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Generalized closed sets in hereditary m-spaces with γ-operations

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ABSTRACT: Let $(X, m, \mathcal{H})$ be a hereditary m-space and $\gamma : m \to P(X)$ be an operation on m. In this paper, a subset A of X is said to be $\mathcal{H}_\gamma g$-*closed* if $\gamma\text{Cl}(A) \setminus U \in \mathcal{H}$ whenever $A \subseteq U$ and U is m-open. We obtain some characterizations and properties of $\mathcal{H}_\gamma g$-closed and $\mathcal{H}_\gamma g$-open sets.

Key Words: hereditary m-space, g-closed, γg-closed, $\mathcal{H}_\gamma g$-closed, $\mathcal{H}_\gamma g$-open.

Contents

1. Introduction

Generalized closed (briefly g-closed) sets in a topological space are introduced by Levine [14]. Since then, many generalizations of g-closed sets are introduced and investigated, for example, refer to Definition 2.4 of [20] and Definition 3.9 of [17]. The second author [20] introduced mg-closed sets in an m-space and unified several types of generalizations of g-closed sets. Ideals and hereditary classes are introduced in [10]. Ogata [22] introduced the notions of γ-operations and γ-open sets in a topological space. The notions $m\gamma$-operations and $m\gamma$-closed sets in an m-space are introduced and investigated in [21].

In this paper, we introduce a generalization of mg-closed sets, called $\mathcal{H}_\gamma g$-closed, in a hereditary m-space with a γ-operation. In Section 3, we obtain characterizations and properties of $\mathcal{H}_\gamma g$-closed sets. In Section 4, we obtain characterizations and properties of $\mathcal{H}_\gamma g$-open sets and a preservation theorem of $\mathcal{H}_\gamma g$-closed sets.

2. Preliminaries

Definition 2.1 A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly m-*structure*) on X if m satisfies the following conditions:
(1) $\emptyset \in m$ and $X \in m$,
(2) The union of any family of subsets belonging to m belongs to m.

A set X with an m-structure m on X is denoted by (X, m) and is called an m-*space*. Each member of m is said to be m-*open* and the complement of an m-open set is said to be m-*closed*. The property (2) in Definition 2.1 is called property $\mathcal{B}$ in [15]. In this paper, the m-structure [23] having property $\mathcal{B}$ [23] is briefly called m-structure.

Definition 2.2 [15] Let (X, m) be an m-space and A a subset of X. The m-*closure* $\text{mCl}(A)$ and the m-*interior* $\text{mInt}(A)$ of A are defined as follows:
(1) $\text{mCl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in m\}$,
(2) $\text{mInt}(A) = \cup\{U \subset X : U \subset A, U \in m\}$.

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Lemma 2.1 [23]. *Let (X, m) be an m-space and A a subset of X.*
(1) $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m(x)$, where $m(x) = \cup\{U : x \in U \in m\}$.
(2) A is m-closed if and only if $\mathrm{mCl}(A) = A$.

Definition 2.3 A nonempty subfamily $\mathcal{H}$ of $\mathcal{P}(X)$ is called a *hereditary class* on X [10] if it satisfies the following properties: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A hereditary class $\mathcal{H}$ is called an *ideal* [13], [24] if it satisfies the additional condition: $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$.

A minimal space (X, m) with a hereditary class $\mathcal{H}$ on X is called a *hereditary minimal space* (briefly *hereditary m-space*) and is denoted by $(X, m, \mathcal{H})$.

In [16], Modak defined m-nowhere dense set in a minimal space as a subset A of an m-space (X, m) is called m-nowhere dense if $mInt(mCl(A)) = \emptyset$. The collection of m-nowhere dense sets forms a Hereditary class but not forms an ideal.

Definition 2.4 Let (X, m) be an m-space. Let $m\gamma : m \to P(X)$ be a function from m into $P(X)$ such that $U \subset m\gamma(U)$ for each $U \in m$. The function $m\gamma$ is called an *$m\gamma$-operation* on m [21] and the image $m\gamma(U)$ is simply denoted by $\gamma(U)$. In this paper, an $m\gamma$-operation is simply called a γ-operation.

Definition 2.5 Let (X, m) be an m-space and $\gamma : m \to P(X)$ be a γ-operation. A subset A of X is said to be *γ-open* [21] if for each $x \in A$ there exists $U \in m$ such that $x \in U \subseteq \gamma(U) \subseteq A$. The complement of a γ-open set is said to be *γ-closed*. The family of all γ-open sets of (X, m) is denoted by $\gamma(X)$. The *γ-closure* of A, $\gamma\mathrm{Cl}(A)$, is defined as follows: $\gamma\mathrm{Cl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in \gamma(X)\}$.

