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α-Product of Product Fuzzy Graphs

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ABSTRACT: In this article, a new operation on product fuzzy graphs (PFGs) is provide; namely $\alpha-$product. Conditions for the $\alpha-$product of two PFGs to be strong are provided and we prove if the $\alpha-$product of two PFGs is complete, then one of them is strong. We also study the unbiased notion of the class of PFGs and also conditions for the $\alpha-$product to be unbiased are given.

Key Words: FG, complete PFG, unbiased PFG, α-product.

Contents

1. Background

Theory of graph has many applications in mathematics and economics. Since most problems of graphs are undetermined, it is necessary to handle these facets via the method of fuzzy logic. Fuzzy relations were made known by Zadeh [24]. Rosenfeld [21] in 1975, defined fuzzy graphs (simply, FG) and some ideas that are generalizations of those of graph's. Now days, this theory is having more and more applications in which the information level immanent in the set of things working together as parts of a mechanism differ with various degrees of accuracy. Fuzzy fashion are convenient as they reduce differences between long-established numerical models of expert systems and symbolic models. Peng and Mordeson [16] defined the conceptualization of FG's complement and conscious FG's operations. In [23], improved complement's definition in order to guarantee the original FG is isomorphic to complement of the complement, which concur with the case of crisp graphs. In addition, self-complementary FGs properties and the complement under FG's join, union and composition (introduced in [16]) were explored. Al-Hawary [6] introduced the concept of balanced in the class of FGs and Al-Hawary and others have deeply explored this ides for many types of FGs. For more on the foregoing concepts and those coming after ones, one can see [1,2,3,4,5,6,7,8,9,10,11,12,13,16,18,19,20,23].

For a non-empty finite set $\H{U}$, a fuzzy subset of $\H{U}$ is a mapping ꝭ:$\H{U}\to [0,1]$ and a fuzzy subset of $\H{U}\times\H{U}$ is called a fuzzy relation ς on ꝭ. We assume that $\H{U}$ is finite and ς is reflexive and symmetric.

Definition 1.1 *[21] A* fuzzy graph (simply, FG), *with $\H{U}$ as the underlying set, is a pair $\Omega : (ꝭ,\varsigma)$ where ꝭ:$\H{U}\to [0,1]$ is a fuzzy subset and $\varsigma : \H{U}\times \H{U}\to [0,1]$ is a fuzzy relation on ꝭ such that $\varsigma(c,s) \leq$ ꝭ$(c)\wedge$ꝭ(s) for all $c,s \in \H{U}$, where by $\wedge$, we mean the minimum. Its classical graph is $\Omega^* : (ꝭ^*,\varsigma^*)$ where $ꝭ^* = \sup c(ꝭ) = \{c \in \H{U}: ꝭ(c) > 0\}$ and $\varsigma^* = \sup c(\varsigma) = \{(c,s) \in \H{U}\times \H{U}: \varsigma(c,s) > 0\}$.*

Definition 1.2 *[21] Two FGs $\Omega_1 : (ꝭ_1,\varsigma_1)$ and $\Omega_2 : (ꝭ_2,\varsigma_2)$ are said to be* isomorphic *providing the existence of a bijective $\tau : \H{U}_1 \to \H{U}_2$ such that $ꝭ_1(c) = ꝭ_2(\tau(c))$ for all $c \in \H{U}_1$ and $\varsigma_1(c,s) = \varsigma_2(\tau(c),\tau(s))$ for all $(c,s) \in F_1$. We then write $\Omega_1 \simeq \Omega_2$ and h is called an* isomorphism.

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Using the operation of product instead of minimum, Ramaswamy and Poornima in [22] established PFGs.

