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Existence of Renormalized Solutions to Non-Linear $p(\cdot)$-Parabolic Problems of Generalized Porous Medium with General Measure Data

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ABSTRACT: In this work, we investigate the existence of renormalized solutions for a nonlinear parabolic problem with variable exponents and general measure data. The solutions are achieved by combining monotone operator theory, Marcinkiewicz estimation, and the truncation method.

Key Words: Measure data, Marcinkiewicz estimation, renormalized solutions.

Contents

1. Introduction

The objective of this paper is to determine some results of the existence for renormalized solution to a nonlinear parabolic equation modeled as follows:

$$(\mathcal{P})\quad \begin{cases} \dfrac{\partial b(z,w)}{\partial t} - div\Big(\Phi(t,z,w,\nabla w)\Big) = \mu & \text{in } Q_T := \Omega \times (0,T),\\ b(z,w)(t=0) = b(z,w_0) & \text{in } \Omega,\\ w = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded open subset with smooth boundary $\partial\Omega$, $-div(\Phi(t,z,w,\nabla w))$ is a Leray Lions operator which verifies the polynomial $p(z)-$growth condition with respect to w and ∇w and $b(z,w)$ is an unbounded function of w. Furthermore, we suppose that $b(z,w_0) \in L^1(\Omega)$ and $\mu \in \mathcal{M}_b(Q_T)$.

The utilization of partial differential equations with variable exponents has found application in various models of fluid dynamics, particularly in the context of electro-rheological and thermo-rheological fluids [3]. Additionally, these equations have been employed in fields such as robotics, fluid dynamics, and image processing (see [16]). Conversely, our interest in investigating problem $(\mathcal{P})$ arises from its relevance in modelling diverse physical phenomena linked to electro-rheological fluids, as noted in Rajagopal's work(see [46]). These fluids possess the unique ability to alter their mechanical properties in response to external electro-magnetic fields. Key domains benefiting from this research encompass continuum mechanics, population dynamics, and image processing (see [53,20]). Notably, the central rationale for introducing the concept of capacity lies in its capacity to yield optimally regular boundary results. Hence, an extended adaptation of this concept is deemed appropriate when addressing generalized Lebesgue-Sobolev spaces.

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From a theoretical approach, the study of nonlinear boundary problems with variable growth conditions allows the development of a new class of functional frameworks. They are called Lebesgue and Sobolev spaces with variable exponent which are denoted $L^{p(z)}(\Omega)$ and $W^{1,p(z)}(\Omega)$, respectively. In this direction, Sharapudinov et. al. in [52] represents an important step as it introduces a focused study of the topology of spaces $L^{p(z)}(E)$, which encompass measurable functions on a set E with $p(z) \geq 1$. For a more complete overview, readers are invited to consult the article by Samko et. al. in [51]. In [20], authors present an overview of various unsolved questions concerning variable exponent spaces. in addition, we recommend the article [28] to gain a basic understanding of spaces of functions with variable exponents and to explore spaces of related functions. Authors are also advised to consult similar articles, such as [26]. By incorporating these references, the authors will provide readers with a solid foundation for understanding the concepts and recent developments in the field of function spaces with variable exponents.

More recently, the study of these problems has aroused increasing interest in recent years. Nevertheless, all the works devoted to the analysis of this type of problem highlight the existence results of the problem as it has been developed by some authors, (see [1,42,43,49,50]). A famous book that we strongly recommend is [45]. It is an excellent and very complete introduction to the study of boundary value problems with variable exponents. In order to develop the analysis, we will review some previous work in which a special case of the problem $(\mathcal{P})$ has been studied. First, we recall some results related to the parabolic equation $(\mathcal{P})$ with the datum μ being a bounded Radon measure on Q_T. In [15], Bouajaja et al. considered the equation $p(z)-$parabolic $(\mathcal{P})$ with $p(z) = p$ is a constant, where b is supposed to be a strictly increasing $\mathcal{C}^1-$function, $\Phi(z, t, w)\nabla w = \Phi(z, t, w, \nabla w)$, and μ is a bounded measure. The authors ensured the existence of a distributive solution to the considered problem, however, because of lack of regularity of the solution, the distributive formulation is not strong enough to ensure the uniqueness. Later, in order to overcome this constraint, the new idea of renormalized solutions was for the first time presented by Di-Perna and Lions in [22]. they investigate the Boltzmann equation, and extended it to the parabolic (and elliptic) equations with L^1 data (see [21,8,9,10,33,18]). Concerning the measure μ (where $p(z) = p$ is a constant), the existence and uniqueness of the renormalized solution of $(\mathcal{P})$ were proved in [23] where $b(z, w) = w,\ w_0 \in L^1(\Omega)$ and for any μ measure which does not load on sets of zero $p-$capacity, this measures so-called diffuse measures or concentrated measures, and we shall employ the symbol $\mu \in \mathcal{M}_0(Q_T)$ to indicate them. The importance of this type of measure was initially remarked in the stable case in [14], and elaborated in the evolving case in [23]. When $b(z, w) = b(w), \mu \in \mathcal{M}_0(Q_T)$ and $w_0 \in L^1(\Omega)$, the same problematic subject was considered in [12], also In the case when $\mu \in \mathcal{M}_0(Q_T))$ and with $b(z, w)$ the existence of the renormalized solution of $(\mathcal{P})$ has been proved in [32], we recall that several authors have approached the same theme under different assumptions and in different contexts, see [35,36,37,38]. In a different situation, Chipot et al. [17] establish, under certain conditions, a proof of an explosion result in the case $b(x, t) = w$.

Our contribution represents original work by extending and generalizing prior findings found in the existing literature [2,48]. This paper is dedicated to investigating the well-posedness of renormalized solutions for problem $(\mathcal{P})$, considering its dependence on parameter s. Our focus encompasses scenarios involving general measures, and our findings contribute novel insights to the treatment of such problems. We develop an approximate series of solutions and establish certain preliminary estimates. Subsequently, we extract a subsequence to arrive at the limiting function, demonstrating its status as a renormalized solution. By averaging both "cut-off" test functions and the "near-far from" approach, we unveil fresh properties that facilitate the treatment of the measure's singular component. Notably, our approach avoids relying on the strong convergence of truncations, and it is extensible to a broader class of non-monotone operators, denoted as Φ, with respect to w.

The organisation of this document can be summarised as follows. In Section 2, we focus on presenting fundamental notions about capacity and the essential characteristics of metrics. In Section 3, we introduce the key assumptions underlying our work, leading to the formulation of an existence theorem. Finally, Section 4 is entirely devoted to the proof of our central result.

2. Preliminaries

2.1. Sobolev spaces with variable exponents

In the study of $(\mathcal{P})$, we employ the theory of generalized Lebesgue-Sobolev spaces $L^{p(z)}(\Omega)$ and $W_0^{1,p(z)}(\Omega)$. For the reader's understanding, we point out only some background facts which will eventually be applied, we send back to [24,55] for more details.
Consider $p : \overline{\Omega} \to [1, +\infty)$ be a continuous function, we have

$$p^- = \min_{z\in\overline{\Omega}} p(z) \ and \ p^+ = \max_{z\in\overline{\Omega}} p(z).$$

In the following, we suppose that

$$\mathcal{C}_+(\Omega) = \big\{p \text{ is a real measurable functions on } \Omega \text{ such that } 1 < p^- \leq p^+ < N\big\},$$

where

$$1 < p^- \leq p(z) \leq p^+ < \infty. \tag{2.1}$$

Denote

$$E = \Big\{\varpi : \varpi \text{ is a measurable function on } \Omega\Big\}.$$

We give the spaces $L^{p(z)}(\Omega)$ as follows

$$L^{p(z)}(\Omega) = \Big\{w \in E : \int_\Omega |w(z)|^{p(z)} dz < +\infty\Big\}.$$

We endow the space $L^{p(z)}(\Omega)$ by the so-called Luxemburg norm

$$\|w\|_{p(z)} = \inf\Big\{\lambda > 0; \int_\Omega \Big|\frac{w(z)}{\lambda}\Big|^{p(z)} dz \leq 1\Big\}.$$

Recall the inequality below, which will be used thereafter.

$$\min\Big\{\|w\|_{p(z)}^{p^-} \ ; \ \|w\|_{p(z)}^{p^+}\Big\} \leq \int_\Omega |w(z)|^{p(z)} dz \leq \max\Big\{\|w\|_{p(z)}^{p^-} \ ; \ \|w\|_{p(z)}^{p^+}\Big\}.$$

By hypothesis(2.1), the space $L^{p(z)}(\Omega)$ becomes a separable and reflexive Banach space. We set the dual space of $L^{p(z)}(\Omega)$by $L^{p'(z)}(\Omega)$ with $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$. Moreover, for each $f \in L^{p(z)}(\Omega)$ and $g \in L^{p'(z)}(\Omega)$, we have here following $p(z)$-Hölder inequality

$$\int_\Omega |fg| dz \leq \Big(\frac{1}{p(z)} + \frac{1}{p'(z)}\Big)\|f\|_{p(z)}\|g\|_{p'(z)}$$

holds true. Now, if $p(z), p'(z) \in C_+(\overline{\Omega})$ where $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$ and for each $a, \ b > 0$, we obtain Young's inequality defined by

$$ab \leq \frac{a^{p(z)}}{p(z)} + \frac{b^{p'(z)}}{p'(z)}.$$

A variable exponent p is extended from $\overline{\Omega} \to [1, +\infty)$ to $\overline{Q_T} = \overline{\Omega} \times [0, T]$ with $p(t, z) := p(z)$, for all $(z, t) \in \overline{Q_T}$.

2.1.1. Functional Setting. The present paragraph aims at presenting the suggested problem solving framework $(\mathcal{P})$. We start by defining the space

$$L^{p(z)}(Q_T)=\Big\{w:Q_T\to\mathbb{R};\text{measurable such that }\int_{Q_T}\Big|w(z,t)\Big|^{p(z)}dzdt<\infty\Big\},$$

equipped with the norm

$$\|w\|_{L^{p(z)}(Q_T)}=\inf\Big\{\lambda>0;\int_{Q_T}\Big|\frac{w(z,t)}{\lambda}\Big|^{p(z)}dzdt<1\Big\}.$$

The space $(L^{p(z)}(Q_T),\|\cdot\|_{p(z)})$ is then a separable reflexive Banach. We now introduce the Sobolev space with variable exponent

$$W^{1,p(z)}(\Omega)=\Big\{w\in L^{p(z)}(\Omega)\ ;\ |\nabla w|\in L^{p(z)}(\Omega)\Big\}.$$

It has the following standard

$$\|w\|_{1,p(z)}=\|w\|_{p(z)}+\|\nabla w\|_{p(z)}.$$

This is recognized as being equivalent to

$$\|w\|_{1,p(z)}=\inf\Big\{\lambda>0;\int_{\Omega}\Big(\Big|\frac{w(z)}{\lambda}\Big|^{p(z)}+\Big|\frac{\nabla w(z)}{\lambda}\Big|^{p(z)}\Big)dz\le 1\Big\}.$$

We also denote by $W_0^{1,p(z)}(\Omega)$ the subspace of $W^{1,p(z)}(\Omega)$ which is the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(z)}$, i.e., $W_0^{1,p(z)}(\Omega)=\overline{C_0^\infty(\Omega)}^{W^{1,p(z)}(\Omega)}$. Moreover, By assuming that $p^->1$, one can say that the spaces $W^{1,p(z)}(\Omega)$ and $W_0^{1,p(z)}(\Omega)$ are separable and reflexive Banach spaces. Additionally, if $0<T<\infty$, we begin by setting the space

$$L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))=\Big\{w\in L^{p(z)}(Q_T):\Big(\int_0^T\|w\|^{p^-}_{W_0^{1,p(z)}(\Omega)}\Big)^{\frac{1}{p^-}}dt<+\infty\Big\}.$$

2.2. Measure and Parabolic p(x)-Capacity

We take back the notion of $p(z)$-capacity for the problem $(\mathcal{P})$. Consider that $Q_T=\Omega\times(0,T)$ for every $T>0$ and let's note here again $\mathcal{V}=W_0^{1,p(z)}(\Omega)\cap L^2(\Omega)$ has its natural norm $\|.\|_{W_0^{1,p(z)}}+\|.\|_{L^2(\Omega)}$, we can define the space $W_{p(z)}(0,T)$ by

$$W_{p(z)}(0,T)=\Big\{\nabla w\in(L^{p(z)}(Q_T))^N;\ w\in L^{p^-}(0,T,\mathcal{V})\text{ and }w_t\in L^{(p')^-}(0,T,\mathcal{V}')\Big\},$$

equipped by

$$\|w\|_{W_{p(z)}(0,T)}=\|w\|_{L^{p^-}(0,T,\mathcal{V})}+\|\nabla w\|_{(L^{p(z)}(Q_T))^N}+\|w_t\|_{L^{(p')^-}(0,T,\mathcal{V}')}.$$

Recall that $W_{p(z)}(0,T)$ is continuously embedded in $\mathcal{C}([0,T],L^2(\Omega))$.
Now, we will define the $p(z)-$parabolic capacity of U where $U\subseteq Q_T$ be an open set as follows

$$cap_{p(z)}(U)=\inf\Big\{\|w\|_{W_{p(z)}(0,T)}:w\in W_{p(z)}(0,T),w\ge\chi_U\ a.e.\text{ in }Q_T\Big\},$$

where $\inf\{\emptyset\}=+\infty$, then for each Borel set $B\subseteq Q_T$, we have

$$cap_{p(z)}(B)=\inf\Big\{cap_{p(z)}(U):U\text{ open subset of }Q_T,B\subseteq U\Big\}.$$

Since we want to use certain regular properties, we will set the space Υ as follows

$$\Upsilon = \Big\{ w \in L^{p^-}(0,T,W_0^{1,p(z)}(\Omega)) \ : \ \nabla w \in (L^{p(z)}(Q_T))^N \\ \text{ and } w_t \in L^{(p')^-}(0,T,W^{1,p'(z)}(\Omega)) + L^1(Q_T) \Big\}.$$

It is endowed by the following standard

$$\|w\|_\Upsilon = \|w\|_{L^{p^-}(0,T,W_0^{1,p(z)}(\Omega))} + \|\nabla w\|_{(L^{p(z)}(Q_T))^N} + \|w_t\|_{L^{(p')^-}(0,T,W^{1,p'(z)}(\Omega))+L^1(Q_T)}.$$

In the rest of this paper, $\mathcal{M}_b(Q_T)$ designates the set of all Radon measures with bounded variation on Q_T, and $\mathcal{M}_0(Q_T)$ denotes

$$\mathcal{M}_0(Q_T) = \Big\{ \mu \in \mathcal{M}_b(Q_T) \ : \ \mu(E) = 0 \text{ for every } E \subset Q_T \text{ such that } cap_{p(z)}(E) = 0 \Big\}.$$

as mentioned earlier, $\mathcal{M}_0(Q_T)$ the set of all bounded total variation measures on Q_T that do not load sets of zero p(x)-capacity, i.e., if $\mu \in \mathcal{M}_0(Q_T)$, then $\mu(E) = 0$, for any $E \subset Q_T$ such that $cap_{p(z)}(E) = 0$. To properly specify the nature of a measure in $\mathcal{M}_0(Q_T)$, we must then detail the structure of the dual space $(W_{p(z)}(0,T))'$.