Theorem 2.1 *[21] Let (X, m) be an m-space and γ be a γ-operation on m, the following properties hold:*

1. *$\emptyset, X \in \gamma(X)$,*
2. *If $A_\alpha \in \gamma(X)$ for each $\alpha \in \Lambda$, then $\cup_{\alpha \in \Lambda} A_\alpha \in \gamma(X)$.*
3. *$\gamma(X) \subseteq m$.*

The following Examples shows that not every m-open sets is γ-open sets.

Example 2.1 *Let $X = \{a, b, c\}$, $m = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : m \to P(X)$ by $\gamma(U) = \begin{cases} U, & \text{if } b \in U; \\ mCl(U), & \text{if } b \notin U. \end{cases}$*

The collection of all γ-open sets are $\emptyset$, X, $\{b\}$, $\{a, b\}$ and $\{a, c\}$. Here $\{a\}$ is an m-open set which is not γ-open.

Example 2.2 *Let $X = \{a, b, c\}$ with $m = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\gamma(A) = mCl(A)$ for any subset A of X. Then, $A = \{a, b\}$ is an m-open set but not γ-open. Because if $a \in A \in m$, then the collection of all m-open sets containing a is $\mathcal{U} = \{\{a\}, \{a, b\}, X\}$. If $U = \{a\}$, then $a \in U \subset \gamma(U) = mCl(U) = \{a, c\}$ and $\gamma(U)$ does not contain in $A = \{a, b\}$. If $U = \{a, b\}$, then $a \in U \subset \gamma(U) = mCl(U) = X$ and hence $\gamma(U)$ does not contain in $A = \{a, b\}$. If $U = X$, then $a \in U \subset \gamma(U) = mCl(U) = X$ and $\gamma(U)$ does not contain in $\{a, b\}$. Therefore, $A = \{a, b\}$ is not γ-open. Note that the collection of all γ-open sets are $\emptyset$, X.*

Definition 2.6 [21] Let (X, m) be an m-space and $\gamma : m \to P(X)$ be a γ-operation. An operation γ is said to be m-regular if for each $x \in X$ and each $U, V \in m$ containing x, there exists $W \in m$ such that $x \in W \subseteq \gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Theorem 2.2 *[21] Let (X, m) be an m-space and γ be a γ-operation on m. Then, $\gamma(X)$ is a topology for X if the operation γ is m-regular.*

Several characterizations of minimal structures with notion of hereditary class were provided in [1-8].

3. $\mathcal{H}_\gamma g$-closed sets

Definition 3.1 Let (X, m) be an m-space and γ be a γ-operation on m. A subset A of X is said to be *γg-closed* if $\gamma\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is m-open.

Definition 3.2 Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A subset A of X is said to be *$\mathcal{H}_\gamma g$-closed* if $\gamma\mathrm{Cl}(A) \setminus U \in \mathcal{H}$ whenever $A \subseteq U$ and U is m-open.

Remark 3.1 Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ a γ-operation on m.

1. Let $\mathcal{H} = \{\emptyset\}$, then an $\mathcal{H}_\gamma g$-closed set is a γg-closed set.
2. Let $\mathcal{H} = \{\emptyset\}$ and $\gamma = mCl$, then a γg-closed set is an mg-closed set.
3. Let (X, m) be a topological space, then an mg-closed set is a g-closed set.

Theorem 3.1 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. Every γg-closed set is $\mathcal{H}_\gamma g$-closed.*

Proof: Suppose that A is γg-closed and let $U \in m$ such that $A \subseteq U$. Then, $\gamma\mathrm{Cl}(A) \subseteq U$ and hence $\gamma\mathrm{Cl}(A) \setminus U = \emptyset \in \mathcal{H}$. Therefore, A is $\mathcal{H}_\gamma g$-closed. □

The following example shows that the converse of the above theorem is in general not true.

Example 3.1 *Let $X = \{a, b, c\}$, $m = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}$. Then, (X, m) is a topological space and let $\gamma = mCl$, so $\gamma(X) = \{\emptyset, X\}$. Then, A is an $\mathcal{H}_\gamma g$-closed set but it is not γg-closed. Let $U = \{a\} \in m$, then $A \subseteq U$ and $\gamma Cl(A) = X$ which is not contained in U. Hence, A is not γg-closed. Next, Let $U = \{a\}$, then $A \subseteq U$ and $\gamma Cl(A) \setminus U = X \setminus \{a\} = \{b, c\} \in \mathcal{H}$. Let $U = \{a, b\}$, then $A \subseteq U$ and $\gamma Cl(A) \setminus U = X \setminus \{a, b\} = \{c\} \in \mathcal{H}$. Let $U = X$, then $A \subseteq U$ and $\gamma Cl(A) \setminus U = X \setminus X = \emptyset \in \mathcal{H}$. Therefore, A is an $\mathcal{H}_\gamma g$-closed set.*

