\\
&= (n+1)v_{n+1} - \sum_{i=1}^{n} \sum_{k=1}^{n-(i-1)} \Big(\sum_{\substack{r_1+r_2+\ldots+r_k\\ = n-(i-1)}} (-1)^{k-1}\, r_1 \langle v_{r_1}, \mathbf{e}\rangle \langle v_{r_2}, \mathbf{e}\rangle \ldots \langle v_{r_k}, \mathbf{e}\rangle \Big) v_i\\
&= (n+1)v_{n+1} - \sum_{i=1}^{n} \Big\langle \sum_{k=1}^{n-(i-1)} \Big(\sum_{\substack{r_1+r_2+\ldots+r_k\\ = n-(i-1)}} (-1)^{k-1}\, r_1 \langle v_{r_1}, \mathbf{e}\rangle \langle v_{r_2}, \mathbf{e}\rangle \ldots \langle v_{r_{k-1}}, \mathbf{e}\rangle v_{r_k} \Big), \mathbf{e} \Big\rangle v_i\\
&= (n+1)v_{n+1} - \sum_{i=1}^{n} \langle A_{n-(i-1)}, \mathbf{e}\rangle v_i = (n+1)v_{n+1} - \sum_{i=0}^{n-1} \langle A_{n-i}, \mathbf{e}\rangle v_{i+1}\\
&= (n+1)v_{n+1} - \sum_{k=0}^{n-1} \langle A_{k+1}, \mathbf{e}\rangle v_{n-k}.
\end{aligned}$$

This proves the validity of equation (i).

Proof of (ii): For $n = 1$ we have $B_1 = w_1$. Suppose that B_k is defined for all $k = 1, 2, \ldots, n$ as equation (ii). Then for $n+1$ we have

$$
\begin{aligned}
&B_{n+1}\\
&=\sum_{k=1}^{n+1}\Big(\sum_{\sum_{j=1}^{k} r_j=n+1}(-1)^{k-1}\ r_k\langle w_{r_1},\mathbf{e}\rangle\langle w_{r_2},\mathbf{e}\rangle\ldots\langle w_{r_{k-1}},\mathbf{e}\rangle w_{r_k}\Big)\\
&=(n+1)w_{n+1}+\sum_{k=2}^{n+1}\Big(\sum_{\sum_{j=1}^{k} r_j=n+1}(-1)^{k-1}\ r_k\langle w_{r_1},\mathbf{e}\rangle\langle w_{r_2},\mathbf{e}\rangle\ldots\langle w_{r_{k-1}},\mathbf{e}\rangle w_{r_k}\Big)\\
&=(n+1)w_{n+1}+\sum_{k=1}^{n}\Big(\sum_{\sum_{j=1}^{k+1} r_j=n+1}(-1)^{k}\ r_{k+1}\langle w_{r_1},\mathbf{e}\rangle\langle w_{r_2},\mathbf{e}\rangle\ldots\langle w_{r_k},\mathbf{e}\rangle w_{r_{k+1}}\Big)\\
&=(n+1)w_{n+1}-\sum_{k=1}^{n}\Big(\sum_{r_1=1}^{n-(k-1)}\langle w_{r_1},\mathbf{e}\rangle\sum_{\sum_{j=2}^{k+1} r_j=n+1-r_1}(-1)^{k-1}\ r_{k+1}\langle w_{r_2},\mathbf{e}\rangle\ldots\langle w_{r_k},\mathbf{e}\rangle w_{r_{k+1}}\Big)\\
&=(n+1)w_{n+1}-\sum_{k=1}^{n}\Big(\sum_{i=1}^{n-(k-1)}\langle w_{i},\mathbf{e}\rangle\sum_{\sum_{j=2}^{k+1} r_j=n+1-i}(-1)^{k-1}\ r_{k+1}\langle w_{r_2},\mathbf{e}\rangle\ldots\langle w_{r_k},\mathbf{e}\rangle w_{r_{k+1}}\Big)\\
&=(n+1)w_{n+1}-\sum_{i=1}^{n}\langle w_{i},\mathbf{e}\rangle\sum_{k=1}^{n-(i-1)}\Big(\sum_{\sum_{j=2}^{k+1} r_j=n-(i-1)}(-1)^{k-1}\ r_{k+1}\langle w_{r_2},\mathbf{e}\rangle\ldots\langle w_{r_k},\mathbf{e}\rangle w_{r_{k+1}}\Big)\\
&=(n+1)w_{n+1}-\sum_{i=1}^{n}\langle w_{i},\mathbf{e}\rangle\sum_{k=1}^{n-(i-1)}\Big(\sum_{\sum_{j=1}^{k} r_j=n-(i-1)}(-1)^{k-1}\ r_{k}\langle w_{r_1},\mathbf{e}\rangle\ldots\langle w_{r_{k-1}},\mathbf{e}\rangle w_{r_{k}}\Big)\\
&=(n+1)w_{n+1}-\sum_{i=1}^{n}\langle w_{i},\mathbf{e}\rangle B_{n-(i-1)}=(n+1)w_{n+1}-\sum_{i=0}^{n-1}\langle w_{i+1},\mathbf{e}\rangle B_{n-i}\\
&=(n+1)w_{n+1}-\sum_{k=0}^{n-1}\langle w_{n-k},\mathbf{e}\rangle B_{k+1}.
\end{aligned}
$$