Lemma 2.1 *[34, Lemma 4.2] Consider $g \in (W_{p(z)}(0,T))'$ then there exists $g_1 \in L^{(p')^-}(0,T,W^{-1,p'(z)}(\Omega))$, $g_2 \in L^{p^-}(0,T,\mathcal{V})$, $F \in (L^{p'(z)}(Q_T))^N$ and $g_3 \in L^{(p')^-}(0,T,L^2(\Omega))$ such that*

$$\ll g,\ w \gg = \int_0^T \Big\langle g_1, w \Big\rangle dt + \int_0^T \Big\langle w_t,\ g_2 \Big\rangle dt + \int_{Q_T} F\nabla w \ dxdt + \int_{Q_T} g_3 w dxdt,$$

for each $w \in W_{p(z)}(0,T)$. Furthermore, we can take (g_1, g_2, F, g_3) such that

$$\|g_1\|_{L^{(p')^-}(0,T;W^{-1,p'(z)}(\Omega))} + \|g_2\|_{L^{p^-}(0,T;\mathcal{V})} + \||F|\|_{(L^{p'(z)}(Q_T))^N} + \|g_3\|_{L(0,T;L^2(\Omega))} \leq C\|g\|_{(W_{p(z)}(0,T))'},$$

given that C is independent of g.

A decomposition result of $\mathcal{M}_0(Q_T)$ is given below

Theorem 2.1 *[34, Theorem 4.4] Let μ be a bounded measure on Q_T. If $\mu \in \mathcal{M}_0(Q_T)$ then there exists $g \in (W_{p(z)}(0,T))'$and $h \in L^1(Q_T)$ such that $\mu = g + h$ in the sense that*

$$\int_{Q_T} \varphi \ d\mu = \ll g,\ \varphi \gg + \int_{Q_T} h\varphi \ dzdt,\ \forall \varphi \in C_c^\infty([0,T] \times \Omega).$$

From the Theorem 2.1 and Lemma 2.1, we have the follows theorem

Theorem 2.2 *[34, Theorem 4.5] Let $\mu \in \mathcal{M}_0(Q_T)$, then there exists a decomposition (F, f, g_1, g_2) with $F \in (L^{p'(z)}(Q_T))^N$, $f \in L^1(Q_T)$, $g_1 \in L^{(p')^-}(0,T;W^{-1,p'(z)}(\Omega))$, $g_2 \in L^{p^-}(0,T;\mathcal{V})$ and*

$$\int_{Q_T} \varphi d\mu = \int_{Q_T} f\varphi dzdt + \int_{Q_T} F\nabla w dzdt + \int_0^T \Big\langle g_1, \varphi \Big\rangle dt - \int_0^T \Big\langle \varphi_t, g_2 \Big\rangle dt, \quad \varphi \in \mathcal{C}_c^\infty([0,T] \times \Omega).$$

such a quadruplet (F, f, g_1, g_2) will be called a decomposition of μ.

It is worth noting that this decomposition of $\mathcal{M}_0(Q_T)$ from the aforementioned theorem is not unique, see [34, Lemma 4.6]. With the help of a decomposition result, see [25, Lemma 2. 1], for each μ in $\mathcal{M}_b(Q_T)$, can then be written as a summation of its absolutely continuous part μ_0 with respect to the capacity $p(z)$ and its singular part λ_s concentrated on a set E of capacity $p(z)$ zero, Therefore, if λ_s is $\mathcal{M}_b(Q_T)$ and by the theorem 2.2, we can get

$$\mu = f - div(F) + \frac{\partial g}{\partial t} + \lambda_s^+ - \lambda_s^-,$$

in the sense of distributions, for $g \in L^{p^-}(0,T;\mathcal{V})$, $f \in L^1(Q_T)$, $F \in (L^{p'(z)}(Q_T))^N$, where $(\lambda_s^+$ is the positive part of μ and $(\lambda_s^-$ is the negative part of μ; It is important to mention that the decomposition of the absolutely continuous part of μ according to Theorem 2.2 is by no means uniquely defined. As we are concerned about density results to show the existence of a solution, we have to give the following introductory result which utilizes a proper approximation of the data.

Proposition 2.1 *Suppose that $\mu \in \mathcal{M}_0(Q_T)$, then there exists $(f, div(H), g)$ of μ in the sense of Theorem 2.2 and an approximation μ^η of μ verifying*

$$\mu^\eta \in \mathcal{C}_c^\infty(Q_T), \quad \|\mu^\eta\|_{L^1(Q_T)} \leq C,$$

and

$$\int_{Q_T} \mu^\eta \varphi \; d\mu = \int_{Q_T} f^\eta \varphi \; dz \; dt + \int_0^T \Big\langle div(H^\eta), \varphi \Big\rangle dzdt - \int_0^T \Big\langle \varphi_t, g^\eta \Big\rangle \; dt,$$

where

$$\begin{cases} f^\eta \in \mathcal{C}_c^\infty(Q_T): \; f^\eta \to f \; in \; L^1(Q_T) \; as \; \eta \to 0, \\ H^\eta \in \mathcal{C}_c^\infty(Q_T): \; H^\eta \to H \; in \; L^{p'(z)}(Q_T)^N \; as \; \eta \to 0, \\ g^\eta \in \mathcal{C}_c^\infty(Q_T): \; g^\eta \to g \; in \; L^{p^-}(0,T,\mathcal{V}) \; as \; \eta \to 0. \end{cases}$$

Proof: See [34, Proposition 2.31]. □

3. Definition of Renormalized Solution and Essential Hypotheses

3.1. Essential Hypotheses

The following Hypotheses are assumed to be true throughout this document:

Hypothesis ($\mathcal{H}_1$)

Assume that $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded open subset and $b : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function where for each $x \in \Omega, b(z,.)$ is a strictly increasing $\mathcal{C}^1$ function such that

$$b(z,0) = 0. \tag{3.1}$$

Then, there exists $\gamma, \; \Lambda > 0$, and a function $B_k \in L^{p(z)}(\Omega)$ where

$$\gamma \leq \frac{\partial b(z,s)}{\partial s} \leq \Lambda \text{ and } \Big|\nabla_z\Big(\frac{\partial b(z,s)}{\partial s}\Big)\Big| \leq B_k(z), \tag{3.2}$$

for a. e. $z \in \Omega$ and any s where $|s| \leq k$, and the gradient of $\partial b(z,s)/\partial s$ is defined in the sense of the distributions by $\nabla_z(\partial b(z,s)/\partial s)$.

Hypothesis ($\mathcal{H}_2$)

$\Phi : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which satisfies classical Leray-Lions hypothesis

$$\Phi(z,t,s,\xi) \cdot \xi \geq \alpha|\xi|^{p(z)}, \tag{3.3}$$

$$\Big|\Phi(z,t,s,\xi)\Big| \leq \beta\Big[\mathcal{L}(z,t) + |\xi|^{p(z)-1} + |s|^{p(z)-1}\Big], \tag{3.4}$$

$$\Big[\Phi(z,t,s,\xi) - \Phi(z,t,s,\eta)\Big](\xi - \eta) > 0, \tag{3.5}$$

for each $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, α and β are positive real number and for a. e. $(x,t) \in Q_T$, and $\mathcal{L}$ is a non-negative function in $L^{p'(z)}(Q_T)$.

$$\mu \in \mathcal{M}_b(Q_T), \tag{3.6}$$

$$w_0 \in L^1(\Omega). \tag{3.7}$$

3.2. Definition of Renormalized Solution

Before stating our results, we now present the definition of renormalized solution of $(\mathcal{P})$ and some essential lemmas that will help us establish the proof of our main result.

Definition 3.1 *Let $(f, div(F), g)$ be a decomposition of $\mu \in \mathcal{M}_0(Q_T)$. A measurable function w defined on Q_T is a renormalized solution of problem $(\mathcal{P})$ if*

$$T_k(b(z,w)-g) \text{ belong to } L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)), \text{ for all } k \geq 0, \qquad b(z,w)-g \text{ belong to } L^\infty(0,T;L^1(\Omega)), \tag{3.8}$$

$$\nabla T_k(b(z,w)-g) \in (L^{p(z)}(Q_T))^N, \text{ for all } k \geq 0, \tag{3.9}$$

for all S belong to $W^{2,\infty}(\mathbb{R})$, that is piecewise $\mathcal{C}^1$, with S' is a function with compact support, we get

$$\frac{\partial S(b(z,w)-g)}{\partial t} - div(S'(b(z,w)-g)\Phi(z,t,w,\nabla w)) + S''(b(z,w)-g)\Phi(z,t,w,\nabla w)\nabla(b(z,w)-g) = fS'(b(z,w)-g) - div(FS'(b(z,w)-g)) + FS''(b(z,w)-g)\nabla(b(z,w)-g) \text{ in } \mathcal{D}'(Q_T), \tag{3.10}$$

for each $\psi \in C(\overline{Q_T})$, we obtain

$$\lim_{m\to\infty} \frac{1}{m}\int_{\{(z,t)\in Q_T : m \leq |b(z,w)-g| < 2m\}} \Phi(z,t,w,\nabla w)\nabla(b(z,w) - g)\psi dz dt = \int_{Q_T} \psi \mathrm{d}\mu_s^+, \tag{3.11}$$

$$\lim_{m\to\infty} \frac{1}{m}\int_{\{(z,t)\in Q_T : -2m < |b(z,w)-g| \leq -m\}} \Phi(z,t,w,\nabla w)\nabla(b(z,w) - g)\psi dz dt = \int_{Q_T} \psi d\mu_s^-. \tag{3.12}$$

and

$$S(b(z,w)-g)(t=0) = S(b(z,w_0)) \text{ in } \Omega. \tag{3.13}$$

Remark 3.1 The essential regularity results derived from the distribution equation (3.10) are considered. It The main regularity results derived from the equation (3.10) in terms of distribution are considered. It is important to note that thanks to our regularity assumptions of S, all terms present in (3.10) are well defined. This is possible because $T_k(b(z,w)-g)$ belongs to $L^{p(z)}(0,T;W_0^{1,p(z)}(\Omega))$ for all positive k, and because S' has compact support. More precisely, by choosing a suitable k such that Supp $(S') \subset]-k,k[$, we ensure that $S'(b(z,w)-g) = S''(b(z,w)-g) = 0$ whenever $|b(z,w)-g| \geq k$. As a result we can replace, everywhere in (3.10), $\nabla(b(z,w)-g)$ by $\nabla T_k(b(z,w)-g) \in L^{p(z)}(Q_T)^N$ and ∇w by $(\frac{\partial b(z,w)}{\partial s})^{-1}(\nabla T_k(r) + (\nabla g - \nabla_z b(z,w))\chi_{\{|r|\leq k\}}) \in L^{p(z)}(Q_T)^N$. Moreover, in view of (3.2)−(3.3) and the definition of ∇w, $(\frac{\partial b(z,w)}{\partial s})^{-1}(\nabla T_k(r) - (\nabla_z b(z,w) - \nabla g)\chi_{\{|r|\leq k\}}) \in L^{p(z)}(Q_T)^N$, we have $\nabla(b(z,w)-g)$ is well defined and since $|w| \leq \gamma^{-1}(k+|g|)$ as soon as $|r| \leq k$, we can also deduce that $|\Phi(t,z,w,\nabla w)\chi(|r| \leq k)| \in L^{p'(z)}(Q_T)^N$. Additionally, for each S as mentioned earlier, we have $S(b(z,w)-g) = S(T_k(b(z,w)-g)) \in L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$ and

$$\begin{cases} S'(b(z,w)-g)\Phi(t,z,w,\nabla w) \in L^{p'(z)}(Q_T)^N; \\ S'(b(z,w)-g)F \in L^{p'(z)}(Q_T)^N; \\ S'(b(z,w)-g)f \in L^1(Q_T); \\ S''(b(z,w)-g)F \cdot \nabla T_k(b(z,w)-g) \in L^1(Q_T); \\ S''(b(z,w)-g)\Phi(t,z,w,\nabla w) \cdot \nabla T_k(b(z,w)-g) \in L^1(Q_T). \end{cases}$$

Therefore, based on (3.10), we can assert that $\frac{\partial S(b(z,w)-g)}{\partial t}$ belongs to the space $L^{(p')^-}(0,T;W^{-1,p'(z)}(\Omega)) + L^1(Q_T)$. Consequently, $S(b(z,w)-g)$ belongs to $C([0,T];L^1(\Omega))$, as stated in [41, Theorem 1.1]. This allows us to conclude that the initial datum is achieved in a weak sense, i.e., $S(b(z,w)-g)(0) = S(b(z,w)(0)-g(0)) = S(b(z,w_0))$ in $L^1(\Omega)$ (noting that g has compact support in Q_T) for every renormalization S. Additionally, it is worth mentioning that since $\frac{\partial S(b(z,w)-g)}{\partial t} \in L^{(p')^-}(0,T;W^{-1,(p'(z)}(\Omega)) + L^1(Q_T)$, we are not limited to using only functions in $C_0^\infty(Q_T)$ in (3.10), but we can also include functions from $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)) \cap L^\infty(Q_T)$.

It is important to mention that the preparation of the renormalized solution is independent of the decomposition of μ, for that we will use the following result

Lemma 3.1 *Let (f,H,g_1,g_2) and $(f,\tilde{H},\tilde{g_1},\tilde{g_2})$ be two different decompositions of μ with $\mu \in \mathcal{M}_0(Q_T)$ by the Theorem 2.1, we get*

$$\int_0^T \Big\langle (g_2 - \tilde{g_2})_t, \varphi \Big\rangle dt = \int_{Q_T} \Big(\tilde{H} - H\Big).\nabla\varphi dz dt + \int_{Q_t} \Big(\tilde{f} - f\Big)\varphi dz dt + \int_0^T \Big\langle g_1 - \tilde{g_1}, \varphi \Big\rangle dt$$

for every $\varphi \in \mathcal{C}_c^\infty([0,T]\times\Omega)$. Moreover $g_2 - \tilde{g_2} \in \mathcal{C}([0,T];L^1(Q_T))$ and $(g_2-\tilde{g_2})(0) = 0$.

Proof: See [34, Lemma 4.6]. □

The following result shows that in the presence of bounded perturbations of the time derivative component of μ, the definition of a renormalized solution is stable.

Proposition 3.1 *[37, Proposition 3] If w is a renormalized solution of $(\mathcal{P})$, then w satisfies (3.8)-(3.12) for any decomposition $(\tilde{f}, div(\tilde{H}), \tilde{g})$ of μ.*

4. Main Result and Proof

In this part, we demonstrate the following main result.

Theorem 4.1 *Under assumptions (3.2)-(3.7) there exists at least a renormalized solution w of $\mathcal{P}$.*

Proof: The complexity of extending the main result when μ is in $\mathcal{M}_0(Q_T)$ is due to the fact that there is a singular part of the data and to a absence of regularity. To overcome these difficulties in the proof of our main result, we will divide the proof into several steps. We start by introducing an approximate problem. Then we will establish some a priori estimates. Finally, we will be showing that u satisfies (3.8)-(3.13) of Definition 3.1.