Theorem 3.2 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. If $m\mathrm{Cl}(\{x\}) \cap A \notin \mathcal{H}$ for every $x \in \gamma\mathrm{Cl}(A)$, then $F \setminus (\gamma\mathrm{Cl}(A) \setminus A) \notin \mathcal{H}$ for any m-closed set F containing x.*

Proof: If possible, let there exist an m-closed set F containing x such that $F \setminus (\gamma\mathrm{Cl}(A) \setminus A) = F \cap [A \cup (X - \gamma\mathrm{Cl}(A))] \in \mathcal{H}$. Then $F \cap A \in \mathcal{H}$. Let $x \in \gamma Cl(A)$, since $m\mathrm{Cl}(\{x\}) \cap A \notin \mathcal{H}$ and $m\mathrm{Cl}(\{x\}) \cap A \subseteq F \cap A$ then, $F \cap A \notin \mathcal{H}$, which is a contradiction. Hence, $F \setminus (\gamma\mathrm{Cl}(A) \setminus A) \notin \mathcal{H}$ for any m-closed set F containing x. □

Theorem 3.3 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A subset A of X is $\mathcal{H}_\gamma g$-closed if $m\mathrm{Cl}(\{x\}) \cap A \notin \mathcal{H}$ holds for every $x \in \gamma\mathrm{Cl}(A)$.*

Proof: Suppose that A is not $\mathcal{H}_\gamma g$-closed. We show that there exists $x \in \gamma Cl(A)$ such that $mCl(\{x\}) \cap A \in \mathcal{H}$. By assumption, there exists an m-open set U such that $A \subseteq U$ and $\gamma Cl(A) \setminus U \notin \mathcal{H}$. Then, $\gamma Cl(A) \setminus U \neq \emptyset$ and there exists $x \in \gamma Cl(A)$ such that $x \notin U$. But U is m-open and $X \setminus U$ is m-closed. Since $x \in X \setminus U$, $mCl(\{x\}) \subseteq X \setminus U$ and hence $mCl(\{x\}) \cap A \subseteq (X \setminus U) \cap A = \emptyset \in \mathcal{H}$ and so $mCl(\{x\}) \cap A \in \mathcal{H}$. Which is a contradiction hence A is $\mathcal{H}_\gamma g$-closed. □

The following example shows that the converse of the above theorem is not true.

Example 3.2 *Let $X = \{a, b, c\}$, $m = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}$. Let $\gamma = mCl$. Then, by Example 3.1, A is an $\mathcal{H}_\gamma g$-closed set. There exists $x = c \in \gamma Cl(A) = X$ such that $mCl(\{x\}) \cap A = \{c\} \cap \{a\} = \emptyset \in \mathcal{H}$.*

Theorem 3.4 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. For each $x \in X$, either $\{x\}$ is m-closed or $X \setminus \{x\}$ is a $\mathcal{H}_\gamma g$-closed set.*

Proof: If $\{x\}$ is m-closed, then we have nothing to prove. Suppose that $\{x\}$ is not m-closed. Then, $X \setminus \{x\}$ is not m-open. Let U be any m-open set such that $X \setminus \{x\} \subseteq U$. Hence, $U = X$. Thus, $\gamma Cl(X \setminus \{x\}) \setminus U = \gamma Cl(X \setminus \{x\}) \setminus X = \emptyset \in \mathcal{H}$ and $X \setminus \{x\}$ is a $\mathcal{H}_\gamma g$-closed. □

Theorem 3.5 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A set A is $\mathcal{H}_\gamma g$-closed in $(X, m, \mathcal{H})$ if and only if $F \in \mathcal{H}$ whenever $F \subseteq \gamma\mathrm{Cl}(A) \setminus A$ and F is m-closed in X.*

Proof: Assume that A is $\mathcal{H}_\gamma g$-closed. Suppose that $F \subseteq \gamma\mathrm{Cl}(A) \setminus A$ and F is m-closed in X. Then, $A \subseteq X \setminus F$. By our assumption, $\gamma\mathrm{Cl}(A) \setminus (X \setminus F) \in \mathcal{H}$. But $F \subseteq \gamma\mathrm{Cl}(A) \setminus (X \setminus F)$ and hence $F \in \mathcal{H}$.