Definition 1.3 *[22] Let $\Omega^* : (\H{U}, \digamma)$ be a graph, ξ be a fuzzy subset of $\H{U}$ and ς be a fuzzy subset of $\H{U} \times \H{U}$. $\Omega : (\xi, \varsigma)$ is called a* product fuzzy graph (simply, PFG) *if $\varsigma(c, s) \leq \xi(c)\xi(s)$ for all $c, s \in \H{U}$.*

Next result is immediate:

Lemma 1.1 *Every PFG is a FG.*

Definition 1.4 *[22] A PFG $\Omega : (\xi, \varsigma)$ is called* complete *if $\varsigma(c, s) = \xi(c)\xi(s)$ for all $c, s \in \H{U}$.*

Definition 1.5 *[22] A PFG $\Omega : (\xi, \varsigma)$ is called* strong *if $\varsigma(c, s) = \xi(c)\xi(s)$ for all $(c, s) \in \digamma$.*

Definition 1.6 *[22] The* complement *of a PFG $\Omega : (\xi, \varsigma)$ is $\Omega^c : (\xi^c, \varsigma^c)$ where $\xi^c = \xi$ and*

$$\begin{aligned} \varsigma^c(c, s) &= \xi^c(c)\xi^c(s) - \varsigma(c, s) \\ &= \xi(c)\xi(s) - \varsigma(c, s). \end{aligned}$$

Lemma 1.2 *[11]If $\Omega : (\xi, \varsigma)$ is a self-complementary PFG, then*

$$\sum_{(c,s)\in\digamma} \varsigma(c, s) = \frac{1}{2} \sum_{(c,s)\in\digamma} \xi(c)\xi(s).$$

Lemma 1.3 *[11]Let $\Omega : (\xi, \varsigma)$ be a PFG such that $\varsigma(c, s) = \frac{1}{2}\xi(c)\xi(s)$ for all $c, s \in \H{U}$. Then Ω is self-complementary.*

Several types of products of two FGs were explored. The notion of $\alpha-$product of FGs was introduced and studied in [17] where the regularity property for this product was the main idea. In Section 2 of this paper, we launch the conception of $\alpha-$product of PFGs. Conditions for the $\alpha-$product of two PFGs to be strong (complete) are established and if at least one factor is a complete PFG, the $\alpha-$product should be complete. Section 3 is devoted to provide equivalent conditions for the $\alpha-$product of two unbiased PFGs to be unbiased.

2. $\alpha-$product of product fuzzy graphs

We begin this section by defining the rooted product of PFGs.

Definition 2.1 *The $\alpha-$*product *of two PFGs $\Omega_1 : (\xi_1, \varsigma_1)$ is defined to be the PFG $\Omega_1 \boxplus_\alpha \Omega_2 : (\xi_1 \boxplus_\alpha \xi_2, \varsigma_1 \boxplus_\alpha \varsigma_2)$ on the vertex set $\H{U}_1 \times \H{U}_2$, where*

$$(\xi_1 \boxplus_\alpha \xi_2)(\H{u}, \ddot{y}) = \xi_1(\H{u})\xi_2(\ddot{y}), \text{ for all } (\H{u}, \ddot{y}) \in \H{U}_1 \times \H{U}_2 \text{ and}$$

$$(\varsigma_1 \boxplus_\alpha \varsigma_2)((\H{u}_1, \ddot{y}_1)(\H{u}_2, \ddot{y}_2)) = \begin{cases} \xi_2(\ddot{y}_1)\xi_2(\ddot{y}_2)\varsigma_1(\H{u}_1\H{u}_2) & \H{u}_1\H{u}_2 \in \digamma_1, \ddot{y}_1\ddot{y}_2 \notin \digamma_2 \\ \xi_1(\H{u}_1)\xi_1(\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2) & \H{u}_1\H{u}_2 \notin \digamma_1, \ddot{y}_1\ddot{y}_2 \in \digamma_2 \\ \xi_1(\H{u}_1)\varsigma_2(\ddot{y}_1\ddot{y}_2) & \H{u}_1 = \H{u}_2, \ddot{y}_1\ddot{y}_2 \in \digamma_2 \\ \xi_2(\ddot{y}_1)\varsigma_1(\H{u}_1\H{u}_2) & \H{u}_1\H{u}_2 \in \digamma_1, \ddot{y}_1 = \ddot{y}_2 \end{cases}$$

.