Step 1: Approximate problem

Let us return to the essential decomposition theorem for measures data. As previously stated, if μ belong to $\mathcal{M}_b(Q_T)$, it can be decomposed as follows:

$$\mu = f - div(F) + \frac{\partial g}{\partial t} + \lambda_+^s - \lambda_-^s.$$

There are several methods for approximating this measure by determining the existence of solutions to $(\mathcal{P})$; we will choose the following.

$$\begin{cases} \mu^\eta \in C_c^\infty(Q_T) \text{ such that } \|\mu^\eta\|_{L^1(\Omega)} \le C, \\ \mu^\eta = f^\eta - div(F^\eta) + \dfrac{\partial g^\eta}{\partial t} + \lambda_+^\eta - \lambda_-^\eta, \end{cases} \tag{4.1}$$

where

$$f^\eta \in \mathcal{C}_c^\infty(Q_T),\ f^\eta \to f \text{ in } L^1(Q_T), \text{ as } \eta \to 0, \tag{4.2}$$

$$F^\eta \in (\mathcal{C}_c^\infty(Q_T))^N,\ F^\eta \to F \text{ in } (L^{p'(z)}(Q_T))^N, \text{ as } \eta \to 0, \tag{4.3}$$

$$g^\eta \in \mathcal{C}_c^\infty(Q_T),\ g^\eta \to g \text{ in } L^{p^-}(0,T,W_0^{1,p(z)}(\Omega)\cap L^2(\Omega)), \text{ as } \eta \to 0, \tag{4.4}$$

$$\lambda_+^\eta \in C_0^\infty(Q_T) \text{ where } \lambda_+^\eta \to \lambda_+^s \text{ in the narrow topology of measures,} \tag{4.5}$$

$$\lambda_-^\eta \in C_0^\infty(Q_T) \text{ where } \lambda_-^\eta \to \lambda_-^s \text{ in the narrow topology of measures.} \tag{4.6}$$

Furthermore, let us

$$b_\eta(z,k) = T_{\frac{1}{\eta}}(b(z,k)) + \eta k, \text{ for all } k \in \mathbb{R} \textit{ and } \eta > 0, \tag{4.7}$$

$$w_0^\eta \in C_c^\infty(\Omega): \ b_\eta(z,w_0^\eta) \to b_\eta(z,w_0) \text{ in } L^1(\Omega) \text{ as } \eta \to 0. \tag{4.8}$$

Consider now the following regularized problem $(\mathcal{P}_\eta)$.

$$(\mathcal{P}_\eta) \quad \begin{cases} \dfrac{\partial b_\eta(z,w^\eta)}{\partial t} - div\Big[\Phi(z,t,w^\eta,\nabla w^\eta)\Big] = \mu^\eta & \text{in } Q_T : \Omega \times (0,T), \\ w^\eta(z,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ b_\eta(z,w^\eta)(t=0) = b_\eta(z,w_0^\eta) & \text{in } \Omega. \end{cases}$$

As a result, it is straightforward to demonstrate the existence of a weak solution $w^\eta \in L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$ of $(\mathcal{P}_\eta)$(for more information, see [5]). This approach gives standard compactness results which we put together in the next step.
Step 2 : By selecting $T_k(b_\eta(z,w))$ as a test function in $\big(\mathcal{P}_\eta\big)$, we obtain

$$\int_\Omega \overline{T_k}(b_\eta(z,w^\eta))(t)dz + \int_0^t \int_\Omega \Phi(t,z,w^\eta,\nabla w_\eta)\cdot \nabla T_k(b_\eta(z,w^\eta))dzdt$$
$$= \int_0^t \int_\Omega T_k(b_\eta(z,w^\eta))d\mu^\eta + \int_\Omega \overline{T_k}(b_\eta(z,w_0^\eta))dz \tag{4.9}$$

where $t \in [0,T]$ and $\overline{T_k}(s)$ the primitive function of $T_k(s)$. Due to (3.3) and the boundedness of $\|b_\eta(z,w^\eta)\|_{L^1(Q_T)}$, it results that

$$\int_\Omega \overline{T_k}(b_\eta(z,w^\eta))(t)dz + \int_{\{|b_\eta(z,w^\eta)|\le k\}} \frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(t,z,w^\eta,\nabla w^\eta)\cdot\nabla w^\eta dzdt$$
$$+ \int_{\{|b_\eta(z,w^\eta)|\le k\}} \Phi(t,z,w^\eta,\nabla w^\eta)\cdot\nabla_z b_\eta(z,w^\eta)dzdt \le k\|\mu\|_{\mathcal{M}_b(Q_T)} + \int_\Omega \overline{T_k}(b_\eta(z,w_0^\eta))dz.$$

Then,

$$\int_\Omega \overline{T_k}(b_\eta(z,w^\eta))(t) + \alpha\int_{E_k} \frac{\partial b_\eta(z,w^\eta)}{\partial s}|\nabla w^\eta|^{p(z)}dzdt \le k\|\mu\|_{\mathcal{M}_b(Q_T)} + \beta\int_{E_k}\mathcal{L}(z,t)\cdot|\nabla_z b_\eta(z,w^\eta)|$$
$$+\beta\int_{E_k}|w^\eta|^{p(z)-1}\cdot|\nabla_z b_\eta(z,w^\eta)dzdt + \frac{\beta}{\gamma}\int_{E_k}\frac{\partial b_\eta(z,w^\eta)}{\partial s}|\nabla w_\eta|^{p(z)-1}\cdot|\nabla_z b_\eta(z,w^\eta)|dzdt + k\|b_\eta(z,w_0^\eta)\|_{L^1(\Omega)}$$

where $E_k = \{(z,t) \ : |b_\eta(z,w^\eta)| \le k\}$, by (3.2) and Young's inequality, we can infer that

$$\beta\int_{E_k}|\nabla w^\eta|^{p(z)-1}\cdot|\nabla_z b_\eta(z,w^\eta)dzdt \le \frac{\beta}{\gamma}\int_{E_k}\frac{\partial b_\eta(z,w^\eta)}{\partial s}|\nabla w^\eta|^{p(z)-1}\cdot|\nabla_z b_\eta(z,w^\eta)dzdt$$
$$\le \frac{\alpha}{2}\int_{E_k}\frac{\partial b_\eta(z,w^\eta)}{\partial s}|\nabla w^\eta|^{p(z)}dzdt$$
$$+ \frac{T(\Lambda+1)}{p^-}\Big(\max\Big(\Big(\frac{2\beta(p')^-}{\alpha\gamma}\Big)^{p^--1}, \Big(\frac{2\beta(p')^-}{\alpha\gamma}\Big)^{p^+-1}\Big)\max\Big(\|B_k\|_{L^{p(z)}(\Omega)}^{p^-}, \|B_k\|_{L^{p(z)}(\Omega)}^{p^+}\Big)$$

and

$$\int_{E_k} |w^\eta|^{p(z)-1} \cdot |\nabla_z b_\eta(z,w^\eta)| dzdt \le \int_{E_k} |\frac{k}{\gamma}|^{p(z)-1} |\nabla_z b_\eta(z,w_\eta)| dzdt$$
$$\le C \max\Big(\|B_k\|^{p^-}_{L^{p(z)}(\Omega)}, \|B_k\|^{p^+}_{L^{p(z)}(\Omega)}\Big).$$

In view of $\overline{T_k}(s) \ge 0$ and $|\overline{T_k}(s)| \ge |s| - 1$, we have

$$\int_\Omega |b_\eta(z,w^\eta)(t)| dz + \frac{\alpha}{2} \int_{E_k} \frac{\partial b_\eta(z,w^\eta)}{\partial s} |\nabla w^\eta|^{p(z)} dzdt \le k(\|\mu\|_{\mathcal{M}_b(Q_T)} + \|b_\eta(z,u^\eta)\|$$
$$+ C\Big(\max\big(\|L\|^{(p')^-}_{L^{p'(z)}(\Omega)}, \|L\|^{(p')^+}_{L^{p'(z)}(\Omega)}\big) + \max\big(\|B_k\|^{p^-}_{L^{p(z)}(\Omega)}, \|B_k\|^{p^+}_{L^{p(z)}(\Omega)}\big)\Big).$$

Finally, we obtain

$$\int_\Omega |b_\eta(z,u^\eta)(t)| dz + \frac{\alpha}{2} \int_0^t \int_\Omega |\nabla T_k(b_\eta(z,w^\eta))|^{p(z)} dzdt \le C(k+1) \quad \forall k > 0, \qquad \text{for all } t \in [0,T].$$

Based on the previously obtained estimates, we can infer that $\|b_\eta(z,w^\eta)\|_{L^\infty(0,T,L^1(\Omega))} \le C$ and

$$\int_{Q_T} |\nabla T_k(b_\eta(z,w^\eta))|^{p(z)} dzdt \le C(k+1).$$

Likewise, by selecting $T_k(r)$ as the test function in $\big(\mathcal{P}_\eta\big)$, we can also derive an estimate on $r^\eta = b_\eta(z,w^\eta) - g^\eta$.

$$\int_\Omega \overline{T_k}(s)(r^\eta)(t) dz + \alpha \int_{E_k} \frac{\partial b_\eta(z,w^\eta)}{\partial s} |\nabla w^\eta|^{p(z)} dzdt \le \int_\Omega \overline{T_k}(s)(b_\eta(z,w_0^\eta)) dz + k\|f\|_{L^1(Q_T)}$$
$$+ \int_{E_k} |F \cdot \nabla T_k(r) dzdt + \beta\Big(\int_{E_k} \mathcal{L}(z,t) |\nabla g^\eta| dzdt + \int_{E_k} |w^\eta|^{p(z)-1} |\nabla g^\eta| dzdt + \int_{E_k} |\nabla w^\eta|^{p(z)-1} dzdt\Big)$$
$$+ \int_{E_k} |\Phi(t,z,w,\nabla w_\eta) \cdot \nabla_z b_\eta(z,w^\eta)| dzdt + \int_{Q_T} T_k(r_\eta) d\lambda_+^\eta - \int_{Q_T} T_k(r_\eta) d\lambda_-^\eta$$

where C is a constant independent on η and $E_k = \{(z,t) \ : \ |b_\eta(z,w^\eta) - g^\eta| \le k\}$.
By utilizing (3.1), (3.2), and applying Young's inequality, we obtain

$$\int_{E_k} |F \cdot \nabla T_k(r_\eta)| dzdt \le \frac{\alpha}{2}\Big(\frac{1}{(p')^+} + \frac{1}{p^-}\Big) \int_{E_k} \frac{\partial b_\eta(z,w^\eta)}{\partial s} |\nabla w^\eta|^{p(z)} dzdt$$
$$+ C\Big(\max\big(\|B_k\|^{p^-}_{L^{p(z)}(\Omega)}, \|B_k\|^{p^+}_{L^{p(z)}(\Omega)}\big) + \max\big(\|F^\eta\|^{(p')^+}_{L^{p'(z)}(Q_T)}, \|F^\eta\|^{(p')^-}_{L^{p'(z)}(Q_T)}\big)$$
$$+ \max\big(\|\nabla g^\eta\|^{p^-}_{L^{p(z)}(Q_T)}, \|\nabla g^\eta\|^{p^+}_{L^{p(z)}(Q_T)}\big)\Big),$$

$$\int_{E_k} |w^\eta|^{p(z)-1} |\nabla g| dzdt \le \int_{E_k} (k + |g^\eta|)^{p'(z)-1} |\nabla g| dzdt$$
$$\le C\Big(\max\big(\|g^\eta\|^{p^-}_{L^{p(z)}(Q_T)}, \|g^\eta\|^{p^+}_{L^{p(z)}(Q_T)}\big) + \max\big(\|\nabla g^\eta\|^{p^-}_{L^{p(z)}(Q_T)}, \|\nabla g^\eta\|^{p^+}_{L^{p(z)}(Q_T)}\big)\Big),$$

$$\int_{E_k} |\Phi(t,z,w^\eta,\nabla w^\eta) \nabla_z b_\eta(z,w^\eta)| dzdt \le \frac{\alpha}{4(p^-)'} \int_{E_k} \frac{\partial b_\eta(z,w^\eta)}{\partial s} |\nabla w^\eta|^{p(z)} dzdt$$
$$+ \frac{T(\Lambda+1)}{p^-} \max\Big(\big(\frac{4\beta}{\alpha\gamma}\big)^{p^- -1}, \big(\frac{4\beta}{\alpha\gamma}\big)^{p^+ -1}\Big) \max\Big(\|B_k\|^{p^-}_{L^{p(z)}(\Omega)}, \|B_k\|^{p^+}_{L^{p(z)}(\Omega)}\Big),$$

and

$$\beta\int_{E_k}|\nabla w^\eta|^{p(z)-1}|\nabla g^\eta|\,dzdt \le \frac{\beta}{\gamma}\int_{E_k}\frac{\partial b_\eta(z,w^\eta)}{\partial s}|\nabla w^\eta|^{p(z)-1}|\nabla g^\eta|^{p(z)}\,dzdt$$
$$\le \frac{\alpha}{4(p')^-}\int_{E_k}\frac{\partial b_\eta(z,w^\eta)}{\partial s}|\nabla w^\eta|^{p(z)}\,dzdt$$
$$+\frac{T(\Lambda+1)}{p^-}\max\Big(\big(\frac{4\beta}{\alpha\gamma}\big)^{p^--1},\big(\frac{4\beta}{\alpha\gamma}\big)^{p^+-1}\Big)\int_{E_k}|\nabla g^\eta|^{p(z)}\,dzdt.$$

Hence,

$$\int_\Omega \overline{T}_k(b_\eta(z,w^\eta)-g^\eta)(t)dz+\frac{\alpha}{2}\int_{E_k}\frac{\partial b_\eta(z,w^\eta)}{\partial s}|\nabla w^\eta|^{p(z)}dzds$$
$$\le C\Big(\max\big\{\|\mathcal{L}\|^{p^-}_{L^{p'(z)}(Q_T)},\|\mathcal{L}\|^{p^+}_{L^{p'(z)}(Q_T)}\big\}+\max\big\{\|\nabla g^\eta\|^{p^-}_{L^{p'(z)}(Q_T)},\|\nabla g^\eta\|^{p^+}_{L^{p'(z)}(Q_T)}\big\}$$
$$+\max\big\{\|g^\eta\|^{p^-}_{L^{p'(z)}(Q_T)},\|g^\eta\|^{p^+}_{L^{p'(z)}(Q_T)}\big\}+\max\big\{\|B_k\|^{p^-}_{L^{p(z)}(\Omega)},\|B_k\|^{p^+}_{L^{p(z)}(\Omega)}\big\}\Big)$$
$$+k\Big(||f^\eta\|+\|F^\eta\|^{p'(z)}_{L^{p'(z)}(Q_T)}+\|b_\eta(z,w_0^\eta)\|_{L^1(\Omega)}+\|\lambda_+^\eta\|_{L^1(Q_T)}+\|\lambda_-^\eta\|_{L^1(Q_T)}\Big),\quad (4.10)$$

as F^η is bounded in $(L^{p'(z)}(Q_T))^N$, g^η is bounded in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$, f^η, λ_+^η, λ_+^η are bounded in $L^1(Q_T)$ and $b_\eta(z,w_0^\eta)$ is bounded in $L^1(\Omega)$, we obtain

$$\int_\Omega \overline{T_k}(r^\eta)(t)dz\le C\quad \forall t\in[0,T],$$

which implies the estimate of r^η in $L^\infty(0,T;L^1(\Omega))$, and also

$$\int_{Q_T}\|\nabla w^\eta\|^{p(z)}\chi_{\{|r^\eta|\le k\}}dzdt\le C(k+1),$$

which yields that $T_k(r^\eta)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$, for any $k>0$ (recall that g^η is itself is bounded in $L^{P^-}(0,T;W_0^{1,p(z)}(\Omega))$).
According to the properties of $\overline{T}_k$ and the fact that $|\mu^\eta|_{L^1(Q_T)}$ and $|b_\eta(z,w^\eta)|_{L^1(\Omega)}$ are bounded, it can be concluded from (4.10) that

$$\int_\Omega |(b_\eta(z,w^\eta)-g^\eta)(t)|dz\le C+1. \quad (4.11)$$

This means that

$$b_\eta(z,w^\eta)-g^\eta \text{ is bounded in } L^\infty(0,T,L^1(\Omega)). \quad (4.12)$$