This proves the validity of equation (ii). Now we have

$$
\begin{aligned}
\sum_{k=0}^{n}\psi_{k+1}\varphi_{n-k}(x)&=\sum_{k=0}^{n}\psi_{k+1}\Big(\sum_{i=0}^{n-k}\langle v_i,\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-k-i}\Big)=\sum_{k=0}^{n}\sum_{i=0}^{n-k}\psi_{k+1}(\langle v_i,\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-k-i})\\
&=\sum_{k=0}^{n}\sum_{i=0}^{n-k}\Big(\langle A_{k+1},\mathbf{e}\rangle\langle v_i,\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-k-i}+\langle v_i,\mathbf{e}\rangle\langle x,\mathbf{e}\rangle\langle w_{n-k-i},\mathbf{e}\rangle B_{k+1}\Big)\\
&=\sum_{k=0}^{n}\sum_{i=0}^{n-k}\Big(\langle A_{k+1},\mathbf{e}\rangle\langle v_i,\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-k-i}+\langle v_{n-k-i},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle\langle w_{i},\mathbf{e}\rangle B_{k+1}\Big).
\end{aligned}
$$

In the summation $\sum_{k=0}^{n}\sum_{i=0}^{n-k}$, we have $0\le k+i\le n$. Thus if we put $k+i=r$, then we can write it as the form $\sum_{r=0}^{n}\sum_{k+i=r}$. Putting $i=r-k$, we indeed have