Next, we show that the above definition is well-defined.

Theorem 2.1 *The $\alpha-$product of two PFGs is a PFG.*

Proof: Let $\Omega_1 : (ţ_1, ς_1)$ and $\Omega_2 : (ţ_2, ς_2)$ be two PFGs.
Case 1: If $ű_1ű_2 \in Ϝ_1, ÿ_{1ÿ2} \notin Ϝ_2$, then

$$\begin{aligned}(ς_1 \boxplus_\alpha ς_2)((ű_1, ÿ_1)(ű_2, ÿ_2)) &= ţ_2(ÿ_1)ţ_2(ÿ_2)ς_1(ű_1ű_2)\\ &\leq ţ_1(ű_1)ţ_1(ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2)\\ &= ((ţ_1 \boxplus_\alpha ţ_2)(ű_1, ÿ_1))((ţ_1 \boxplus_\alpha ţ_2)(ű_2, ÿ_2)).\end{aligned}$$

Case 2: If $ű_1ű_2 \notin Ϝ_1, ÿ_1ÿ_2 \in Ϝ_2$, this case is similar to Case 1.
Case 3: If $ű_1 = ű_2, ÿ_1ÿ_2 \in Ϝ_2$, then

$$\begin{aligned}(ς_1 \boxplus_\alpha ς_2)((ű_1, ÿ_1)(ű_2, ÿ_2)) &= ţ_1(ű_1)ς_2(ÿ_1ÿ_2)\\ &\leq ţ_1(ű_1)ţ_1(ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2)\\ &= ((ţ_1 \boxplus_\alpha ţ_2)(ű_1, ÿ_1))((ţ_1 \boxplus_\alpha ţ_2)(ű_2, ÿ_2)).\end{aligned}$$

Case 4: If $ű_1ű_2 \in Ϝ_1, ÿ_1 = ÿ_2$, this case is similar to Case 3. □

Theorem 2.2 *If $\Omega_1 : (ţ_1, ς_1)$ and $\Omega_2 : (ţ_2, ς_2)$ are strong PFGs, then $\Omega_1 \boxplus_\alpha \Omega_2$ is a strong PFG.*

Proof: Let $\Omega_1 : (ţ_1, ς_1)$ and $\Omega_2 : (ţ_2, ς_2)$ be two strong PFGs.
Case 1: If $ű_1ű_2 \in Ϝ_1, ÿ_{1ÿ2} \notin Ϝ_2$, then as Ω_1 is strong,

$$\begin{aligned}(ς_1 \boxplus_\alpha ς_2)((ű_1, ÿ_1)(ű_2, ÿ_2)) &= ţ_2(ÿ_1)ţ_2(ÿ_2)ς_1(ű_1ű_2)\\ &= ţ_1(ű_1)ţ_1(ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2)\\ &= ((ţ_1 \boxplus_\alpha ţ_2)(ű_1, ÿ_1))((ţ_1 \boxplus_\alpha ţ_2)(ű_2, ÿ_2)).\end{aligned}$$

Case 2: If $ű_1ű_2 \notin Ϝ_1, ÿ_1ÿ_2 \in Ϝ_2$, is similar to Case 1.
Case 3: If $ű_1 = ű_2, ÿ_1ÿ_2 \in Ϝ_2$, then as Ω_2 is strong,

$$\begin{aligned}(ς_1 \boxplus_\alpha ς_2)((ű_1, ÿ_1)(ű_2, ÿ_2)) &= ţ_1(ű_1)ς_2(ÿ_1ÿ_2)\\ &= ţ_1(ű_1)ţ_1(ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2)\\ &= ((ţ_1 \boxplus_\alpha ţ_2)(ű_1, ÿ_1))((ţ_1 \boxplus_\alpha ţ_2)(ű_2, ÿ_2)).\end{aligned}$$

Case 4: If $ű_1ű_2 \in Ϝ_1, ÿ_1 = ÿ_2$, this case is similar to Case 3.
Thus, $\Omega_1 \boxplus_\alpha \Omega_2$ is a strong PFG. □

Corollary 2.1 *If $\Omega_1 : (ţ_1, ς_1)$ and $\Omega_2 : (ţ_2, ς_2)$ are fuzzy complete (strong) FGs, then $\Omega_1 \boxplus_\alpha \Omega_2$ is a strong FG.*

We remark that if both PFGs are complete, then their $\alpha-$product need not be a complete PFG.