In addition, using the Hölder inequality and (3.2) and from (4.10), we conclude that

$$\frac{\alpha}{2b_1^\kappa}\int_{Q_T}|\nabla b_\eta(z,w^\eta)|^{p(z)}\chi_{\{|b_\eta(z,w^\eta)-g^\eta|<k\}}dzds\le C,$$

with $\Big(\dfrac{1}{b_1}\Big)^\kappa=\min\big\{(\frac{1}{b_1})^{p^--1};(\frac{1}{b_1})^{p^+-1}\big\}$ and as g^η is bounded in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$, we get

$$\int_{Q_T}|\nabla T_k(b_\eta(z,w^\eta)-g^\eta)|^{p(z)}dzds\le C, \quad (4.13)$$

thus

$$T_k(b_\eta(z,w^\eta)-g^\eta) \text{ is bounded in the space } L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)), \quad (4.14)$$

for each $k \geq 0$ and independently of η. As a result, for every $S \in W^{2,\infty}(\mathbb{R})$ where S' has a compact support i.e., $supp(S') \subset [-k,k]$, we get

$$S(b_\eta(z,w^\eta) - g^\eta) \text{ is bounded in the space } L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)), \tag{4.15}$$

$$\frac{\partial S(b_\eta(z,w^\eta) - g^\eta)}{\partial t} \text{ is bounded in } L^1(Q_T) + L^{(p')^-}(0,T;W^{-1,p'(z)}(\Omega)). \tag{4.16}$$

In fact, from (4.14) and thanks to the famous Stampacchia Theorem, we can obtain (4.15). In order to prove (4.16), we do a multiplication of the equation $(\mathcal{P}_\eta)$ by $S'(b_\eta(z,w^\eta) - g^\eta)$ to obtain

$$\begin{aligned}\frac{\partial S(b_\eta(z,w^\eta) - g^\eta)}{\partial t} &= div(S'(b_\eta(z,w^\eta) - g^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)) \\ &\quad - \Phi(z,t,w^\eta,\nabla w^\eta)\nabla S'(b_\eta(z,w^\eta) - g^\eta) + f^\eta S'(b_\eta(z,w^\eta) - g^\eta) \\ &\quad - div(F^\eta S'(b_\eta(z,w^\eta) - g^\eta)) + F^\eta \nabla S'(b_\eta(z,w^\eta) - g^\eta) \\ &\quad + \lambda_+^\eta S'(b_\eta(z,w^\eta) - g^\eta) - \lambda_-^\eta S'(b_\eta(z,w^\eta) - g^\eta) \quad \text{in } \mathcal{D}'(Q_T).\end{aligned} \tag{4.17}$$

Since $b_\eta(z,w^\eta) - g^\eta \in L^\infty(0,T,L^1(\Omega)) \subset L^1(Q_T)$, thus $S(b_\eta(z,w^\eta) - g^\eta) \in L^1(Q_T)$ and $\dfrac{\partial S(b_\eta(z,w^\eta) - g^\eta)}{\partial t} \in \mathcal{D}'(Q_T)$. Then we get

$$\begin{cases} f^\eta S'(b_\eta(z,w^\eta) - g^\eta) \in L^1(Q_T), & F^\eta S'(b_\eta(z,w^\eta) - g^\eta) \in (L^{p'(z)}(Q_T))^N, \\ \lambda_+^\eta S'(b_\eta(z,w^\eta) - g^\eta) \in L^1(Q_T), & \lambda_-^\eta S'(b_\eta(z,w^\eta) - g^\eta) \in L^1(Q_T). \end{cases}$$

Given that $supp(S') \subset [-k,k]$ and $supp(S'') \subset [-k,k]$, w^η can be replaced by $(b_\eta(z,w^\eta))^{-1}(r^\eta - \nabla_z b_\eta(z,w^\eta) + g^\eta)$ in $\{|b_\eta(z,w^\eta) - g^\eta| \leq k\}$, where $r^\eta := b_\eta(z,w^\eta) - g^\eta$, In fact we obtain

$$\begin{aligned}\Big|S'(b_\eta(z,w^\eta) - g^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\Big| &\leq \beta\|S\|_{L^\infty(\mathbb{R})} + \Big|\Big(\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Big)^{-1}\Big(T_k(r^\eta) - \nabla_z b_\eta(z,w) + g^\eta\Big)\Big|^{p(z)-1}\Big] \\ &\leq \beta\|S\|_{L^\infty(\mathbb{R})}\Big[\mathcal{L}(z,t) + |(b_\eta(z,w^\eta))^{-1})(r^\eta - \nabla_z b_\eta(z,w) + g^\eta)|^{p(z)-1} \\ &\quad + \rho|\nabla T_k(r^\eta) - \nabla_z b_\eta(z,w) + \nabla g^\eta|^{p(z)-1}\Big]\end{aligned} \tag{4.18}$$

where $\rho = \max((\frac{1}{\gamma})^{p^+-1}, (\frac{1}{\gamma})^{p^--1})$, this means that $S'(b_\eta(z,w^\eta) - g^\eta)\Phi(z,t,w^\eta,\nabla w^\eta) \in (L^{p'(z)}(Q_T))^N$ and we have from (4.10) that $\nabla(b_\eta(z,w^\eta) - g^\eta) \in L^{p(z)}(Q_T)$.
Furthermore, we get

$$\begin{cases} S''(b_\eta(z,w^\eta) - g^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla(b_\eta(z,w^\eta) - g^\eta) \in L^1(Q_T) \\ S''(b_\eta(z,w^\eta) - g^\eta)F^\eta\nabla(b_\eta(z,w^\eta) - g^\eta) \in L^1(Q_T). \end{cases}$$

As a consequence, we conclude that

$$\frac{\partial S(b_\eta(z,w^\eta) - g^\eta)}{\partial t} \in L^1(Q_T) + L^{(p')^-}(0,T;W^{-1,p'(z)}(\Omega)).$$

Step 3 : Reasoning again as before. Thanks to (4.12), we have for a subsequence always indexed by η,

$$b_\eta(z,w^\eta) - g^\eta \text{ converges to } b(z,w) - g \text{ a.e in } Q_T. \tag{4.19}$$

through $r^\eta := b_\eta(z,w^\eta) - g^\eta$, which gives that $w^\eta = \big(b_\eta(z,w^\eta)\big)^{-1}(r^\eta - \nabla_z b_\eta(z,w^\eta) + g^\eta)$, using(4.4) and (4.8), we have

$$w^\eta \to w \quad \text{almost everywhere in } Q_T, \tag{4.20}$$

by (4.14), we get

$$T_k(r^\eta) \text{ converges to } T_k(r) \text{ weakly in } L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)). \tag{4.21}$$

We deduce from (4.18) that for each $k>0$,

$$\Phi(z,t,w^\eta,\nabla w^\eta)\chi_{\{|b_\eta(z,w^\eta)-g^\eta|\le k\}} \rightharpoonup \sigma_k \text{ in } (L^{p'(x)}(Q_T))^N \text{ as } \eta\to 0, \tag{4.22}$$

and

$$\sigma_k \in (L^{p'(z)}(Q_T))^N,\ \forall k>0.$$

By using (4.2), (4.8), (4.20) and by Lebesgue's convergence theorem we obtain

$$\nabla_z b(z,w^\eta) \to \nabla_z b(z,w) \text{ strongly in } (L^{p(z)}(Q_T))^N.$$

as η tends to zero, for any $k>0$.
We will now demonstrate that $b(z,w)-g\in L^\infty(0,T,L^1(\Omega))$, to do this we use (4.10) and the fact that $|\overline{T}_k(s)|\ge |s|-1$, which leads to

$$\begin{aligned}\int_\Omega |b_\eta(z,w^\eta)-g^\eta|(t)dz \le C\Big(&\max\big\{\|B_k\|^{(p')^-}_{L^{p(z)}(Q_T)};\ \|B_k\|^{(p')^+}_{L^{p(z)}(Q_T)}\big\}\\ &+\max\big\{\|\mathcal{L}\|^{(p')^-}_{L^{p(z)}(Q_T)};\ \|\mathcal{L}\|^{(p')^+}_{L^{p(z)}(Q_T)}\big\}+\max\big\{\|\nabla g^\eta\|^{p^-}_{L^{p(z)}(Q_T)};\|\nabla g^\eta\|^{p^+}_{L^{p(z)}(Q_T)}\big\}\Big)\\ &+k\|\mu^\eta\|_{L^1(Q_T)}+k\|b_\eta(z,w_0^\eta)\|_{L^1(\Omega)}+meas(\Omega),\end{aligned} \tag{4.23}$$

almost everywhere in $(0,T)$, where C is a constant independent of η. From (4.2)-(4.8) and using(4.19) we deduced that $b(z,w)-g\in L^\infty(0,T,L^1(\Omega))$.
Step 4:
The time regularization of $T_k(w)$ is defined as follows: Let $(v_0^v)_v$ be a sequence of functions in Ω, such that

$$v_0^v\in L^\infty(\Omega)\cap W_0^{1,p(z)}(\Omega),\quad \|v_0^v\|_{L^\infty(\Omega)}\le k,\ \forall\ v>0, \tag{4.24}$$

and

$$\begin{cases} v_0^v \text{ converge to } T_k(w_0) \quad \text{almost everywhere in } \Omega,\\ \frac{1}{v}\|(v_0^v)_t\|_{L^{p(z)}(\Omega)} \text{ converge to } 0 \text{ as } v \text{ go to } \infty.\end{cases} \tag{4.25}$$

For each $k>0$ and all $v>0$, $T_k(w)_v$ is a the unique solution of the monotonous problem

$$\frac{\partial (T_k(w))_v}{\partial t}+v(T_k(w)_v-T_k(w))=0 \qquad \text{in } \mathcal{D}'(\Omega), \tag{4.26}$$

$$T_k(w)_v(t=0)=v_0^v \qquad \text{in } \Omega. \tag{4.27}$$

Thus,

$$T_k(v)_v\in L^\infty(Q_T)\cap L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)) \text{ and } \frac{\partial (T_k(w))_v}{\partial t}\in L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)).$$

Note that(4.24)-(4.27) give the following convergence result

$$T_k(v)_v\to T_k(v) \text{ in } L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)) \text{ and almost everywhere in } Q_T, \tag{4.28}$$

as v converge to $+\infty$, with $\|T_k(v)_v\|_{L^\infty(Q_T)}\le k,$ for all $v>0$. In addition, we state a fundamental result on the usefulness of the approximate capacities, which will be indispensable for dealing the measure (singular part). □

Lemma 4.1 *[37, Lemma 5] Assume that $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q_T)$ with μ_s^- and μ_s^+ are concentrated respectively on two disjoint sets E^- and E^+ of zero $p(z)$-capacity. Then, for all $\delta > 0$, there exist $K_\delta^- \subseteq E^-$ and $K_\delta^+ \subseteq E^+$ compact sets where*

$$\mu_s^+(E^+ \backslash K_\delta^+) \leq \delta, \quad \mu_s^-(E^- \backslash K_\delta^-) \leq \delta, \tag{4.29}$$

and there exist two functions $\psi_\delta^-, \psi_\delta^+ \in \mathcal{C}_0^1(Q_T)$, where

$$\psi_\delta^- \equiv \text{ on } K_\delta^- \text{ and } \psi_\delta^+ \equiv 1 \text{ on } K_\delta^+, \tag{4.30}$$

$$0 \leq \psi_\delta^- \leq 1, \ 0 \leq \psi_\delta^+ \leq 1, \tag{4.31}$$

$$\sup(\psi_\delta^-) \cap s\sup(\psi_\delta^+) \equiv \emptyset, \tag{4.32}$$

moreover

$$\|\psi_\delta^-\|_\Gamma \leq \delta, \ \|\psi_\delta^+\|_\Gamma \leq \delta, \tag{4.33}$$

thus, there exists a particular decomposition of $(\psi_\delta^-)_t$ and $(\psi_\delta^+)_t$ where

$$\|(\psi_\delta^-)_t^1\|_{L^{(p^-)'}(0,T;W^{-1,p'(z)}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q_T)} \leq \frac{\delta}{3}, \tag{4.34}$$

$$\|(\psi_\delta^+)_t^1\|_{L^{(p^-)'}(0,T;W^{-1,p'(z)}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q_T)} \leq \frac{\delta}{3}, \tag{4.35}$$

with ψ_δ^- and ψ_δ^+ converge to zero weakly* in $L^\infty(Q_T)$, in $L^1(Q_T)$, and up to subsequences, almost everywhere as δ disappears. Furthermore, if λ_+^η and λ_-^η are as in Theorem 4.1, we get

$$\int_{Q_T} \psi_\delta^- d\lambda_+^\eta = \ell(\eta, \delta) \ , \quad \int_{Q_T} \psi_\delta^- d\mu_+^\eta \leq \delta, \tag{4.36}$$

$$\int_{Q_T} \psi_\delta^+ d\lambda_-^\eta = \ell(\eta, \delta) \ , \quad \int_{Q_T} \psi_\delta^+ d\mu_-^\eta \leq \delta, \tag{4.37}$$

$$\int_{Q_T} (1 - \psi_\delta^+ \psi_\varepsilon^+) d\lambda_+^\eta = \ell(\eta, \delta, \varepsilon) \ , \quad \int_{Q_T} (1 - \psi_\delta^+ \psi_\varepsilon^+) d\mu_+^\eta \leq \delta + \varepsilon, \tag{4.38}$$

$$\int_{Q_T} (1 - \psi_\delta^- \psi_\epsilon^-) d\lambda_-^\eta = \ell(\eta, \delta, \varepsilon) \ , \quad \int_{Q_T} (1 - \psi_\delta^- \psi_\varepsilon^-) d\mu_-^\eta \leq \delta + \varepsilon. \tag{4.39}$$

In the following, we will again refer to the subsequences of ψ_δ^+ and ψ_δ^-, which all satisfy the convergence results given in Lemma 4.1. Next, we will establish the key result in the proof of the main theorem.

Theorem 4.2 *Consider $r^\eta = b(z, w^\eta) - g^\eta$ and $r = b(z, w) - g$. Then for each $k > 0$*

$$T_k(r^\eta) \to T_k(r) \text{ strongly in } L^{p^-}(0, T; W_0^{1,p(z)}(\Omega)) \text{ as } \eta \text{ go to } 0.$$

The proof is done in several steps.