$$\begin{aligned}
&\sum_{k=0}^{n}\psi_{k+1}\varphi_{n-k}(x)\\
&=\sum_{r=0}^{n}\sum_{k=0}^{r}\Big(\langle A_{k+1},\mathbf{e}\rangle\langle v_{r-k},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-r}+\langle v_{n-r},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle\langle w_{r-k},\mathbf{e}\rangle B_{k+1}\Big)\\
&=\sum_{r=0}^{n}\Big(\Big\langle\sum_{k=0}^{r}\langle A_{k+1},\mathbf{e}\rangle v_{r-k},\mathbf{e}\Big\rangle\langle x,\mathbf{e}\rangle w_{n-r}+\langle v_{n-r},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle\sum_{k=0}^{r}\langle w_{r-k},\mathbf{e}\rangle B_{k+1}\Big)\\
&=\sum_{r=0}^{n}\Big(\langle(r+1)v_{r+1},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-r}+(r+1)\langle v_{n-r},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{r+1}\Big)\\
&=\sum_{r=0}^{n}(r+1)\langle v_{r+1},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-r}+\sum_{r=0}^{n}(r+1)\langle v_{n-r},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{r+1}\\
&=(n+1)\langle v_{n+1},\mathbf{e}\rangle x+\sum_{r=0}^{n-1}(r+1)\langle v_{r+1},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-r}\\
&\quad+(n+1)\langle x,\mathbf{e}\rangle w_{n+1}+\sum_{r=0}^{n-1}(r+1)\langle v_{n-r},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{r+1}\\
&=(n+1)\langle v_{n+1},\mathbf{e}\rangle x+\sum_{r=0}^{n-1}(r+1)\langle v_{r+1},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-r}\\
&\quad+(n+1)\langle x,\mathbf{e}\rangle w_{n+1}+\sum_{r=0}^{n-1}(n-r)\langle v_{r+1},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-r}\\
&=(n+1)\langle v_{n+1},\mathbf{e}\rangle x+(n+1)\sum_{r=0}^{n-1}\langle v_{r+1},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n-r}+(n+1)\langle x,\mathbf{e}\rangle w_{n+1}\\
&=(n+1)\langle v_{n+1},\mathbf{e}\rangle x+(n+1)\sum_{r=1}^{n}\langle v_{r},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n+1-r}+(n+1)\langle x,\mathbf{e}\rangle w_{n+1}\\
&=(n+1)\sum_{r=0}^{n+1}\langle v_{r},\mathbf{e}\rangle\langle x,\mathbf{e}\rangle w_{n+1-r}\\
&=(n+1)\varphi_{n+1}(x).
\end{aligned}$$

Thus $\{\psi_n\}_{n=1}^{\infty}$ is the corresponding sequence of derivations to the higher derivation $\{\varphi_n\}_{n=0}^{\infty}$. Since $\psi_n(1_{\mathcal{M}})=0$ for all $n\in\mathbb{N}$, it follows that $A_n+B_n=0$ or $B_n=-A_n$ for all $n\in\mathbb{N}$. Thus $\psi_n(x)=\langle A_n,\mathbf{e}\rangle x-\langle x,\mathbf{e}\rangle A_n$ is an inner derivation for each $n\in\mathbb{N}$. This completes the proof. □

Example 2.2 *Using Proposition 2.2, the four terms of sequence of inner derivations* $\{\psi_n\}_{n=1}^{\infty}$ *are defined* *on* $(\mathcal{M}, \pi_{\boldsymbol{e}})$ *as follows:*

$$\begin{aligned}
\psi_1(x) &= \langle v_1, \boldsymbol{e}\rangle x + \langle x, \boldsymbol{e}\rangle w_1,\\
\psi_2(x) &= \Big\langle 2v_2 - \langle v_1, \boldsymbol{e}\rangle v_1, \boldsymbol{e}\Big\rangle x + \langle x, \boldsymbol{e}\rangle \Big(2w_2 - \langle w_1, \boldsymbol{e}\rangle w_1\Big),\\
\psi_3(x) &= \Big\langle 3v_3 - 2\langle v_2, \boldsymbol{e}\rangle v_1 - \langle v_1, \boldsymbol{e}\rangle v_2 + \langle v_1, \boldsymbol{e}\rangle\langle v_1, \boldsymbol{e}\rangle v_1, \boldsymbol{e}\Big\rangle x\\
&\quad + \langle x, \boldsymbol{e}\rangle \Big(3w_3 - \langle w_2, \boldsymbol{e}\rangle w_1 - 2\langle w_1, \boldsymbol{e}\rangle w_2 + \langle w_1, \boldsymbol{e}\rangle\langle w_1, \boldsymbol{e}\rangle w_1\Big),\\
\psi_4(x) &= \Big\langle 4v_4 - 3\langle v_3, \boldsymbol{e}\rangle v_1 - 2\langle v_2, \boldsymbol{e}\rangle v_2 - \langle v_1, \boldsymbol{e}\rangle v_3 + 2\langle v_2, \boldsymbol{e}\rangle\langle v_1, \boldsymbol{e}\rangle v_1\\
&\qquad + \langle v_1, \boldsymbol{e}\rangle\langle v_2, \boldsymbol{e}\rangle v_1 + \langle v_1, \boldsymbol{e}\rangle\langle v_1, \boldsymbol{e}\rangle v_2 - \langle v_1, \boldsymbol{e}\rangle\langle v_1, \boldsymbol{e}\rangle\langle v_1, \boldsymbol{e}\rangle v_1, \boldsymbol{e}\Big\rangle x\\
&\quad + \langle x, \boldsymbol{e}\rangle \Big(4w_4 - \langle w_3, \boldsymbol{e}\rangle w_1 - 2\langle w_2, \boldsymbol{e}\rangle w_2 - 3\langle w_1, \boldsymbol{e}\rangle w_3 + \langle w_2, \boldsymbol{e}\rangle\langle w_1, \boldsymbol{e}\rangle w_1\\
&\qquad + \langle w_1, \boldsymbol{e}\rangle\langle w_2, \boldsymbol{e}\rangle w_1 + 2\langle w_1, \boldsymbol{e}\rangle\langle w_1, \boldsymbol{e}\rangle w_2 - \langle w_1, \boldsymbol{e}\rangle\langle w_1, \boldsymbol{e}\rangle\langle w_1, \boldsymbol{e}\rangle w_1\Big).
\end{aligned}$$