Example 2.1 Consider $\Omega_1 : (ţ_1, ς_1)$ where $ţ_1(ű) = .2, ţ_1(ÿ) = .4, ς_1(ű, Ű) = .08$ and $\Omega_2 : (ţ_2, ς_2)$ where $ţ_2(c) = .1 = ţ_2(s)$ and $ς_2(c, s) = .01$. Then both are are complete PFGs while $\Omega_1 \boxplus_\alpha \Omega_2$ is not a complete PFG since $(ς_1 \boxplus_\alpha ς_2)((ű, c)(ű, s)) = .002 \neq (.02)(.02) = .0004 = (ţ_1 \boxplus_\alpha ţ_2)(ű, c)(ţ_1 \boxplus_\alpha ţ_2)(ű, s)$.

A nice property of complement is the following:.

Theorem 2.3 *If $\Omega_1 : (\mathfrak{t}_1, \varsigma_1)$ and $\Omega_2 : (\mathfrak{t}_2, \varsigma_2)$ are complete PFGs, then $(\Omega_1 \boxplus_\alpha \Omega_2)^c \simeq \Omega_1^c \boxplus_\alpha \Omega_2^c$.*

Proof: Let $\Omega : (\mathfrak{t}, \overline{\varsigma}) = (\Omega_1 \boxplus_\alpha \Omega_2)^c$, $\overline{\varsigma} = (\varsigma_1 \boxplus_\alpha \varsigma_2)^c$, $(\Omega^*)^c = (\H{U}, F^c)$, $\overline{\Omega_1} : (\mathfrak{t}_1, \varsigma_1^c)$, $(\Omega_1^*)^c = (\H{U}_1, F_1^c)$, $\Omega_2^c : (\mathfrak{t}_2, \varsigma_2^c)$, $(\Omega_2^*)^c = (\H{U}_2, (F_2)^c)$ and $\Omega_1^c \boxplus_\alpha \Omega_2^c : (\mathfrak{t}_1 \boxplus_\alpha \mathfrak{t}_2, \varsigma_1^c \boxplus_\alpha \varsigma_2^c)$. We only need to show $(\varsigma_1 \boxplus_\alpha \varsigma_2)^c = \varsigma_1^c \boxplus_\alpha \varsigma_2^c$. Given an arc ϵ joining nodes of $\H{U}$, $\H{u}_1\H{u}_2 \in F_1, \acute{y}_1\acute{y}_2 \notin F_2$ and $\H{u}_1\H{u}_2 \notin F_1, \acute{y}_1\acute{y}_2 \in F_2$ are not possible to occur as both Ω_1 and Ω_2 are complete. If $\H{u}_1 = \H{u}_2, \acute{y}_1\acute{y}_2 \in F_2$, then as Ω_1 is complete, $\varsigma_1^c(\epsilon) = 0$. But $(\varsigma_1 \boxplus_\alpha \varsigma_2)^c(\epsilon) = 0$ since $\H{u}_1\H{u}_2 \notin F_1^c$ and $\acute{y}_1 \notin F_2^c$. The case $\H{u}_1\H{u}_2 \in F_1, \acute{y}_1 = \acute{y}_2$ is similar.

In all cases $(\varsigma_1 \boxplus_\alpha \varsigma_2)^c = \varsigma_1^c \boxplus_\alpha \varsigma_2^c$ and therefore, $(\Omega_1 \boxplus_\alpha \Omega_2)^c \simeq \Omega_1^c \boxplus_\alpha \Omega_2^c$. □

Next, we prove if the $\alpha-$product of two PFGs is complete, then both can not be not complete.