Proof: The proof will easily follow from an asymptotic estimate

$$\overline{\lim_{\eta \to 0}} \int_{Q_T} \Phi(t, z, w^\eta, \nabla w^\eta) \nabla T_k(r^\eta) dz dt \leq \int_{Q_T} \Phi(t, z, w, \nabla w) \nabla T_k(r) dz dt, \tag{4.40}$$

in the sequel, first let us present the next function that, we shall employ throughout the rest of this proof

$$F_m(s) = \begin{cases} 1, & if\ |s| \leq m, \\ \dfrac{2m-s}{n}, & if\ m < s \leq 2m, \\ \dfrac{2m+s}{m}, & if\ -2m < s \leq -m, \\ 0, & if\ |s| \geq 2m. \end{cases} \tag{4.41}$$

Step 4.1: Near and farm from E
Let E^+ and E^- be the sets, where respectively, λ_s^+ and λ_s^- are concentrated and for each $\delta, \varepsilon > 0$, let $\psi_\delta^+, \psi_\varepsilon^+, \psi_\delta^-$ and ψ_ε^- as in Lemma 4.1. Let's set $\varphi_{\delta,\varepsilon} = \psi_\delta^+ \psi_\eta^+ + \psi_\delta^- \psi_\varepsilon^-$ we get

$$\begin{aligned}\int_{Q_T} & \Phi(z,t,w^\eta,\nabla w^\eta)\nabla(T_k(r^\eta) - T_k(r)_r)F_m(r^\eta)dzdt \\ &= \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla(T_k(r^\eta) - T_k(r)_r)F_m(r^\eta)\varphi_{\delta,\varepsilon}dzdt \\ &\quad + \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla(T_k(r^\eta) - T_k(r)_r)F_m(r^\eta)(1-\varphi_{\delta,\varepsilon})dxdt. \end{aligned} \tag{4.42}$$

On the other side, if $m > k$, knowing that $\Phi(z,t,w^\eta,\nabla w^\eta \chi_{\{|r^\eta| \leq 2m\}})\nabla T_k(r)_r$ is weakly compact in $L^1(Q_T)$ when η tends to 0, $F_m(r^\eta)$ converges to $F_m(r)$ in the weak* topology of $L^\infty(Q_T)$ and a. e. in Q_T, by Lebesgue convergence theorem we get

$$\begin{aligned}&\overline{\lim_{\eta\to 0}} \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla(T_k(r^\eta) - T_k(r)_r)F_m(r^\eta)\varphi_{\delta,\varepsilon}dzdt \\ &= \overline{\lim_{\eta\to 0}} \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla(T_k(r^\eta))\varphi_{\delta,\varepsilon}dzdt - \int_{Q_T} \sigma_{2n}\nabla T_k(r)_r F_m(r^\eta)\varphi_{\delta,\varepsilon}dzdt \\ &= \overline{\lim_{\eta\to 0}} \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla(T_k(r^\eta))\varphi_{\delta,\varepsilon}dzdt - \int_{Q_T} \sigma_{2n}\nabla T_k(r)\varphi_{\delta,\varepsilon}dzdt + \ell(\eta), \end{aligned} \tag{4.43}$$

as δ go to zero, then $\varphi_{\delta,\eta} \to 0$ weakly* in $L^\infty(Q_T)$, we get

$$\int_{Q_T} \sigma_{2n}\nabla T_k(r)\varphi_{\delta,\varepsilon}dzdt = \ell(\delta),$$

as a consequence, if we show that

$$\overline{\lim_{\eta\to 0}}\ \overline{\lim_{\delta\to 0}}\ \overline{\lim_{\varepsilon\to 0}} \int_{Q_T} \Phi(z,t,w,\nabla w^\eta)\nabla T_k(r^\eta)\varphi_{\delta,\varepsilon}dzdt \leq 0, \tag{4.44}$$

from (4.43) thus, we can conclude that

$$\overline{\lim_{\eta\to 0}}\ \overline{\lim_{\delta\to 0}}\ \overline{\lim_{\varepsilon\to 0}} \int_{Q} \Phi(z,t,w,\nabla w^\eta)\nabla(T_k(r^\eta) - T_k(r)_r)F_n(r^\eta)\varphi_{\delta,\varepsilon}dzdt \leq 0. \tag{4.45}$$

Step 4.2: Near to E. To prove (4.44), we must verify the following result. □

Lemma 4.2 *Let φ_+^ε, φ_-^ε are two non-negative functions in $\mathcal{C}_c^\infty(Q_T)$ and ε is a positive real number such that*

$$\begin{cases} 0 \leq \varphi_+^\varepsilon \leq 1,\ 0 \leq \varphi_-^\varepsilon \leq 1, \\ 0 \leq \displaystyle\int_{Q_T} \varphi_-^\varepsilon d\mu_s^+ \leq \varepsilon,\ 0 \leq \int_{Q_T} \varphi_+^\eta \mathrm{d}\mu_s^- \leq \varepsilon. \end{cases} \tag{4.46}$$

Let w^η be a solution of $(\mathcal{P}_\eta)$, we get

$$\frac{1}{m}\int_{\{-2m<r^\eta\leq -m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla b_\eta(z,w^\eta)\varphi^\varepsilon_+ dzdt=\ell(\eta,m,\varepsilon), \tag{4.47}$$

$$\frac{1}{m}\int_{\{m<r^\eta\leq 2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla b_\eta(z,w^\eta)\varphi^\varepsilon_- dzdt=\ell(\eta,m,\varepsilon). \tag{4.48}$$

Proof: To establish the proof of (4.48), assume that $\beta_m(s)=\beta_m(s^+)$, we consider $\beta_m(r^\eta)\varphi^\varepsilon_-$ as test function in $(\mathcal{P}_\eta)$ and by easily changing the order of all terms, we obtain

$$\begin{aligned}\frac{1}{m}\int_{\{m<r^\eta\leq 2m\}}&\Phi(z,t,w^\eta,\nabla w^\eta)\nabla b_\eta(z,w^\eta)\varphi^\varepsilon_- dzdt+\int_{Q_T}\beta_m(r^\eta)\varphi^\varepsilon_- d\lambda^\eta_-\\
&=\frac{1}{m}\int_{\{m<r^\eta\leq 2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla g^\eta\varphi^\varepsilon_- dzdt+\int_{Q_T}\overline{\beta}_m(r^\eta)\frac{d\varphi^\eta_-}{dt}dzdt\\
&\quad-\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla\varphi^\varepsilon_-\beta_m(r^\eta)dzdt+\int_{Q_T}\beta_m(r^\eta)\varphi^\varepsilon_- d\lambda^\eta_+\\
&\quad+\int_{Q_T}f^\eta\beta_m(r^\eta)\varphi^\varepsilon_- dzdt-\int_0^T\Big\langle div(F^\eta),\ \beta_m(r^\eta)\varphi^\varepsilon_-\Big\rangle dt,\end{aligned} \tag{4.49}$$

with $\overline{\beta}_m(s)=\int_0^s\beta_m(r)dr$. Since $\int_{Q_T}\beta_m(r^\varepsilon)\varphi^\varepsilon_- d\lambda^\eta_-\geq 0$ and using (3.3), (3.4) and Young's inequality we have

$$\begin{aligned}\frac{1}{m}\int_{\{m<r^\eta\leq 2m\}}&\Phi(z,t,w^\eta,\nabla w^\eta)\nabla b_\eta(z,w^\eta)\varphi^\varepsilon_- dzdt\leq\frac{C}{m}\int_{Q_T}(|\nabla g^\eta|^{p(z)}+|B_k|^{p(z)}+|\mathcal{L}|^{p'(z)})dzdt\\
&+\int_{Q_T}\overline{\beta}_m(r^\eta)\frac{d\varphi^\eta_-}{dt}dzdt-\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla\varphi^\varepsilon_-\beta_m(r^\eta)dzdt+\int_{Q_T}\beta_m(r^\eta)\varphi^\varepsilon_- d\lambda^\eta_+\\
&+\int_{Q_T}f^\eta\beta_m(r^\eta)\varphi^\varepsilon_- dzdt-\int_0^T\Big\langle div(H^\eta),\ \beta_m(r^\eta)\varphi^\varepsilon_-\Big\rangle dt.\end{aligned} \tag{4.50}$$

By means (4.12) and (4.14) and to the fact that $\varphi^\varepsilon_-\in\mathcal{C}^\infty_c(Q_T),\beta_m(r^\eta)$ converges to $\beta_m(r)$ *a.e.* in Q_T and in $L^\infty(Q_T)$ *weak** as η tends to 0 and $\beta_m(r)$ converges to 0 a.e. in Q_T and in $L^\infty(Q_T)$ *weak** as m tends to $+\infty$ and thanks to Lebesgue convergence theorem, we get

$$\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla\varphi^\varepsilon_-\beta_m(r^\eta)dzdt=\ell(\eta,m)\ . \tag{4.51}$$

As $\overline{\beta}_m(r^\eta)$ converges to $\overline{\beta}_m(r)$ in $L^1(Q_T)$ when η tends to 0 and $\overline{\beta}_m(r)$ converges to 0 in $L^1(Q_T)$ when m tends to $+\infty$, we have

$$\int_{Q_T}\overline{\beta}_m(\mu^\eta)\frac{d\varphi^\eta_-}{dt}dzdt=\ell(\eta,m)\ .$$

Furthermore, thanks to Lebesgue's convergence theorem and f^ηconverges to f weakly in $L^1(Q_T)$, we deduce that

$$\int_{Q_T}f^\eta\beta_m(r^\eta)\varphi^\varepsilon_- dzdt=\ell(\eta,m).$$

By exploiting the fact that $div(F^\eta)$ converges to $div(F)$ strongly in $L^{(p^-)'}(0,T;W^{-1,p'(z)}(\Omega))$ and $\beta_m(r^\eta)$ converges to $\beta_m(r)$ weakly in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$ to zero strongly in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$ (this fact is an simple result of the estimate on the truncates of w^η in (4.12) and (4.14)), we have

$$\int_0^T \Big\langle div(F^\eta), \beta_m(r^\eta)\varphi_-^\varepsilon \Big\rangle dt = \ell(\eta, m).$$

At last, as $\beta_m(r^\eta)$ is non-negative and bounded and according to (4.46) and φ_-^ε is continuous, we obtain

$$\int_{Q_T} \beta_m(r^\eta)\varphi_-^\eta d\lambda_+^\eta \leq \int_{Q_T} \varphi_-^\varepsilon d\mu_s^+ + \ell(\eta) = \ell(\eta,\varepsilon).$$

From all the above, we obtain (4.48) and we can show (4.47) analogously by taking $\beta_m(s) = \beta_m(s^-)$ and $\beta_m(r^\eta)\varphi_+^\varepsilon$ as a function of the test in $(\mathcal{P}_\eta)$. □

Let us now verify (4.44): for $k>0$ fixed, we consider $(k - T_k(r^\eta))H_m(r^\eta)\psi_\delta^+\psi_\varepsilon^+$ as the test function in $(\mathcal{P}_\eta)$, by defining $\Lambda_{m,k}(s) = \int_0^s (k - T_k(r))H_m(r)dr$ and using integration by part, we get

$$\begin{aligned}
&-\int_{Q_T} \Lambda_{m,k}(r^\eta)\frac{d}{dt}(\psi_\delta^+\psi_\varepsilon^+)dzdt + \int_{Q_T} (k - T_k(r^\eta))H_m(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla(\psi_\delta^+\psi_\eta^+)dzdt \\
&\quad + \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla H_m(r^\eta)(k - T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+ dzdt - \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta)F_m(r^\eta)\psi_\delta^+\psi_\varepsilon^+ dzdt \\
&\quad = \int_{Q_T} f^\eta H_m(r^\eta)(k - T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+ dzdt - \int_0^T \Big\langle div(F^\eta), H_m(r^\eta)(k - T_k(r^\eta))\psi_\delta^+\psi_\epsilon^+ \Big\rangle dt \qquad (4.52)\\
&\qquad + \int_{Q_T} H_m(r^\eta)(k - T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+ d\lambda_+^\eta - \int_{Q_T} H_m(r^\eta)(k - T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+ d\lambda_-^\eta.
\end{aligned}$$

For $m>k$, we obtain

$$H_m(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\chi_{\{|r^\eta|\leq k\}} = \Phi(z,t,w^\eta,\nabla w^\eta)\chi_{\{|r^\eta|\leq k\}} \text{ a.e. in } Q_T, \qquad (4.53)$$

thus, we have

$$\begin{aligned}
&\int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta)\psi_\delta^+\psi_\varepsilon^+ dzdt + \int_{Q_T} H_m(r^\eta)(k - T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+ d\lambda_+^\eta \\
&\quad = -\int_{Q_T} \Lambda_{m,k}(r^\eta)\frac{d}{dt}(\psi_\delta^+\psi_\eta^+)dzdt + \frac{2k}{m}\int_{\{-2m<r\leq -m\}} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta\psi_\delta^+\psi_\varepsilon^+ dzdt \\
&\quad + \int_{Q_T} (k - T_k(r^\eta))H_m(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla(\psi_\delta^+\psi_\varepsilon^+)dzdt - \int_{Q_T} f^\eta(k - T_k(r^\eta))H_m(r^\eta)\psi_\delta^+\psi_\varepsilon^+ dzdt \\
&\quad - \int_0^T \Big\langle div(F^\eta), H_m(r^\eta)(k - T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+ \Big\rangle dt + \int_{Q_T} (k - T_k(r^\eta))H_m(r^\eta)\psi_\delta^+\psi_\varepsilon^+ d\lambda_-^\eta. \qquad (4.54)
\end{aligned}$$

Let us examine term by term the right side of (4.54). First, according to (4.12), we get the weak convergence in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$ of $\Lambda_{m,k}(r^\eta)$ to $\Lambda_{m,k}(r)$ and since $\Lambda_{m,k}(r^\eta) \in L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)) \cap L^\infty(Q_T)$, we deduce that

$$\begin{aligned}
\int_{Q_T} \Lambda_{m,k}(r^\eta)\frac{d}{dt}(\psi_\delta^+\psi_\varepsilon^+)dzdt &= \int_{Q_T} \Lambda_{m,k}(r^\eta)\frac{d\psi_\delta^+}{dt}\psi_\varepsilon^+ dzdt \\
&\quad + \int_{Q_T} \Gamma_{m,k}(r^\eta)\frac{d\psi_\varepsilon^+}{dt}\psi_\delta^+ dzdt + \ell(\eta) = \ell(\varepsilon,\delta). \qquad (4.55)
\end{aligned}$$

By combining (4.12), (4.14), Lebesgue convergence Theorem, Lemma 4.1 and using the weak convergence in $L^\infty(Q_T)$ of $(k-T_k(r^\eta))H_m(r^\eta)$ to $(k-T_k(r))H_m(r)$ and a.e. in Q_T, we conclude that

$$\int_{Q_T}(k-T_k(r^\eta))H_m(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla(\psi_\delta^+\psi_\varepsilon^+)dzdt$$
$$=\int_{Q_T}(k-T_k(r))H_m(r)\sigma_{2m}\nabla(\psi_\delta^+\psi_\eta^+)dzdt+\ell(\eta)=\ell(\eta,\delta). \quad (4.56)$$

Furthermore, $(k-T_k(r^\eta))H_m(r^\eta)\psi_\delta^+\psi_\varepsilon^+$ converges weakly to $(k-T_k(r))H_m(r)\psi_\delta^+\psi_\varepsilon^+$ in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$ and in $L^\infty(Q_T)$ weak* topology, then thanks to Lemma 4.1, we have

$$\int_0^T\Big\langle div(F^\eta),\ H_m(r^\eta)(k-T_k(r^\eta))\psi_\delta^+\psi_\eta^+\Big\rangle dt=\ell(\eta,\delta) \text{ and } \int_{Q_T}f^\eta(k-T_k(r^\eta))H_m(r^\eta)\psi_\delta^+\psi_\varepsilon^+dzdt=\ell(\varepsilon,\delta).$$

applying Young's inequality, hypotheses (3.3)-(3.4) and since $0\le\psi_\delta^+\le 1$, we have

$$\Big|\frac{1}{m}\int_{\{-2m<r^\eta\le -m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta\psi_\delta^+\psi_\varepsilon^+dzv\Big|$$
$$\le\frac{1}{m}\int_{\{-2m<r^\eta\le -m\}}\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w^\eta\psi_\varepsilon^+dzdt+\frac{C}{m}\int_{Q_T}(|\nabla g^\eta|^{p(z)}+|B_k|^{p(z)}+|\mathcal{L}|^{p'(z)})dzdt,$$

using Lemma 4.2 for $\varphi_+^\varepsilon=\psi_\varepsilon^+$, we get

$$\frac{1}{m}\int_{\{-2m<r^\eta\le -m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta\psi_\eta^+dzdt=\ell(\eta,m,\varepsilon).$$

Thanks to (4.37) in Lemma 4.1, we obtain

$$\Big|\int_{Q_T}H_m(r^\eta)(k-T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+d\lambda_-^\eta\Big|\le 2k\int_{Q_T}\psi_\delta^+\psi_\varepsilon^+d\lambda_-^\eta=2k\int_{Q_T}\psi_\delta^+\psi_\varepsilon^+d\lambda_s^-+\ell(\eta)=\ell(\eta,\delta).$$

Putting all the previous results together, we obtain

$$\int_{Q_T}H_m(r^\eta)(k-T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+d\lambda_+^\eta+\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta)\psi_\delta^+\psi_\varepsilon^+dzdt=\ell(\eta,m,\delta,\varepsilon),$$

and according to $\displaystyle\int_{Q_T}H_m(r^\eta)(k-T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+d\lambda_+^\eta\ge 0$, we obtain $\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta)\psi_\delta^+\psi_\varepsilon^+dzdt\le\ell(\eta,\delta,\varepsilon)$. For the same reason as before and taking $(k+T_k(r^\eta))H_m(r^\eta)\psi_\delta^-\psi_\varepsilon^-$ as a test function, we get $\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta)\psi_\delta^-\psi_\varepsilon^-dzdt\le\ell(\eta,\delta,\varepsilon)$. Then we have (4.44) which gives (4.45).