Suppose that $F \in \mathcal{H}$ whenever $F \subseteq \gamma Cl(A) \setminus A$ and F is m-closed in X. Let $A \subseteq U$ and $U \in m$. Then, $\gamma\mathrm{Cl}(A) \setminus U = \gamma\mathrm{Cl}(A) \cap (X \setminus U)$ is an m-closed set in X, that is contained in $\gamma\mathrm{Cl}(A) \setminus A$. By assumption $\gamma\mathrm{Cl}(A) \setminus U \in \mathcal{H}$. This implies A is $\mathcal{H}_\gamma g$-closed. □

Theorem 3.6 *Let $(X, m, \mathcal{H})$ be an ideal m-space and γ be a γ-operation on m. If A and B are $\mathcal{H}_\gamma g$-closed in $(X, m, \mathcal{H})$, then $A \cup B$ is $\mathcal{H}_\gamma g$-closed.*

Proof: Suppose A and B are $\mathcal{H}_\gamma g$-closed sets in $(X, m, \mathcal{H})$. If $A \cup B \subseteq U$ and U is m-open, then $A \subseteq U$ and $B \subseteq U$. By assumption, $\gamma\mathrm{Cl}(A) \setminus U \in \mathcal{H}$ and $\gamma\mathrm{Cl}(B) \setminus U \in \mathcal{H}$ and hence $\gamma\mathrm{Cl}(A \cup B) \setminus U = [\gamma\mathrm{Cl}(A) \cup \gamma\mathrm{Cl}(B)] \setminus U = [\gamma\mathrm{Cl}(A) \setminus U] \cup [\gamma\mathrm{Cl}(B) \setminus U] \in \mathcal{H}$. Thus, is $A \cup B$ is $\mathcal{H}_\gamma g$-closed. □

The following example shows that the intersection of a $\mathcal{H}_\gamma g$-closed is not $\mathcal{H}_\gamma g$-closed.

Example 3.3 *Let $X = \{a, b, c, d\}$, $m = \{\emptyset, X, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$ be not an ideal. If γ = identity, then*

- *$A = \{a, c, d\}$ the collection of all m-open sets containing A is $\mathcal{U} = \{X, \{a, c, d\}\}$ and $\gamma Cl(A) = X$. Therefore, $\gamma Cl(A) \setminus U \in \mathcal{H}$ for all $U \in \mathcal{U}$ so, A is an $\mathcal{H}_\gamma g$-closed set.*
- *$B = \{b, c, d\}$ the collection of all m-open sets containing B is $\mathcal{U} = \{X, \{b, c, d\}\}$ and $\gamma Cl(B) = X$. Therefore, $\gamma Cl(B) \setminus U \in \mathcal{H}$ for all $U \in \mathcal{U}$ so, B is an $\mathcal{H}_\gamma g$-closed set.*
- *$A \cap B = \{c, d\}$ the collection of all m-open sets containing $A \cap B$ is $\mathcal{U} = \{X, \{a, c, d\}, \{b, c, d\}, \{c, d\}\}$ and $\gamma Cl(A \cap B) = X$. It is clear that, $\gamma Cl(A \cap B) \setminus U \notin \mathcal{H}$ for some $U \in \mathcal{U}$ so, $A \cap B$ is not an $\mathcal{H}_\gamma g$-closed set.*

Theorem 3.7 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. If A is $\mathcal{H}_\gamma g$-closed and $A \subseteq B \subseteq \gamma\mathrm{Cl}(A)$ in $(X, m, \mathcal{H})$, then B is $\mathcal{H}_\gamma g$-closed.*

Proof: Suppose A is $\mathcal{H}_\gamma g$-closed and $A \subseteq B \subseteq \gamma\mathrm{Cl}(A)$ in $(X, m, \mathcal{H})$. Suppose $B \subseteq U$ and U is m-open. Then, $A \subseteq U$. Since A is $\mathcal{H}_\gamma g$-closed, we have $\gamma\mathrm{Cl}(A) \setminus U \in \mathcal{H}$. Since $B \subseteq \gamma\mathrm{Cl}(A)$, $\gamma\mathrm{Cl}(B) \subseteq \gamma\mathrm{Cl}(A)$ and $\gamma\mathrm{Cl}(B) \setminus U \subseteq \gamma\mathrm{Cl}(A) \setminus U \in \mathcal{H}$, then $\gamma\mathrm{Cl}(B) \setminus U \in \mathcal{H}$. Hence, B is $\mathcal{H}_\gamma g$-closed. □

Theorem 3.8 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. If $A \subseteq Y \subseteq X$ and suppose that A is $\mathcal{H}_\gamma g$-closed in X. Then, A is $\mathcal{H}_\gamma g$-closed relative to the subspace $m_Y = \{B = U \cap Y : U \in m\}$ with respect to the hereditary $\mathcal{H}_Y = \{F \subseteq Y : F \in \mathcal{H}\}$.*