Theorem 2.4 *If $\Omega_1 : (\mathfrak{t}_1, \varsigma_1)$ and $\Omega_2 : (\mathfrak{t}_2, \varsigma_2)$ are PFGs where $\Omega_1 \boxplus_\alpha \Omega_2$ is complete, then one PFG is complete.*

Proof: Suppose to the contrary that both PFGs are not complete. Thus there exist $\H{u}_1, \H{u}_2 \in \H{U}_1$ and $\acute{y}_1, \acute{y}_2 \in \H{U}_2$ with

$$\begin{aligned} \varsigma_1(\H{u}_1\H{u}_2) &< \mathfrak{t}_1(\H{u}_1)\mathfrak{t}_1(\H{u}_2)) \text{ and} \\ \varsigma_2(\acute{y}_1\acute{y}_2) &< \mathfrak{t}_2(\acute{y}_1)\mathfrak{t}_2(\acute{y}_2)). \end{aligned}$$

Case 1: If $\H{u}_1\H{u}_2 \in F_1, \acute{y}_{1\acute{y}2} \notin F_2$, then $(\varsigma_1 \boxplus_\alpha \varsigma_2)((\H{u}_1, \ddot{y}_1)(\H{u}_2, \ddot{y}_2)) = \varsigma_1(\H{u}_1\H{u}_2)\mathfrak{t}_2(\mathrm{v}_1)\mathfrak{t}_2(\ddot{y}_2)$ and as $\Omega_1 \boxplus_\alpha \Omega_2$ is complete,

$$\begin{aligned} (\varsigma_1 \boxplus_\alpha \varsigma_2)((\H{u}_1, \ddot{y}_1)(\H{u}_2, \ddot{y}_2)) &= (\mathfrak{t}_1 \boxplus_\alpha \mathfrak{t}_2)((\H{u}_1, \ddot{y}_1))(\mathfrak{t}_1 \boxplus_\alpha \mathfrak{t}_2)((\H{u}_2, \mathrm{v}_2)) \\ &> \mathfrak{t}_1(\H{u}_1)\mathfrak{t}_1(\H{u}_2)\mathfrak{t}_2(\ddot{y}_1)\mathfrak{t}_2(\ddot{y}_2) \\ &= \varsigma_1(\H{u}_1\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2), \end{aligned}$$

which is a contradiction.

Case2: $\H{u}_1\H{u}_2 \notin F_1, \ddot{y}_{1\ddot{y}2} \in F_2$ is similar.

Case 3: If $\H{u}_1 = \H{u}_2, \ddot{y}_{1\ddot{y}2} \in F_2$, then as $\Omega_1 \boxplus_\alpha \Omega_2$ is complete,

$$\begin{aligned} (\varsigma_1 \boxplus_\alpha \varsigma_2)((\H{u}_1, \ddot{y}_1)(\H{u}_2, \mathrm{v}_2)) &= (\mathfrak{t}_1 \boxplus_\alpha \mathfrak{t}_2)((\H{u}_1, \mathrm{v}_1))(\mathfrak{t}_1 \boxplus_\alpha \mathfrak{t}_2)((\H{u}_2, \ddot{y}_2)) \\ &= \mathfrak{t}_1(\H{u}_1)\mathfrak{t}_1(\H{u}_2)\mathfrak{t}_2(\ddot{y}_1)\mathfrak{t}_2(\ddot{y}_2) \\ &> \varsigma_1(\H{u}_1\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2), \end{aligned}$$

which is a contradiction.

Case 4: If $\H{u}_1\H{u}_2 \in F_1, \ddot{y}_1 = \ddot{y}_2$, this case is similar to Case 3. □

3. Unbiased product fuzzy graphs

We begin this section by recalling the definition of unbiased (balanced) PFGs from [11] and then proving the following Lemma 3.1 to make it possible characterize unbiased $\alpha-$product of two unbiased PFGs.