The next corollary follows from Proposition 2.2.

Corollary 2.3 *Let* $(\mathcal{M}, \pi_{\boldsymbol{e}})$ *be unital, with the identity element* $1_{\mathcal{M}}$ *and* $\{\varphi_n\}_{n=0}^{\infty}$ *be an inner higher derivation on* $(\mathcal{M}, \pi_{\boldsymbol{e}})$ *defined by*

$$\varphi_n(x) = \sum_{i=0}^{n} \langle v_i, \boldsymbol{e}\rangle\langle x, \boldsymbol{e}\rangle w_{n-i},$$

in which $\{v_n\}_{n=0}^{\infty}$ *and* $\{w_n\}_{n=0}^{\infty}$ *are sequences in* $\mathcal{M}$ *such that* $v_0 = w_0 = 1_{\mathcal{M}}$ *and*

$$\sum_{i=0}^{n} \langle v_i, \boldsymbol{e}\rangle w_{n-i} = \sum_{i=0}^{n} \langle w_i, \boldsymbol{e}\rangle v_{n-i} = 0,$$

for all $n \in \mathbb{N}$. *Then*

$$\sum_{i=1}^{n} \Big(\sum_{\sum_{j=1}^{i} r_j = n} (-1)^{i-1} \ \big(r_1 \langle v_{r_1}, \boldsymbol{e}\rangle\langle v_{r_2}, \boldsymbol{e}\rangle \ldots \langle v_{r_{i-1}}, \boldsymbol{e}\rangle v_{r_i} + r_i \langle w_{r_1}, \boldsymbol{e}\rangle\langle w_{r_2}, \boldsymbol{e}\rangle \ldots \langle w_{r_{i-1}}, \boldsymbol{e}\rangle w_{r_i}\big)\Big) = 0,$$

for all $n \in \mathbb{N}$.

Proof: Putting $x = 1_{\mathcal{M}}$ in (2.4), we get the result. □

Example 2.3 *Let* $M_n(\mathbb{C})$ *be the* C^**-algebra of all* $n \times n$ *complex matrices,* $E \in M_n(\mathbb{C})$ *with* $\|E\| = 1$ *and* $A \in M_n(\mathbb{C})$ *be an arbitrary matrix. Define the sequences* $\boldsymbol{V} = \{V_n\}_{n=0}^{\infty}$ *and* $\boldsymbol{W} = \{W_n\}_{n=0}^{\infty}$ *in* $M_n(\mathbb{C})$ *by* $V_0 = W_0 = I_n$ *(the* $n \times n$ *identity matrix) and*

$$V_n = \tfrac{(-1)^n}{n!} \underbrace{\langle A, E\rangle \ldots \langle A, E\rangle}_{n-1 \text{ times}} A,$$