Proof: Suppose $A \subseteq U \cap Y$ and $U \in m$, then $A \subseteq U$. Since A is $\mathcal{H}_\gamma g$-closed in X we have $\gamma\mathrm{Cl}(A) \setminus U \in \mathcal{H}$. Now $(\gamma\mathrm{Cl}(A) \cap Y) \setminus (U \cap Y) = (\gamma\mathrm{Cl}(A) \setminus U) \cap Y \in \mathcal{H}_Y$, whenever $A \subseteq U \cap Y$ and $U \in m$. Hence, A is $\mathcal{H}_\gamma g$-closed relative to the subspace m_Y. □

Theorem 3.9 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. If A is $\mathcal{H}_\gamma g$-closed in X and F is γ-closed, then $A \cap F$ is $\mathcal{H}_\gamma g$-closed in X.*

Proof: Let $A \cap F \subseteq U$ and U is m-open. Then $A \subseteq U \cup (X \setminus F)$. Since A is $\mathcal{H}_\gamma g$-closed, we have $\gamma\mathrm{Cl}(A) \setminus (U \cup (X \setminus F)) \in \mathcal{H}$. Now, $\gamma\mathrm{Cl}(A \cap F) \subseteq \gamma\mathrm{Cl}(A) \cap F = (\gamma\mathrm{Cl}(A) \cap F) \setminus (X \setminus F)$. Therefore,

$$\begin{aligned}\gamma\mathrm{Cl}(A \cap F) \setminus U &\subseteq (\gamma\mathrm{Cl}(A) \cap F) \setminus (U \cap (X \setminus F)) \\ &\subseteq \gamma\mathrm{Cl}(A) \setminus (U \cup (X \setminus F)) \\ &\in \mathcal{H}.\end{aligned}$$

Hence, $A \cap F$ is $\mathcal{H}_\gamma g$-closed in X. □

Let $(X, m, \mathcal{H})$ be a hereditary m-space. If for each $H_1 \in \mathcal{H}$ there exists $H_2 \in \mathcal{H} \cap m$ such that $H_1 \subseteq H_2$, then m is said to be saturated by $\mathcal{H}$.

Theorem 3.10 *Let $(X, m, \mathcal{H})$ be an ideal m-space and γ be a γ-operation on m. Let $B \subseteq A \subseteq X$ and B be $\mathcal{H}_\gamma g$-closed relative to A and A be a $\mathcal{H}_\gamma g$-closed subset of X. If m is saturated by $\mathcal{H}$, then B is $\mathcal{H}_\gamma g$-closed in X.*

Proof: Let m be saturated by $\mathcal{H}$. Let $B \subseteq U$ and U be m-open in X. Then $B \subseteq U \cap A$. Since B is $\mathcal{H}_\gamma g$-closed relative to A, we have $\gamma\mathrm{Cl}_{\mathrm{A}}(B) \subseteq (U \cap A) \cup H_1$ for some $H_1 \in \mathcal{H}$. By assumption, there exists $H_2 \in \mathcal{H} \cap m$ such that $A \cap \gamma\mathrm{Cl}(B) \subseteq (U \cap A) \cup H_2$. So $A \subseteq (U \cup H_2) \cup [X \setminus \gamma\mathrm{Cl}(B)]$. Since A is $\mathcal{H}_\gamma g$-closed and $(U \cup H_2) \cup [X \setminus \gamma\mathrm{Cl}(B)] \in m$, $\gamma\mathrm{Cl}(A) \subseteq (U \cup H_2) \cup [X \setminus \gamma\mathrm{Cl}(B)] \cup H_3$ for some $H_3 \in \mathcal{H}$. By assumption, there exist $H_4 \in \mathcal{H} \cap m$ such that $\gamma\mathrm{Cl}(A) \subseteq (U \cup H_2) \cup [X \setminus \gamma\mathrm{Cl}(B)] \cup H_4$. Since $B \subseteq A$, we have $\gamma\mathrm{Cl}(B) \subseteq \gamma\mathrm{Cl}(A) \subseteq (U \cup H_2) \cup [X \setminus \gamma\mathrm{Cl}(B)] \cup H_4$. Hence, $\gamma\mathrm{Cl}(B) \subseteq U \cup (H_2 \cup H_4)$ for some $H_2, H_4 \in \mathcal{H}$. Therefor, $\gamma Cl(B) \setminus U \subseteq (H_2 \cup H_4)$. This shows that B is $\mathcal{H}_\gamma g$-closed in X. □

Definition 3.3 Let (X, m) be an m-space. For a subset A of X, $\Lambda_m(A)$ [9] is defined as follows: $\Lambda_m(A) = \cap\{U : A \subseteq U \in m\}$.