Definition 3.1 *[11]. The* compactness degree *of a PFG is* $cd(\Omega) = \dfrac{2\sum\limits_{\H{u}\ddot{y} \in F} (\varsigma(\H{u}\ddot{y}))}{\sum\limits_{\H{u}, \ddot{y} \in \H{U}} (\mathfrak{t}(\H{u}) \wedge \mathfrak{t}(\ddot{y}))}$. *$\Omega$ is unbiased if $cd(\Omega) \geq cd(H)$ for any non-empty PFS H of Ω.*

Lemma 3.1 *Let Ω_1 and Ω_2 be PFGs. Then $cd(\Omega_1 \boxplus_\alpha \Omega_2) \geq cd(\Omega_1)$ and $cd(\Omega_2) \leq cd(\Omega_1 \boxplus_\alpha \Omega_2)$ iff $cd(\Omega_1) = cd(\Omega_2) = cd(\Omega_1 \boxplus_\alpha \Omega_2)$.*

Proof: If $cd(\Omega_1) \leq cd(\Omega_1 \boxplus_\alpha \Omega_2)$ and $cd(\Omega_2) \leq cd(\Omega_1 \boxplus_\alpha \Omega_2)$, then

$$\begin{aligned}
cd(\Omega_1) &= 2(\sum_{ű_1,ű_2 \in Ű_1} \varsigma_1(ű_1ű_2))/(\sum_{ű_1,ű_2 \in Ű_1} (ţ_1(ű_1) \wedge ţ_1(ű_2))) \\
&\geq 2(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ \ddot{y}_1,\ddot{y}_2 \in Ű_2}} \varsigma_1(ű_1ű_2)ţ_2(\ddot{y}_1)ţ_2(\ddot{y}_2))/(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ \ddot{y}_1,\ddot{y}_2 \in Ű_2}} (ţ_1(ű_1)ţ_1(ű_2)ţ_2(\ddot{y}_1)ţ_2(\ddot{y}_2))) \\
&\geq 2(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ \ddot{y}_1,\ddot{y}_2 \in Ű_2}} \varsigma_1(ű_1ű_2)\varsigma_2(\ddot{y}_1\ddot{y}_2))/(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ \ddot{y}_1,\ddot{y}_2 \in Ű_2}} (ţ_1(ű_1)ţ_1(ű_2)ţ_2(\ddot{y}_1)ţ_2(\ddot{y}_2))) \\
&\geq 2(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ \ddot{y}_1,\ddot{y}_2 \in Ű_2}} \varsigma_1 \boxplus_\alpha \varsigma_2((ű_1\ddot{y}_1)(ű_2\ddot{y}_2))/(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ \ddot{y}_1,\ddot{y}_2 \in Ű_2}} (ţ_1 \boxplus_\alpha ţ_2((ű_1,\ddot{y}_1)(ű_2,\ddot{y}_2))) \\
&= cd(\Omega_1 \boxplus_\alpha \Omega_2).
\end{aligned}$$

Hence in all cases $cd(\Omega_1) \geq cd(\Omega_1 \boxplus_\alpha \Omega_2)$ and thus $cd(\Omega_1) = cd(\Omega_1 \boxplus_\alpha \Omega_2)$. Similarly, $cd(\Omega_2) = cd(\Omega_1 \boxplus_\alpha \Omega_2)$. Therefore, $cd(\Omega_1) = cd(\Omega_2) = cd(\Omega_1 \boxplus_\alpha \Omega_2)$. The converse is trivial. □

Theorem 3.1 *For two unbiased PFGs Ω_1 and Ω_2, $\Omega_1 \boxplus_\alpha \Omega_2$ is unbiased if and only if $cd(\Omega_1) = cd(\Omega_2) = cd(\Omega_1 \boxplus_\alpha \Omega_2)$.*

Proof: If $\Omega_1 \boxplus_\alpha \Omega_2$ is unbiased, then $cd(\Omega_1) \leq cd(\Omega_1 \boxplus_\alpha \Omega_2)$ and $cd(\Omega_2) \leq cd(\Omega_1 \boxplus_\alpha \Omega_2)$ and by Lemma 3.1, $cd(\Omega_1) = cd(\Omega_2) = cd(\Omega_1 \boxplus_\alpha \Omega_2)$.