Remark 4.1 As shown above, we obtain

$$\int_{Q_T}H_m(r^\eta)(k-T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+d\lambda_+^\eta+\int_{Q_T}\frac{\partial b(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\chi_{\{|r^\eta|\le k\}}\nabla w^\eta\psi_\delta^+\psi_\eta^+dzdt$$
$$+\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\chi_{\{|r^\eta|\le k\}}\nabla_z b_\eta(z,w^\eta)\psi_\delta^+\psi_\varepsilon^+dzdt-\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\chi_{\{|r^\eta|\le k\}}\nabla g^\eta\psi_\delta^+\psi_\varepsilon^+dzdt$$
$$=\ell(\eta,m,\delta,\varepsilon).$$

From (3.3), (4.12), (4.14) and by means Lemma 4.1, we have

$$\int_{Q_T}H_m(r^\eta)(k-T_k(r^\eta))\psi_\delta^+\psi_\varepsilon^+d\lambda_+^\eta=\ell(\eta,m,\delta,\varepsilon).$$

In a similarly, we obtain

$$\int_{Q_T}H_m(r^\eta)(k+T_k(r^\eta))\psi_\delta^-\psi_\varepsilon^-d\lambda_-^\eta=\ell(\eta,m,\delta,\varepsilon)\ .$$

Step 4.3: Far-from E First, let us show a result that will be fundamental for dealing with the second term of the right-hand side of (4.42).

Lemma 4.3 *Let $m \geq 1$ be fixed. Let choose $r^\eta = b_\eta(z, w^\eta) - g^\eta$, we get*

$$\varlimsup_{r\to+\infty} \lim_{\eta\to 0} \int_0^T \Big\langle \frac{\partial r^\eta}{\partial t}; H_m(r^\eta)(T_k(r^\eta) - T_k(r)_r)(1-\phi_{\delta,\varepsilon})\Big\rangle dsdt \geq 0, \tag{4.57}$$

where is the duality pairing between $L^1(\Omega) + W^{-1,p'(z)}(\Omega)$ and $L^\infty(\Omega) \cap W_0^{1,p(z)}(\Omega)$ and the function H_m is obtained by (4.41).

Proof: Assume that $m \geq 1$ be fixed, let $\overline{H}_m(u) = \displaystyle\int_0^u H_m(s)ds \in \mathcal{C}^1(\mathbb{R}), \overline{H'}_m = H_m$ and $supp(H_m) \subset [-2m, 2m]$ thanks to (4.15) and (4.16), the function $\overline{H}_m(r^\eta) \in L^{p^-}(0,T,W_0^{1,p(z)}(\Omega))$ and $\dfrac{\partial \overline{H}_m(r^\eta)}{\partial t} \in L^1(Q_T) + L^{(p')^-}(0,T,W^{-1,p'(z)}(\Omega))$.
Furthermore, for each $k \leq m$, we obtain $T_k(\overline{H}_m(r^\eta)) = T_k(r^\eta)$ a.e. in Q_T and $T_k(\overline{H}_m(r))_r = (T_k(r))_r$ a.e. in Q_T, for all $r > 0$, moreover

$$\begin{aligned}
&\int_0^T \int_0^t \langle \frac{\partial r^\eta}{\partial t}; H_m(r^\eta)(T_k(r^\eta) - T_k(r)_r)(1-\phi_{\delta,\varepsilon})\rangle dsdt \\
&= \int_0^T \int_0^t \langle \frac{\partial \overline{H}_m(r^\eta)}{\partial t}; (T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)(1-\phi_{\delta,\varepsilon})\rangle dsdt \\
&= \int_0^T \int_0^t \Big\langle \frac{\partial(T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)}{\partial t}; (T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)(1-\phi_{\delta,\varepsilon})\Big\rangle dsdt \\
&+ \int_0^T \int_0^t \Big\langle \frac{\partial T_k(\overline{H}_m(r))_r}{\partial t}; (T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)(1-\phi_{\delta,\eta})\Big\rangle dsdt \\
&+ \int_0^T \int_0^t \Big\langle \frac{\partial(\overline{H}_m(r^\eta) - T_k(\overline{H}_m(r^\eta)))}{\partial t}; (T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)(1-\phi_{\delta,\varepsilon})\Big\rangle dsdt \\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned} \tag{4.58}$$

Integrating by parts, we get

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^T \int_0^t \Big\langle \frac{\partial(T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)}{\partial t}; (T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)(1-\phi_{\delta,\varepsilon})\Big\rangle dsdt \\
&= \frac{1}{2}\int_{Q_T} \Big|T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r\Big|^2 (1-\phi_{\delta,\varepsilon})dtdz - \frac{T}{2}\int_\Omega \Big|T_k(\overline{H}_m(r_0^\eta)) - T_k(\overline{H}_m(r))_r(0)\Big|^2 dz \\
&\qquad + \frac{1}{2}\int_{Q_T} \Big|T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r\Big|^2 \frac{\partial \phi_{\delta,\varepsilon}}{\partial t} dtdz.
\end{aligned} \tag{4.59}$$

In view of (4.8), (4.19), (4.24) and (4.28), we have that

$$\mathcal{I}_1 = \ell(\eta, r). \tag{4.60}$$

Applying (4.26) with $T_k(\overline{H}_m(r))_r$, we obtain

$$\begin{aligned}
\mathcal{I}_2 &= \int_0^T \int_0^t \Big\langle \frac{\partial T_k(\overline{H}_m(r))_r}{\partial r}; (T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)(1-\phi_{\delta,\varepsilon})\Big\rangle dsdt \\
&= r\int_0^T \int_0^t (T_k(\overline{H}_m(r))_- T_k(\overline{H}_m(r))_r)(T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)(1-\phi_{\delta,\varepsilon}) dsdtdz.
\end{aligned} \tag{4.61}$$

Let δ, ε and m fixed, and as $T_k(\overline{H}_m(r^\eta))(1-\phi_{\delta,\varepsilon})$ converges weakly* to $T_k(\overline{H}_m(r))(1-\phi_{\delta,\varepsilon})$ *in* $L^\infty(Q_T)$ and a.e. in Q_T, we get

$$\mathcal{I}_2 = r\int_0^T \int_0^t (T_k(\overline{H}_m(r))_- T_k(\overline{H}_m(r))_r)(T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)(1-\phi_{\delta,\varepsilon}) dsdtdz \geq w(\eta). \tag{4.62}$$

Let us know that, for k fixed, we have $G_k(s) = s - T_k(s)$, then

$$\begin{aligned}\mathcal{I}_3 &= \int_0^T \int_0^t \langle \frac{\partial(\overline{H}_m(r^\eta) - T_k(\overline{H}_m(r^\eta)))}{\partial t}; (T_k(\overline{H}_m(r^\eta)) - T_k(\overline{H}_m(r))_r)(1-\phi_{\delta,\varepsilon})\rangle dsdt \\ &= \int_0^T \int_0^t \langle \frac{\partial G_k(\overline{H}_m(r^\eta))}{\partial t}; T_k(\overline{H}_m(r^\eta))(1-\phi_{\delta,\varepsilon})\rangle dsdt - \int_0^T \int_0^t \langle \frac{\partial G_k(\overline{H}_m(r^\eta))}{\partial t}; T_k(\overline{H}_m(r))_r(1-\phi_{\delta,\varepsilon})\rangle dsdt \\ &= \mathcal{I}_{3.1} + \mathcal{I}_{3.2}.\end{aligned} \tag{4.63}$$

According to (4.8),(4.19) and applying the integration by parts formula, we deduced that

$$\begin{aligned}\mathcal{I}_{3.1} &= \int_0^T \int_0^t \Big\langle \frac{\partial G_k(\overline{H}_m(r^\eta))}{\partial t}; T_k(\overline{H}_m(r^\eta))(1-\phi_{\delta,\varepsilon})\Big\rangle dsdt = k\int_0^T \int_0^t \Big\langle \frac{\partial |G_k(\overline{H}_m(r^\eta))|}{\partial t}; (1-\phi_{\delta,\varepsilon})\Big\rangle dsdt \\ &= k\int_{Q_T} |G_k(\overline{H}_m(r^\eta))|dzdt - Tk\int_\Omega |G_k(\overline{H}_m(r^\eta))(0)|dz + k\int_{Q_T} |G_k(\overline{H}_m(r^\eta))|\frac{\partial \phi_{\delta,\varepsilon}}{\partial t}dzdt \\ &= k\int_{Q_T} |G_k(\overline{H}_m(r))|(1-\phi_{\delta,\varepsilon})dzdt - Tk\int_\Omega |G_k(\overline{H}_m(r))(0)|dz + k\int_{Q_T} |G_k(\overline{H}_m(r))|\frac{\partial \phi_{\delta,\varepsilon}}{\partial t}dzdt + \ell(\eta).\end{aligned} \tag{4.64}$$

Similarly, by applying integration by parts and by the definition of $T_k(\overline{H}_m(r))_r$, we have

$$\begin{aligned}\mathcal{I}_{3.2} &= -\int_0^T \int_0^t \Big\langle \frac{\partial G_k(\overline{H}_m(r^\eta))}{\partial t}; T_k(\overline{H}_m(r))_r(1-\phi_{\delta,\varepsilon})\Big\rangle dsdt \\ &= -\int_{Q_T} G_k(\overline{H}_m(r^\eta))T_k(\overline{H}_m(r))_r(1-\phi_{\delta,\varepsilon})dzdt + T\int_\Omega G_k(\overline{H}_m(r^\eta(0)))T_k(\overline{H}_m(r))_r(0)dz \\ &\quad + r\int_{Q_T}\int_0^t (T_k(\overline{H}_m(r)) - T_k(\overline{H}_m(r))_r)G_k(\overline{H}_m(r^\eta))(1-\phi_{\delta,\varepsilon})dsdtdz \\ &\quad - \int_{Q_T}\int_0^t G_k(\overline{H}_m(r))T_k(\overline{H}_m(r))_r\frac{\partial \phi_{\delta,\varepsilon}}{\partial t}dsdzdt.\end{aligned} \tag{4.65}$$

From (4.8), (4.19), (4.24), (4.28) and the fact that $(T_k(\overline{H}_m(r)) - T_k(\overline{H}_m(r))_r)G_k(\overline{H}_m(r)) \geq 0$ a.e. in Q_T, it is straightforward to verify that

$$\begin{aligned}\mathcal{I}_{3.2} &= -\int_{Q_T} G_k(\overline{H}_m(r))T_k(\overline{H}_m(r))_r(1-\phi_{\delta,\varepsilon})dzdt + T\int_\Omega G_k(\overline{H}_m(r(0)))T_k(\overline{H}_m(r))_r(0)dz \\ &\quad + r\int_{Q_T}\int_0^t (T_k(\overline{H}_m(r)) - T_k(\overline{H}_m(r))_r)G_k(\overline{H}_m(r))(1-\phi_{\delta,\varepsilon})dsdtdz \\ &\quad - \int_{Q_T}\int_0^t G_k(\overline{H}_m(r))T_k(\overline{H}_m(r))_r\frac{\partial \phi_{\delta,\varepsilon}}{\partial t}dsdzdt + \ell(\eta) \\ &\geq -\int_{Q_T} G_k(\overline{H}_m(r))T_k(\overline{H}_m(r))(1-\phi_{\delta,\varepsilon})dzdt + T\int_\Omega G_k(\overline{H}_m(r(0)))T_k(\overline{H}_m(r))(0)dz \\ &\quad - \int_{Q_T}\int_0^t G_k(\overline{H}_m(r))T_k(\overline{H}_m(r))\frac{\partial \phi_{\delta,\varepsilon}}{\partial t}dsdzdt + \ell(\eta, r).\end{aligned} \tag{4.66}$$

Due to $G_k(s)T_k(s) = k|G_k(s)|$ for each $s \in \mathbb{R}$, it is possible to conclude from (4.66) that

$$\begin{aligned}\mathcal{I}_{3.2} \geq &-k\int_{Q_T} |G_k(\overline{H}_m(r))|^2(1-\phi_{\delta,\varepsilon})dzdt + Tk\int_\Omega |G_k(\overline{H}_m(r(0)))|^2dz \\ &- \ k\int_{Q_T}\int_0^t |G_k(\overline{H}_m(r))|^2\frac{\partial \phi_{\delta,\varepsilon}}{\partial t}dsdzdt + \ell(\eta, r).\end{aligned} \tag{4.67}$$

Note that

$$\mathcal{I}_{3.1} + \mathcal{I}_{3.2} \geq \ell(\eta, r). \tag{4.68}$$

As a result of the previous convergence results, we can deduce from (4.58), (4.60), (4.62), (4.63) and (4.68) that (4.57) is valid and that the proof of Lemma 4.3 is finished. □

Step 5: Here, we will verify that the limit $\sigma_k = \Phi(z,t,w,\nabla w)\chi_{\{|r|\le k\}}$ and we'll demonstrate the weak convergence in $L^1(Q_T)$ of $\Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta)$ as η tends to zero.