Theorem 3.11 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. If $\gamma\mathrm{Cl}(A) \setminus \Lambda_m(A) \in \mathcal{H}$, then A is $\mathcal{H}_\gamma g$-closed.*

Proof: Let $\gamma\mathrm{Cl}(A) \setminus \Lambda_m(A) \in \mathcal{H}$ and V be any m-open set containing A. Then

$$\begin{aligned}\gamma\mathrm{Cl}(A) \setminus V &\subseteq \bigcup_{U \in m} \{\gamma\mathrm{Cl}(A) \setminus U : A \subseteq U\} \\ &= \gamma\mathrm{Cl}(A) \setminus \bigcap_{U \in m} \{U : A \subseteq U\} \\ &= \gamma\mathrm{Cl}(A) \setminus \Lambda_m(A) \in \mathcal{H}\end{aligned}$$

Thus, $\gamma\mathrm{Cl}(A) \setminus V \in \mathcal{H}$ and hence A is $\mathcal{H}_\gamma g$-closed set. □

The following example shows that the converse of the above theorem is not true.

Example 3.4 *Let $X = \{a, b, c, d\}$, $m = \{\emptyset, X, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{c\}\}$. Let $\gamma = mCl$ thus, $\gamma(X) = \{\emptyset, X, \{a, d\}, \{b, c\}\}$. Then, for $A = \{b, d\}$ the collection of all m-open sets containing A is $\mathcal{U} = \{X, \{a, b, d\}, \{b, c, d\}\}$ and $\gamma Cl(A) \setminus U \in \mathcal{H}$ for all $U \in \mathcal{U}$ so, A is an $\mathcal{H}_\gamma g$-closed set. But, $\Lambda_m(A) = \cap\{U : A \subseteq U \in m\} = \{b, d\}$ and $\gamma Cl(A) \setminus \Lambda_m(A) = X \setminus \{b, d\} \notin \mathcal{H}$.*

4. $\mathcal{H}_\gamma g$-open sets

Definition 4.1 Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A subset A of X is said to be $\mathcal{H}_\gamma g$-*open* if $X \setminus A$ is $\mathcal{H}_\gamma g$-closed.

For a m-space (X, m) and a γ-operation on m, the interior operator γInt is associated with closure operator γCl i.e. $\gamma Cl \sim^X \gamma Int$ (see [19]). Therefore, $\gamma Cl(A) = X \setminus \gamma Int(X \setminus A)$ for all $A \subseteq X$.

Theorem 4.1 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A set A is $\mathcal{H}_\gamma g$-open in $(X, m, \mathcal{H})$ if and only if $F \setminus U \subseteq \gamma\mathrm{Int}(A)$ for some $U \in \mathcal{H}$, whenever $F \subseteq A$ and F is m-closed.*

Proof: Suppose A is $\mathcal{H}_\gamma g$-open. Suppose $F \subseteq A$ and F is m-closed. We have $X \setminus A \subseteq X \setminus F$. By assumption, $\gamma\mathrm{Cl}(X \setminus A) \setminus (X \setminus F) \in \mathcal{H}$ and $\gamma\mathrm{Cl}(X \setminus A) \subseteq (X \setminus F) \cup U$ for some $U \in \mathcal{H}$. This implies $X \setminus ((X \setminus F) \cup U) \subseteq X \setminus \gamma\mathrm{Cl}(X \setminus A)$ and hence $F \setminus U \subseteq \gamma\mathrm{Int}(A)$.

Conversely, assume that $F \subseteq A$ and F is m-closed imply $F \setminus U \subseteq \gamma\mathrm{Int}(A)$ for some $U \in \mathcal{H}$. Consider, an m-open set G such that $X \setminus A \subseteq G$. Then $X \setminus G \subseteq A$. By assumption, $(X \setminus G) \setminus U \subseteq \gamma\mathrm{Int}(A) = X \setminus \gamma\mathrm{Cl}(X \setminus A)$. This gives that $X \setminus (G \cup U) \subseteq X \setminus \gamma\mathrm{Cl}(X \setminus A)$. Then, $\gamma\mathrm{Cl}(X \setminus A) \subseteq G \cup U$ for some $U \in \mathcal{H}$. This shows that $\gamma\mathrm{Cl}(X \setminus A) \setminus G \in \mathcal{H}$. Hence, $X \setminus A$ is $\mathcal{H}_\gamma g$-closed and A is $\mathcal{H}_\gamma g$-open. □

Recall that the sets A and B are said to be γ-separated [11] if $\gamma\mathrm{Cl}(A) \cap B = \emptyset$ and $A \cap \gamma\mathrm{Cl}(B) = \emptyset$. Let (X, m) be a topological space and $\mathcal{H}$ is an ideal on X. Then, for $\gamma = ()^*$, γ-separated will be $*_*$-separated [18].