and

$$W_n = \tfrac{1}{n!} \underbrace{\langle A, E\rangle \ldots \langle A, E\rangle}_{n-1 \text{ times}} A,$$

for each positive integer n. *Then*

$$\begin{aligned}
(\boldsymbol{V} * \boldsymbol{W})_n &= \sum_{i=0}^{n} \langle V_i, E\rangle W_{n-i} = \sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!} \underbrace{\langle A, E\rangle \ldots \langle A, E\rangle}_{n-1 \text{ times}} A\\
&= \underbrace{\langle A, E\rangle \ldots \langle A, E\rangle}_{n-1 \text{ times}} A \sum_{i=0}^{n} \frac{(-1)^i}{n!} \binom{n}{i} = \frac{1}{n!} \underbrace{\langle A, E\rangle \ldots \langle A, E\rangle}_{n-1 \text{ times}} A \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.
\end{aligned}$$

Similarly, $(\boldsymbol{W} * \boldsymbol{V})_n = \sum_{i=0}^{n} \langle W_i, E\rangle V_{n-i} = 0$. *Then by Corollary 2.2, the sequence of mappings* $\{\varphi_n\}_{n=0}^{\infty}$ *defined on* $(M_n(\mathbb{C}), \pi_E)$ *by* $\varphi_n(X) = \sum_{i=0}^{n} \langle V_i, E\rangle \langle X, E\rangle W_{n-i}$ *for all* $X \in M_n(\mathbb{C})$*, is an inner higher derivation on* $(M_n(\mathbb{C}), \pi_E)$*. Also by Proposition 2.2, the corresponding sequence of inner derivations* $\{\psi_n\}_{n=1}^{\infty}$*, defined on* $(M_n(\mathbb{C}), \pi_E)$ *by* $\psi_1(X) = \langle X, E\rangle A - \langle A, E\rangle X$ *and* $\psi_k = 0$ *for all* $k \geq 2$.

References

1. M. J. Atteya, *Notes on the higher derivations of prime rings*, Boletim da Sociedade Paranaense de Matemática, 37(4), 61–68, (2019).
2. S. Kh. Ekrami, *Approximate orthogonally higher ring derivations*, Control Optimization App. Math., 7(1), 93–106, (2022). doi: 10.30473/coam.2021.59727.1166
3. S. Kh. Ekrami, *Characterization of Hilbert C*-module higher derivations*, Georgian Mathematical Journal, 31(3), 397–403, (2024).
4. S. Kh. Ekrami, *Jordan higher derivations, a new approach*, J. Algebraic Systems, 10(1), 167–177, (2022).
5. P. E. Bland, *Higher derivations on rings and modules*, Int. J. Math. Math. Sci., 15, 2373–2387, (2005).
6. W. Cortes and C. Haetinger, *On Jordan generalized higher derivations in rings*, Turk. J. Math., 29, 1–10 (2005).
7. N. P. Jewell, *Continuity of module and higher derivations*, Pacific J. Math., 68, 91–98, (1977).
8. I. Kaplansky, *Modules over operator algebras*, Amer. J. Math., 75, 839–858, (1953).
9. M. Mirzavaziri, *Inner higher derivations on algebras*, Kragujevac J. Mathematics, 43(2), 267–273, (2019).
10. A. Nowicki, *Inner derivations of higher orders*, Tsukuba J. Math., 8, 219–225, (1984).
11. A. Roy and R. Sridharan, *Higher derivations and central simple algebras*, Nagoya Math. J., 32, 21–30, (1968).
12. H. Saidi, A. R. Janfada, and M. Mirzavaziri, *Kinds of derivations on Hilbert C*-modules and their operator algebras*, Miskolc Mathematical Notes, 16(1), 453–461, (2015).
13. S. Xu and Z. Xiao, *Jordan higher derivation revisited*, Gulf J. Math., 2(1), 11–21, (2014).

Sayed Khalil Ekrami,

Department of Mathematics,

Payame Noor University, P.O. Box 19395-3697,

Tehran, Iran.

E-mail address: ekrami@pnu.ac.ir, khalil.ekrami@gmail.com