Lemma 4.4 *The subsequence of w^η given in Step 3 verifies for every $k \ge 0$,*

$$\overline{\lim_{\eta\to 0}} \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta)dzdt \le \int_{Q_T} \sigma_k \nabla T_k(r)dzdt. \tag{4.69}$$

Proof: For any $k \ge 0$, let $W_r^\eta := (T_k(r^\eta) - T_k(r)_r)$ and using $H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})$ as test function in $(\mathcal{P}_\eta)$, we obtain

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial r^\eta}{\partial t}; H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})\Big\rangle dt + \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla W_r^\eta H_m(r^\eta)(1-\phi_{\delta,\varepsilon})dxdt \\
&\qquad + \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})dzdt - \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla\phi_{\delta,\varepsilon}H_m(r^\eta)W_r^\eta dzdt \\
&\quad = \int_{Q_T} f^\eta F_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})dzdt - \int_0^T \Big\langle div(F^\eta); H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})\Big\rangle dt \\
&\qquad + \int_{Q_T} H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})d\lambda_+^\eta - \int_{Q_T} H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})d\lambda_-^\eta,
\end{aligned} \tag{4.70}$$

thus passing to the limit in (4.70) as η tends to 0, r tends to $+\infty$ and m tends to $+\infty$, the real number $k \ge 0$ is fixed and according to Lemma 4.3, we find

$$\lim_{r\to+\infty}\lim_{\eta\to 0}\int_0^T \Big\langle \frac{\partial r^\eta}{\partial t},\ H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})\Big\rangle dt \ge 0.$$

As W_r^η converges to $(T_k(r) - T_k(r)_r)$ weakly in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$ which compactly embedded into $L^1(Q_T)$, then W_r^η converges to $(T_k(r)-T_k(r)_r)$ a.e in Q_T and weakly* in $L^\infty(Q_T)$ as $\eta \to 0$ and for each $r > 0$.
Using Lebesgue convergence Theorem, for all $r > 0$, for every $m \ge 1$ and the properties of $T_k(r)_r$, we obtain

$$\lim_{r\to+\infty}\lim_{\eta\to 0}\int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla\phi_{\delta,\varepsilon}H_m(r^\eta)W_r^\eta dzdt = 0.$$

Due to the convergence of $div(F^\eta)$ to $div(F)$ strongly in $L^{(p')^-}(0,T;W^{-1,p'(z)}(\Omega))$ and using (4.12), (4.14), and the characteristics of $T_k(r)_r$, we obtain

$$\lim_{r\to+\infty}\lim_{\eta\to 0}\int_0^T \Big\langle div(F^\eta), H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})\Big\rangle dt = 0.$$

The weak convergence of f^η to f in $L^1(Q_T)$, the almost everywhere convergence of $H_m(r^\eta)W_r^\eta$ to $H_m(r)(T_k(r) - T_k(r)_r)$ in Q_T and weakly* in $L^\infty(Q_T)$, as $\eta \to 0$ and for every $r > 0$, and according to Lebesgue's dominated convergence Theorem and properties of $T_k(r)_r$, this leads to f^η converges to f weakly in $L^1(Q_T)$, and $H_m(r^\eta)W_r^\eta$ conveys to $H_m(r)(T_k(r)-T_k(r)_r)$ almost everywhere in Q_T and weakly* in $L^\infty(Q_T)$, when $\eta \to 0$, we conclude that

$$\lim_{r\to+\infty}\lim_{\eta\to 0}\int_{Q_T} f^\eta H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})dzdt = 0.$$

Since $\|H_m(r^\eta)W_r^\eta\|_{L^\infty(Q_T)} \leq 2k$ and by means Lemma 4.1, we get

$$|\int_{Q_T} H_m(r^\eta)W_r^\eta(1 - \phi_{\delta,\varepsilon})d\lambda_+^\eta| \leq 2k\int_{Q_T}(1 - \psi_\delta^+\psi_\eta^+)d\lambda_+^\eta + 2k\int_{Q_T}(1 - \psi_\delta^-\psi_\varepsilon^-)d\lambda_+^\eta,$$

and

$$\int_{Q_T} H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})d\lambda_+^\eta = \ell(\eta,\delta,\varepsilon).$$

Similarly,

$$\int_{Q_T} H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})d\lambda_-^\eta = \ell(\eta,\delta,\varepsilon).$$

Next, we prove that $\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla H_m(r^\eta)W_r^\eta(1-\phi_{\delta,\varepsilon})dzdt = \ell(\eta,m,\delta,\varepsilon)$. For this purpose, we have

$$\begin{aligned}
&\Big|\frac{1}{m}\int_{\{m\leq|r^\eta|<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta W_r^\eta(1-\phi_{\delta,\varepsilon})dzdt\Big|\\
&\leq \frac{2k}{m}\int_{\{m\leq|r^\eta|<2m\}}\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w^\eta(1-\phi_{\delta,\varepsilon})dzdt\\
&\qquad+\frac{C}{m}\int_{Q_T}(|\mathcal{L}|^{p'(z)}+|B_k|^{p'(z)}+|\nabla g^\eta|^{p(z)})dzdt\\
&\leq \frac{2k}{m}\int_{\{m\leq|r^\eta|<2m\}}\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w^\eta(1-\phi_{\delta,\varepsilon})dzdt\\
&\qquad+\frac{C}{m}(\rho(\mathcal{L})+\rho(B_k)+\rho(\nabla g^\eta)) = \mathcal{I}_1+\mathcal{I}_2,
\end{aligned}$$

where C is a constant independent of m and

$$\begin{cases}
\rho(\mathcal{L}) \leq \max\Big\{\|\mathcal{L}\|_{L^{p'(z)}(Q_T)}^{(p')^-},\|\mathcal{L}\|_{L^{p'(z)}(Q_T)}^{(p')^+}\Big\},\\
\rho(B_k) \leq \max\Big\{\|B_k\|_{L^{p'(z)}(Q_T)}^{(p')^-},\|B_k\|_{L^{p'(z)}(Q_T)}^{(p')^+}\Big\},\\
\rho(\nabla g^\eta) \leq \max\Big\{\|\nabla g^\eta\|_{L^{p(z)}(Q_T)}^{p^-},\|\nabla g^\eta\|_{L^{p(z)}(Q_T)}^{p^+}\Big\}.
\end{cases}$$

As $\mathcal{I}_2 = \ell(m)$, we obtain

$$\begin{aligned}
\mathcal{I}_1 = &\frac{2k}{m}\int_{\{m\leq|r^\eta|<2m\}}\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w^\eta(1-\psi_\delta^+\psi_\varepsilon^+)dzdt\\
&-\frac{2k}{m}\int_{\{m\leq|r^\eta|<2m\}}\frac{\partial b_\eta(z,<^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w^\eta\psi_\delta^-\psi_\varepsilon^-dzdt\\
&+\frac{2k}{m}\int_{\{m\leq|r^\eta|<2m\}}\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w^\eta(1-\psi_\delta^-\psi_\varepsilon^-)dzdt\\
&-\frac{2k}{m}\int_{\{m\leq|r^\eta|<2m\}}\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w^\eta\psi_\delta^+\psi_\varepsilon^+dxdt.
\end{aligned}$$

Taking $\varphi_-^{\delta,\varepsilon} = 1-\psi_\delta^+\psi_\varepsilon^+$, and from Lemmas 4.2, 4.1 we infer $\int_{Q_T}\varphi_-^{\delta,\varepsilon}d\mu_s^+ \leq \delta+\varepsilon$, thus, as $\varphi^{\delta,\varepsilon}$ verifies (4.44), and by means Lemma 4.2, we find

$$\frac{2k}{m}\int_{\{m\leq|r^\eta|<2m\}}\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w^\eta(1-\psi_\delta^+\psi_\varepsilon^+)dzdt \leq \ell(\eta,m)+\delta+\varepsilon = \ell(\eta,,\delta,\varepsilon).$$

Similarly, we arrive at the same result for the other terms. Therefore, we make our estimate far-from E :

$$\int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla W_r^\eta H_m(r^\eta)(1-\phi_{\delta,\varepsilon})dxdt \le \ell(\eta,r,m,\delta,\varepsilon). \tag{4.71}$$

Putting together (4.42), (4.45) and (4.71), and again choosing $m > k$, we get

$$\overline{\lim_{\eta\to 0}} \int_{Q_T} \Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta)dzdt \le \int_{Q_T} \sigma_k \nabla T_k(r)dzdt.$$

□

Lemma 4.5 *Let $k \ge 0$, the subsequence of w^η established in step 3 verifies the following conditions*

$$\lim_{\eta\to 0}\int_{Q_T} \frac{\partial b_\eta(z,w^\eta)}{\partial s}\Big[\Phi(z,t,w^\eta,\nabla w^\eta\chi_{\{|r^\eta|\le k\}}) - \Phi(z,t,w,\nabla w\chi_{\{|r|\le k\}})\Big] \tag{4.72}$$
$$\times\Big[\nabla w^\eta\chi_{\{|r^\eta|\le k\}} - \nabla w\chi_{\{|r|\le k\}}\Big]dzdt = 0.$$

Proof: For fixed $k > 0$, due to (3.2) and (3.5), we obtain

$$\mathcal{J}^\eta = \int_{Q_T} \frac{\partial b_\eta(z,w^\eta)}{\partial s}\Big(\Phi(z,t,w^\eta,\nabla w^\eta\chi_{\{|r^\eta|\le k\}} - \Phi(z,t,w,\nabla w\chi_{\{|r|\le k\}}\Big) \tag{4.73}$$
$$\times\Big(\nabla w^\eta\chi_{\{|r^\eta|\le k\}} - \nabla w\chi_{\{|r|\le k\}}\Big)dzdt \ge 0.$$

We write $\mathcal{J}^\eta = \mathcal{J}_1^\eta + \mathcal{J}_2^\eta + \mathcal{J}_3^\eta$ with

$$\begin{cases} \mathcal{J}_1^\eta = -\int_{Q_T} \frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w\chi_{\{|r^\eta|\le k\}}dzdt, \\ \mathcal{J}_2^\eta = -\int_{Q_T} \frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla w\chi_{\{|r^\eta|\le k\}}dzdt, \\ \mathcal{J}_3^\eta = -\int_{Q_T} \frac{\partial b_\eta(z,w^\eta)}{\partial s}\Phi(z,t,w,\nabla w)\chi_{\{|r|\le k\}}(\nabla w^\eta\chi_{\{|r^\eta|\le k\}} - \nabla w\chi_{\{|r|\le k\}})dzdt. \end{cases}$$

Observe that $r^\eta = b_\eta(z,w^\eta) - g^\eta$ and $\frac{\partial b_\eta(z,w^\eta)}{\partial s}\nabla w^\eta\chi_{\{|r^\eta|\le k\}} = \big(\nabla T_k(r^\eta) + (g^\eta - \nabla_z b(z,w^\eta))\chi_{\{|r^\eta|\le k\}}\big)$ almost everywhere in Q_T. We can consider that k where $\chi_{\{|r^\eta|<k\}}$ converges to $\chi_{\{|r|\le k\}}$ almost everywhere (see [14, Lemma 3.2]). We can achieve the following equation by passing to the limit in $\mathcal{J}_1^\eta, \mathcal{J}_2^\eta$ and $\mathcal{J}_3^\eta$ as η tends to zero

$$\lim_{\eta\to 0}\mathcal{J}_1^\eta = \lim_{\eta\to 0}\Big(\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta)dzdt + \int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\chi_{\{|r^\eta|\le k\}}\nabla g^\eta dzdt\Big)$$
$$-\lim_{\eta\to 0}\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla_z b_\eta(z,w^\eta)dzdt.$$

Thanks to (4.69), we get

$$\lim_{\eta\to 0}\mathcal{J}_1^\eta \le \int_{Q_T}\sigma_k\nabla T_k(r)dzdt - \int_{Q_T}\sigma_k\nabla_z b(z,w)dzdt + \int_{Q_T}\sigma_k\nabla g dzdt$$

From (4.21)-(4.22), we obtain

$$\lim_{\eta\to 0}\mathcal{J}_2^\eta = -\lim_{\eta\to 0}\int_{Q_T}\Phi(z,t,w^\eta,\nabla w^\eta)\chi_{\{|r^\eta|\le k\}}\frac{\partial b_\eta(z,w^\eta)}{\partial s}$$
$$\Big(\frac{\partial b(z,w)}{\partial s}\Big)^{-1}\Big(\nabla T_k(r) + (\nabla g - \nabla_z b(z,w^\eta))\chi_{\{|r^\eta|\le k\}}\Big)dzdt = -\int_{Q_T}\sigma_k(\nabla T_k(r) - \nabla_z b(z,w) + \nabla g)dzdt.$$

As a result of (4.4) and (4.21), we get for all $k \geq 0$

$$\lim_{\eta\to 0}\mathcal{J}_3^\eta = -\lim_{\eta\to 0}\int_{Q_T}\Big(\Phi(z,t,w,\nabla w)\chi_{\{|r|\leq k\}}\Big(\nabla T_k(r)-(\nabla g-\nabla_z b(z,w^\eta))\chi_{\{|r^\eta|\leq k\}}$$
$$-\frac{\partial b(z,w^\eta)}{\partial s}\big(\frac{\partial b(z,w)}{\partial s}\big)^{-1}\Big(\nabla T_k(r)+(\nabla g-\nabla_z b(z,w^\eta))\chi_{\{|r^\eta|\leq k\}}\Big)dzdt=0,$$

then, by passing to the limit in (4.73) as η tends to 0, we obtain (4.72). □

Corollary 4.1 *For fixed $k \geq 0$, the subsequence w^η setting in Step 3 satisfies*

$$\Phi(z,t,w^\eta,\nabla w^\eta)\nabla T_k(r^\eta) \rightharpoonup \sigma_k\nabla T_k(r) \text{ weakly in } L^1(Q_T), \text{ as } \eta\to 0, \tag{4.74}$$

with

$$\sigma_k = \Phi(z,t,w,\nabla w)_{\chi_{\{|r|\leq k\}}} \text{ almost everywhere in } Q_T. \tag{4.75}$$

Proof: Observe that
$r^\eta = b_\eta(z,w^\eta)-g^\eta$ and $\frac{\partial b_\eta(z,w^\eta)}{\partial s}\nabla w^\eta\chi_{\{|r^\eta|\leq k\}} = \Big(\nabla T_k(r^\eta)+\big(g^\eta-\nabla_z b(z,w^\eta)\big)\chi_{\{|r^\eta|\leq k\}}\Big)$ a.e. in Q_T and consider k where $\chi_{\{|r^\eta|<k\}}$ converges a.e. to $\chi_{\{|r|\leq k\}}$ (see [14, Lemma 3.2]).
We note that w^η converges to w almost everywhere in Q_T, and $T_k(r^\eta)$ converges weakly to $T_k(r)$ in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$, $\Phi(z,t,w^\eta,\nabla w^\eta)\chi_{\{|r^\eta|\leq k\}}$ converges weakly to σ_k in $(L^{p'(z)}(Q_T))^N$. According to (4.69), (4.72) and due to

$$\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Big[\Phi(z,t,w^\eta,\nabla w^\eta\chi_{\{|r^\eta|\leq k\}})-\Phi(z,t,w,\nabla w\chi_{\{|r|\leq k\}}\Big]\Big[\nabla w^\eta\chi_{\{|r^\eta|\leq k\}}-\nabla w\chi_{\{|r|\leq k\}}\Big] \text{ converges to } 0,$$

strongly in $L^1(Q_T)$ as η tends to 0.
Combining the results of the convergence results given previously and the application of the usual Minty argument, we can affirm that (4.74) and (4.75) are true.
Observe that $\nabla w^\eta = \Big(\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Big)^{-1}(\nabla T_k(r^\eta)-\nabla_z b_\eta(z,w^\eta)+g^\eta)$ a.e. in Q_t, g^η converges strongly to g in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$, and according to (4.74)-(4.75), we get

$$\int_{Q_T}\Phi\Big(z,t,w^\eta,\Big(\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Big)^{-1}(\nabla T_k(r^\eta)-\nabla_z b_\eta(z,w^\eta)+g^\eta)\Big)(\nabla T_k(r^\eta)-\nabla_z b_\eta(z,w^\eta)+g^\eta)dzdt$$
$$=\int_{Q_T}\Phi\Big(z,t,w,\Big(\frac{\partial b(z,w)}{\partial s}\Big)^{-1}(\nabla T_k(r)-\nabla_z b(z,w)+g)\Big)(\nabla T_k(r)-\nabla_z b(z,w)+g)dzdt+\ell(\eta). \tag{4.76}$$