Theorem 4.2 *Let $(X, m, \mathcal{H})$ be an ideal m-space and γ be a γ-operation on m. If A and B are γ-separated $\mathcal{H}_\gamma g$-open sets in X, Then, $A \cup B$ is $\mathcal{H}_\gamma g$-open.*

Proof: Suppose A and B are γ-separated $\mathcal{H}_\gamma g$-open sets in X and F is an m-closed subset of $A \cup B$. Then, since A and B are γ-separated, we have $F \cap \gamma\mathrm{Cl}(A) \subseteq A$ and $F \cap \gamma\mathrm{Cl}(B) \subseteq B$. Now, $F \cap \gamma\mathrm{Cl}(A)$ is m-closed by assumption, $(F \cap \gamma\mathrm{Cl}(A)) \setminus U_1 \subseteq \gamma\mathrm{Int}(A)$ and $(F \cap \gamma\mathrm{Cl}(B)) \setminus U_2 \subseteq \gamma\mathrm{Int}(B)$ for some $U_1, U_2 \in \mathcal{H}$. This mean that $(F \cap \gamma\mathrm{Cl}(A)) \setminus \gamma\mathrm{Int}(A) \in \mathcal{H}$ and $(F \cap \gamma\mathrm{Cl}(B)) \setminus \gamma\mathrm{Int}(B) \in \mathcal{H}$. Then, $[(F \cap \gamma\mathrm{Cl}(A)) \setminus \gamma\mathrm{Int}(A)] \cup [(F \cap \gamma\mathrm{Cl}(B)) \setminus \gamma\mathrm{Int}(B)] \in \mathcal{H}$. Hence, $[F \cap (\gamma\mathrm{Cl}(A) \cup \gamma\mathrm{Cl}(B))] \setminus [\gamma\mathrm{Int}(A) \cup \gamma\mathrm{Int}(B)] \in \mathcal{H}$. But $F = F \cap (A \cup B) \subseteq F \cap \gamma\mathrm{Cl}(A \cup B)$, and we have

$$\begin{aligned} F \setminus \gamma\mathrm{Int}(A \cup B) &\subseteq [F \cap \gamma\mathrm{Cl}(A \cup B)] \setminus \gamma\mathrm{Int}(A \cup B) \\ &\subseteq [F \cap \gamma\mathrm{Cl}(A \cup B)] \setminus [\gamma\mathrm{Int}(A) \cup \gamma\mathrm{Int}(B)] \in \mathcal{H}. \end{aligned}$$

Hence, $F \setminus U \subseteq \gamma\mathrm{Int}(A \cup B)$ for some $U \in \mathcal{H}$. Then, $A \cup B$ is $\mathcal{H}_\gamma g$-open. □

Corollary 4.1 *Let $(X, m, \mathcal{H})$ be an ideal m-space and γ be a γ-operation on m. If $X \setminus A$ and $X \setminus B$ are γ-separated and A, B are $\mathcal{H}_\gamma g$-closed sets in X, Then, $A \cap B$ is $\mathcal{H}_\gamma g$-closed.*

By Theorem 3.6, we have the following corollary:

Corollary 4.2 *Let $(X, m, \mathcal{H})$ be an ideal m-space and γ be a γ-operation on m. If A and B are $\mathcal{H}_\gamma g$-open sets in X, Then, $A \cap B$ is $\mathcal{H}_\gamma g$-open.*

Theorem 4.3 *Let $(X, m, \mathcal{H})$ be an ideal m-space and γ be a γ-operation on m such that the operation γ is m-regular. If $A \subseteq B \subseteq X$ and A is $\mathcal{H}_\gamma g$-open relative to B and B is $\mathcal{H}_\gamma g$-open in X. Then, A is $\mathcal{H}_\gamma g$-open in X.*

Proof: Let A be $\mathcal{H}_\gamma g$-open relative to B and B be $\mathcal{H}_\gamma g$-open relative to X. Suppose $F \subseteq A$ and F is m-closed. Since A is $\mathcal{H}_\gamma g$-open relative to B, by Theorem 4.1, we have $F \setminus U_1 \subseteq \gamma\mathrm{Int}_{\mathrm{B}}(A)$ for some $U_1 \in \mathcal{H}$. This implies that there exists an γ-open set G_1 such that $F \setminus U_1 \subseteq G_1 \cap B \subseteq A$ for some $U_1 \in \mathcal{H}$. Since B is $\mathcal{H}_\gamma g$-open in X, $F \subseteq B$ and F is m-closed, we have $F \setminus U_2 \subseteq \gamma\mathrm{Int}(B)$ for some $U_2 \in \mathcal{H}$. This implies that there exists an γ-open set G_2 such that $F \setminus U_2 \subseteq G_2 \subseteq B$ for some $U_2 \in \mathcal{H}$. Now, $F \setminus (U_1 \cup U_2) \subseteq (F \setminus U_1) \cap (F \setminus U_2) \subseteq G_1 \cap G_2 \subseteq G_1 \cap B \subseteq A$. This is implies that $F \setminus (U_1 \cup U_2) \subseteq \gamma\mathrm{Int}(A)$ for some $U_1 \cup U_2 \in \mathcal{H}$ and hence A is $\mathcal{H}_\gamma g$-open in X. □