If $cd(\Omega_1) = cd(\Omega_2) = cd(\Omega_1 \boxplus_\alpha \Omega_2)$ and K is a product FS of $\Omega_1 \boxplus_\alpha \Omega_2$, then we can find product FSs K_i of Ω_i for $i = 1, 2$ with $K \approx K_1 \boxplus_\alpha K_2$. As Ω_1 and Ω_2 are unbiased and $cd(\Omega_1) = cd(\Omega_2) = m_1/k_1$, then $cd(K_1) = a_1/b_1 \leq m_1/k_1$ and $cd(K_2) = a_2/b_2 \leq m_1/k_1$. Thus $a_1k_1 + a_2k_1 \leq b_1m_1 + b_2m_1$ and hence $cd(K) \leq (a_1 + a_2)/(b_1 + b_2) \leq m_1/k_1 = cd(\Omega_1 \boxplus_\alpha \Omega_2)$. Therefore, $\Omega_1 \boxplus_\alpha \Omega_2$ is unbiased. □

We end this section with the following result which states that unbiased notion is preserved under isomorphism:

Theorem 3.2 *Let Ω_1 and Ω_2 be isomorphic PFGs. If one of them is unbiased, then the other is unbiased.*

Proof: Suppose Ω_2 is unbiased and let $\epsilon : Ű_1 \to Ű_2$ be a bijection such that $ţ_1(ű) = ţ_2(\epsilon(ű))$ and $\varsigma_1(ű\ddot{y}) = \varsigma_2(\epsilon(ű)\epsilon(\ddot{y}))$ for all $ű, \ddot{y} \in Ű_1$. Now $\sum_{ű \in Ű_1} ţ_1(ű) = \sum_{ű \in Ű_2} ţ_2(ű)$ and $\sum_{ű\ddot{y} \in F_1} \varsigma_1(ű\ddot{y}) = \sum_{ű\ddot{y} \in F_2} \varsigma_2(ű\ddot{y})$. If $K_1 = (ţ_1, \varsigma_1)$ is a product FS of Ω_1 with underlying set W, then $K_2 = (ţ_2, \varsigma_2)$ is a product FS of Ω_2 with underlying set $\epsilon(W)$ where $ţ_2(\epsilon(ű)) = ţ_1(ű)$ and $\varsigma_2(\epsilon(ű)\epsilon(\ddot{y})) = \varsigma_1(ű\ddot{y})$ for all $ű, \ddot{y} \in W$. Since Ω_2 is unbiased, $cd(K_1) \leq cd(\Omega_2)$ and so $2\frac{\sum_{ű\ddot{y} \in F_1} \varsigma_2(\epsilon(ű)\epsilon(\ddot{y}))}{\sum_{ű,\ddot{y} \in Ű_1}(ţ_2(ű) \wedge ţ_2(\ddot{y}))} \leq 2\frac{\sum_{ű\ddot{y} \in F_1} \varsigma_2(ű\ddot{y})}{\sum_{ű,\ddot{y} \in Ű_1}(ţ_2(ű) \wedge ţ_2(\ddot{y}))}$. Hence

$$2\frac{\sum_{ű\ddot{y} \in F_1} \varsigma_1(ű\ddot{y})}{\sum_{ű,\ddot{y} \in Ű_1}(ţ_2(ű) \wedge ţ_2(\ddot{y}))} \leq 2\frac{\sum_{ű\ddot{y} \in F_1} \varsigma_1(ű\ddot{y})}{\sum_{ű,\ddot{y} \in Ű_1}(ţ_2(ű) \wedge ţ_2(\ddot{y}))}.$$

Therefore, Ω_1 is unbiased. □

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