Hence, $\Phi\Big(z,t,w^\eta,\Big(\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Big)^{-1}(\nabla T_k(r^\eta)-\nabla_z b_\eta(z,w^\eta)+g^\eta)\Big)(\nabla T_k(r^\eta)-\nabla_z b_\eta(z,w^\eta)+g^\eta)$ converges strongly to $\Phi\Big(z,t,w,\Big(\frac{\partial b(z,w)}{\partial s}\Big)^{-1}(\nabla T_k(r)-\nabla_z b(z,w)+g)\Big)(\nabla T_k(r)-\nabla_z b(z,w)+g)$ in $L^1(Q_T)$, then by coercivity argument

$$\min\Big\{(\frac{\alpha}{\gamma})^{p^- -1};(\frac{\alpha}{\gamma})^{p^+ -1}\Big\}\Big|\nabla T_k(r^\eta)+(g^\eta-\nabla_z b_\eta(z,w^\eta))\Big|^{p(z)}$$
$$\leq\Phi\Big(z,t,w^\eta,\Big(\frac{\partial b_\eta(z,w^\eta)}{\partial s}\Big)^{-1}(\nabla T_k(r^\eta)-\nabla_z b_\eta(z,w^\eta)+g^\eta)\Big)$$

almost everywhere in Q_T, since g^η converges strongly to g in $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$ and using Vitali's theorem, we conclude that

$$T_k(r^\eta)\to T_k(r) \text{ strongly in } L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)).$$

This implies the proof of Theorem 4.2. □

Step 6: Assume that k be a positive real number and let S be a function in $W^{2,\infty}(\mathbb{R})$ where S' has compact support such that $supp(S') \subset [-k,k]$ and $\varphi \in \mathcal{C}_c^\infty(Q_T)$.
Let w satisfies (3.10), (3.13), (3.11) and (3.12), if we take $S'(r^\eta)\varphi$ as test function in $\big(\mathcal{P}_\eta\big)$, we get

$$\begin{aligned}\int_0^T \Big\langle \varphi_t, S(r^\eta)\Big\rangle dt + \int_{Q_T} S'(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla\varphi dzdt + \int_{Q_T} S''(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta \varphi dzdt \quad (4.77)\\ = \int_{Q_T} f^\eta S'(r^\eta)\varphi dzdt + \int_{Q_T} F^\eta S'(r^\eta)\nabla\varphi dzdt + \int_{Q_T} S''(r^\eta)F^\eta\nabla r^\eta\varphi dzdt + \int_{Q_T} S'(r^\eta)\varphi d\lambda_+^\eta\\ - \int_{Q_T} S'(r^\eta)\varphi d\lambda_-^\eta,\end{aligned}$$

with $r^\eta = b(z,w^\eta) - g^\eta$. Due to Theorem 4.2, we can pass to the limit in all terms of (4.77), when $\eta \to 0$ except the last two terms which pose some complications, we can write according to the arguments of [39] that

$$\int_{Q_T} S'(r^\eta)\varphi d\lambda_+^\eta = \int_{Q_T} S'(r^\eta)\varphi\psi_\delta^+ d\lambda_+^\eta + \int_{Q_T} S'(r^\eta)\varphi(1-\psi_\delta^+)d\lambda_+^\eta. \qquad (4.78)$$

Note that ψ_δ^+ is defined the same way as in Lemma 4.1, then we get

$$\Big|\int_{Q_T} S'(r^\eta)\varphi(1-\psi_\delta^+)d\lambda_+^\eta\Big| \le C\int_{Q_T}(1-\psi_\delta^+)d\lambda_+^\eta = \ell(\eta,\delta),$$

by taking $S'(r^\eta)\varphi\psi_\delta^+$ in $\big(\mathcal{P}_\eta\big)$, we obtain

$$\begin{aligned}\int_{Q_T} S'(r^\eta)\varphi\psi_\delta^+ d\lambda_+^\eta &= -\int_{Q_T} f^\eta S'(r^\eta)\varphi\psi_\delta^+ dzdt - \int_{Q_T} F^\eta S'(r^\eta)\nabla(\varphi\psi_\delta^+)dzdt\\ &- \int_{Q_T} F^\eta S''(r^\eta)\nabla r^\eta\varphi\psi_\delta^+ dzdt + \int_{Q_T} S'(r^\eta)\varphi\psi_\delta^+ d\lambda_-^\eta - \int_{Q_T} S(r^\eta)(\varphi\psi_\delta^+)_\eta dzdt\\ &+ \int_{Q_T} S'(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla(\varphi\psi_\delta^+)dzdt + \int_{Q_T} S''(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta\psi_\delta^+\varphi dzdt.\end{aligned}$$

Thanks to (4.2)-(4.3) and properties of ψ_δ^+, we get

$$\int_{Q_T} f^\eta S'(r^\eta)\varphi\psi_\delta^+ dzdt = \ell(\eta,\delta), \ \int_{Q_T} F^\eta S'(r^\eta)\nabla(\varphi\psi_\delta^+)dzdt = \ell(\eta,\delta).$$

By Lemma 4.1, we have

$$\Big|\int_{Q_T} S'(r^\eta)\varphi\psi_\delta^+ d\lambda_-^\eta\Big| \le C\int_{Q_T}\psi_\delta^+ d\lambda_-^\eta = \ell(\eta,\delta),$$

as $S(r) \in L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)) \cap L^\infty(Q_T)$, we obtain $\int_{Q_T} S(r^\eta)(\varphi)\psi_\delta^+)_t dzdt = \ell(\eta,\delta)$.
According to Theorem 4.2 and Lemma 4.1, we infer $\displaystyle\int_{Q_T} S'(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla(\psi_\delta^+\varphi)\ dzdt = \ell(\eta,\delta)$, and $\int_{Q_T} S''(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta\psi_\delta^+\varphi\ dzdt = \ell(\eta,\delta)$.
Next, using (4) we conclude

$$\int_{Q_T} S'(r^\eta)\varphi\ d\lambda_+^\eta = \ell(\eta). \qquad (4.79)$$

In the same way, we can also show that

$$\int_{Q_T} S'(r^\eta)\varphi \, d\lambda_-^\eta = \ell(\eta). \tag{4.80}$$

Due to the convergence findings given previously, one can pass to the limit as η tends to zero in (4.77) and to deduced that w satisfies (3.10). It is now necessary to show that $S(r)$ satisfies (3.13). In the beginning, it is crucial to remember that because we have $S(b_\eta(z,w^\eta)-g^\eta)\to S(b(z,w)-g)$ a.e. in Q_T, and

$$S(b_\eta(z,w^\eta)-g^\eta) \text{ is bounded in } L^{p^-}(0,T,W_0^{1,p(z)}(\Omega))\cap L^\infty(Q_T), \tag{4.81}$$

secondly, we consider the convergence of the terms of (4.77), we deduce that

$$\frac{\partial S(b_\eta(z,w^\eta)-g^\eta)}{\partial t} \text{ is bounded in } L^1(Q_T)+L^{(p')^-}(0,T,W^{-1,p'(z)}(\Omega)). \tag{4.82}$$

Thanks to (4.81)-(4.82), and by Aubin's type lemma we conclude (see e.g., [41,44], the proof of this Corollary is identical to the corresponding result in the case of a constant exponent p) that $S(b_\eta(z,w^\eta)-g^\eta)$ is in a compact set of $\mathcal{C}([0,T];L^1(\Omega))$. On the one hand, we have $S(b_\eta(z,w^\eta)-g^\eta)(t=0)$ converges to $S(b(z,w)-g)(t=0)$ in $L^1(\Omega)$. On the other side, the smoothness of S give that $S(b_\eta(z,w^\eta)-g^\eta)(t=0)=S((b_\eta(z,w_0^\eta))$ converges strongly to $S(b(z,w_0))$ in $L^q(Q_T)$ for all $q<+\infty$, according to (4.8), we deduced that (3.13) is true.
Now, we consider $\beta_m(r^\eta)$ as test function in $\left(\mathcal{P}_\eta\right)$ where $\varphi\in\mathcal{C}_c^\infty(Q_T)$, we obtain

$$\begin{aligned}
-\int_0^T \left\langle \varphi_t\,;\,\overline{\beta}_m(r^\eta)\right\rangle dt+\int_{Q_T}\beta_m(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla\varphi dzdt+\frac{1}{m}\int_{\{m\le r^\eta<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta\varphi dzdt\\
=\int_{Q_T} f^\eta\beta_m(r^\eta)\varphi dzdt-\int_0^T\left\langle div(F^\eta)\,;\,\beta_m(r^\eta)\varphi\right\rangle dzdt\\
+\int_{Q_T}\beta_m(r^\eta)\varphi\, d\lambda_+^\eta-\int_{Q_T}\beta_m(r^\eta)\varphi d\lambda_-^\eta.
\end{aligned} \tag{4.83}$$

We reason as before, in particular as in the proof of Lemma 4.2 to obtain

$$\int_0^T\left\langle\varphi_t\,;\,\overline{\beta}_m(r^\eta)\right\rangle dt=\ell(\eta,m),\qquad \int_{Q_T}\beta_m(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla\varphi dzdt=\ell(\eta,m),$$

and

$$\int_{Q_T} f^\eta\beta_m(r^\eta)\varphi dzdt=\ell(\eta,m),\int_0^T\left\langle div(F^\eta);\beta_m(r^\eta)\varphi\right\rangle dzdt=\ell(\eta,m).$$

In light of the Theorem 4.2 we get

$$\frac{1}{m}\int_{\{m\le r^\eta<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta\varphi dzdt \quad = \quad \frac{1}{m}\int_{\{m\le r<2m\}}\Phi(z,t,w,\nabla w)\nabla r\varphi dzr \quad + \quad \ell(\eta).$$

We now examine the last two terms on the right side of (4.83), for this we have

$$\int_{Q_T}\beta_m(r^\eta)\varphi d\lambda_+^\eta=\int_{Q_T}\phi_m(r^\eta)d\lambda_+^\eta+\int_{Q_T}\varphi d\lambda_+^\eta,$$

where $\phi_m(s)=H_m(s^+)$. By construction of λ_+^η, we have

$$\int_{Q_T}\varphi d\lambda_+^\eta=\int_{Q_T}\varphi d\lambda_s^+ +\ell(\eta).$$

Adopting the same reasoning as in (4.77) -(4.78) by considering $\phi_m(r^\eta)=S'(r^\eta)$ we get

$$\int_{Q_T}\phi_m(r^\eta)\varphi d\lambda_+^\eta=\ell(\eta).$$

If we can show that

$$\int_{Q_T}\beta_m(r^\eta)\varphi\ d\lambda_-^\eta=\ell(\eta), \tag{4.84}$$

thus, we get for each $\varphi\in\mathcal{C}_c^\infty(Q_T)$

$$\lim_{m\to\infty}\frac{1}{m}\int_{\{m\leq r<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r\varphi dzdt=\int_{Q_T}\varphi d\lambda_s^+. \tag{4.85}$$

Therefore, we have

$$\int_{Q_T}\beta_m(r^\eta)\varphi d\lambda_-^\eta=\int_{Q_T}\beta_m(r^\eta)\varphi\psi_\delta^- d\lambda_-^\eta+\int_{Q_T}\beta_m(r^\eta)\varphi(1-\psi_\delta^-)d\lambda_-^\eta,$$

thanks to Lemma 4.1, we get

$$\int_{Q_T}\beta_m(r^\eta)\varphi(1-\psi_\delta^-)\ d\lambda_-^\eta=\ell(\eta,\delta).$$

Using $\beta_m(r^\eta)\varphi\psi_\delta^-$ as test function in the formulation of w^η, we obtain

$$\begin{aligned}\int_{Q_T}\beta_m(r^\eta)\varphi\psi_\delta^- d\lambda_-^\eta&=\int_0^T\Big\langle(\varphi\psi_\delta^-)_t;\overline{\beta}_m(r^\eta)\Big\rangle dt-\int_{Q_T}\beta_m(r^\eta)\Phi(z,t,w^\eta,\nabla w^\eta)\nabla(\varphi\psi_\delta^-)_t)dzdt\\&-\frac{1}{m}\int_{\{m\leq r^\eta<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta\varphi\psi_\delta^- dzdt+\int_{Q_T}f^\eta\beta_m(r^\eta)\varphi\psi_\delta^- dzdt\\&+\int_{Q_T}F^\eta\beta_m(r^\eta)\nabla(\varphi\psi_\delta^-)_t)dzdt+\frac{1}{m}\int_{\{m\leq r^\eta<2m\}}F^\eta\nabla r^\eta\varphi\psi_\delta^- dzdt+\int_{Q_T}\beta_m(r^\eta)\varphi\psi_\delta^- d\lambda_+^\eta.\end{aligned}$$

By (4.20)-(4.21), Lemmas 4.1, 4.2 and the famous Lebesgue's convergence Theorem lead to (4.84). Hence, we have (4.85) for each $\varphi\in\mathcal{C}_c^\infty(Q_T)$.
If $\varphi\in\mathcal{C}^\infty(\overline{Q_T})$, we can separate

$$\begin{aligned}\frac{1}{m}\int_{\{m\leq r<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r\varphi dzdt&=\frac{1}{m}\int_{\{m\leq r<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r\varphi\psi_\delta^+ dxdt\\&+\frac{1}{m}\int_{\{m\leq r<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r\varphi(1-\psi_\delta^+)dzdt,\end{aligned} \tag{4.86}$$

from (4.85), we obtain

$$\lim_{m\to\infty}\frac{1}{m}\int_{\{m\leq r<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r\varphi\psi_\delta^+ dzdt=\int_{Q_T}\varphi d\mu_s^++\ell(\delta),$$

using Lemma 4.2, we obtain

$$\frac{1}{m}\int_{\{m\leq r^\eta<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r^\eta\varphi(1-\psi_\delta^+)dzdt=\ell(\eta,m,\delta).$$

Due to Theorem 4.2, we conclude that

$$\frac{1}{m}\int_{\{m\leq r<2m\}}\Phi(z,t,w^\eta,\nabla w^\eta)\nabla r\varphi(1-\psi_\delta^+)dzdt=\ell(m,\delta).$$

By taking all of the previous factors into account, we have (3.11) for all
$\varphi\in\mathcal{C}^\infty(\overline{Q_T})$ and using a density argument (3.11) holds for all $\varphi\in\mathcal{C}(\overline{Q_T})$. To obtain (3.12) we can proceed as before using ψ_δ^+ instead of ψ_δ^- and the other way around, which completes the proof of Theorem 4.1.

Declarations

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The data used to support the findings of this study are included in the references within the article.

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