Theorem 4.4 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. If $\gamma Int(A) \subseteq B \subseteq A$ and A is $\mathcal{H}_\gamma g$-open set, then B is $\mathcal{H}_\gamma g$-open in X.*

Proof: Suppose $\gamma Int(A) \subseteq B \subseteq A$ and A is $\mathcal{H}_\gamma g$-open. Then, $X \setminus A \subseteq X \setminus B \subseteq X \setminus \gamma Int(A) = \gamma Cl(X \setminus A)$ and $X \setminus A$ is $\mathcal{H}_\gamma g$–closed. By Theorem 3.7, $X \setminus B$ is $\mathcal{H}_\gamma g$-closed and hence B is $\mathcal{H}_\gamma g$-open. □

Theorem 4.5 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A subset A is $\mathcal{H}_\gamma g$-closed in X if and only if $\gamma Cl(A) \setminus A$ is $\mathcal{H}_\gamma g$-open.*

Proof: Suppose $F \subseteq \gamma Cl(A) \setminus A$ and F is m-closed. Then, $X \setminus [\gamma Cl(A) \cap (X \setminus A)] = (X \setminus \gamma Cl(A)) \cup A \subset X \setminus F$ and hence $A \subset X \setminus F \in m$. Since A is $\mathcal{H}_\gamma g$-closed, $F = \gamma Cl(A) \cap F = \gamma Cl(A) \setminus (X \setminus F) \in \mathcal{H}$. This implies that $F \setminus U = \emptyset$, for some $U \in \mathcal{H}$. Clearly, $F \setminus U \subseteq \gamma Int(\gamma Cl(A) \setminus A)$. By Theorem 4.1, $\gamma Cl(A) \setminus A$ is $\mathcal{H}_\gamma g$-open.

Conversely, Suppose $A \subseteq G$ and G is m-open in X. Then, $[\gamma Cl(A) \cap (X \setminus G)] \subseteq [\gamma Cl(A) \cap (X \setminus A)] = \gamma Cl(A) \setminus A$. By hypothesis, $[\gamma Cl(A) \cap (X \setminus G)] \setminus U \subseteq \gamma Int[\gamma Cl(A) \setminus A] = \emptyset$, for some $U \in \mathcal{H}$. This implies that $[\gamma Cl(A) \cap (X \setminus G)] \subseteq U \in \mathcal{H}$ and hence $\gamma Cl(A) \setminus G \in \mathcal{H}$. Thus, A is $\mathcal{H}_\gamma g$-closed. □

Definition 4.2 Let γ (resp. δ) be an operation on m (resp. n). A function $f : (X, m) \to (Y, n)$ is said to be (γ, δ)-*closed* if $f(V)$ is δ-closed in Y for each γ-closed set V of X.

Definition 4.3 [23] A function $f : (X, m) \to (Y, n)$ is said to be (m, n)-*continuous* if for each n-open set V of Y $f^{-1}(V)$ is m-open in X.

Theorem 4.6 *Let $f : (X, m, \mathcal{H}) \to (Y, n)$ be an (m, n)-continuous and (γ, δ)-closed function. If $A \subseteq X$ is $\mathcal{H}_\gamma g$-closed in X, then $f(A)$ is $f(\mathcal{H})_\delta g$-closed in Y, where $f(\mathcal{H}) = \{f(U) : U \in \mathcal{H}\}$.*

Proof: Let A be an $\mathcal{H}_\gamma g$-closed subset of X and $f(A) \subseteq G$, where G is n-open. Then, $A \subseteq f^{-1}(G)$ and $f^{-1}(G)$ is an m-open set in X. Then, by definition of $\mathcal{H}_\gamma g$-closed, $\gamma Cl(A) \setminus f^{-1}(G) \in \mathcal{H}$ and hence $f(\gamma Cl(A)) \setminus G \in f(\mathcal{H})$. Since f is (γ, δ)-closed, $\delta Cl(f(A)) \subseteq \delta Cl(f(\gamma Cl(A))) = f(\gamma Cl(A))$. Then, $\delta Cl(f(A)) \setminus G \subseteq f(\gamma Cl(A)) \setminus G \in f(\mathcal{H})$ and hence $f(A)$ is $f(\mathcal{H})_\gamma g$-closed in Y. □

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