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# Almost Ricci-Yamabe Solitons in $f$-Kenmotsu Manifolds

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ABSTRACT: The purpose of this paper is to charecterize $f$-Kenmotsu manifolds admitting almost Ricci-Yamabe soliton and gradient Ricci-Yamabe soliton. We deduce the ncessary condition for the potential function $u$ is constant. Further, a relation between $\lambda$ and the potential function $u$ has been established. Finally, a sufficient condition is proved for a Ricci-Yamabe soliton to be a gradient Ricci-Yamabe soliton and a characterization of the soliton in terms of shrinking, steady or expanding has been done.

Key Words: $f$-Kenmotsu manifold, Ricci-Yamabe soliton, Gradient Ricci-Yamabe soliton.

**Contents**



## 1. Introduction

In 1972, Kenmotsu [8] introduced a contact metric manifold, later called the Kenmotsu manifold. A study on $f$-Kenmotsu manifold (an almost contact metric manifold which is normal and locally conformal almost cosymplectic) was carried by Olszaka and Rosca [13]. The two concepts - Ricci flow(RF) and Yamabe flow(YF) were introduced in the year 1988 by Hamilton [6]. The Ricci soliton and Yamabe soliton, respectively, emerge as the limits of the solutions of the RF and YF. RF and YF have been deliberated by many geometers (See [1,9,10]). Guler and Crasmareanu have studied Ricci-Yamabe flow(RYF) [5] and many researchers have worked on the same (See [2,15,17]). Some related developments can be found in [3,4,11,12,18-26].

For the metrics on the Riemannian manifold $\mathcal{M}$, the R-YF is defined as

$$\frac{\partial}{\partial \mathfrak{t}} g(\mathfrak{t}) = -2p\ Ric(\mathfrak{t}) + q\, r(\mathfrak{t}) g(\mathfrak{t}),\ g_0 = g(0). \tag{1.1}$$

A soliton to the RYF is called Ricci-Yamabe soliton ($RYS$) and it is precised on Riemannain manifold $(g, V, \lambda, p, q)$ satisfying the equation

$$\mathfrak{L}_V g + 2p\ S + (2\lambda - qr) g = 0, \tag{1.2}$$

where $S$ is the Ricci tensor, $r$ is the scalar curvature, $\mathfrak{L}_V$ is the Lie derivative along the vector field and $p, q$ are scalars. We call $(\mathcal{M}, g)$ as $RYS$ expanding, $RYS$ shrinking and $RYS$ steady, respectively, if $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$. Eq.(1.2) as $RYS$ of type $(p, q)$ is said to be $p$-Ricci soliton and $q$-Yamabe soliton, respectively, when $q = 0$ and $p = 0$.

If $V$ is a gradient of a smooth function $u$ on the manifold $\mathcal{M}$, then the above notion is called gradient Ricci-Yamabe solition (GRYS) and then Eq.(1.2) reduces to

$$Hess^u(U_1, U_2) + pS(U_1, U_2) + (\lambda - \frac{1}{2} qr) g(U_1, U_2) = 0, \tag{1.3}$$

where $Hess^u$ is the Hessian operator of $u$. A RYS (or GRYS) is said to be an almost RYS (or GRYS) if $\lambda$, $p$ and $q$ are smooth functions on $\mathcal{M}$.

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## 2. Basics

A smooth manifold $\mathcal{M}$ of odd dimension is an almost contact metric manifold, if there exist a vector field $\varsigma$, an $(1,1)$ tensor field $\phi$, a 1-form $\eta$ and a Riemannian metric $g$. Here $\varsigma$ is the characteristic vector field or the Reeb vector field so that

$$\phi^2 = -I + \eta \otimes \varsigma, \quad \eta(\varsigma) = 1, \quad \phi\varsigma = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

Every almost contact structure $(\phi, \varsigma, \eta)$ on an odd dimensional manifold $\mathcal{M}$ admits a Riemannian metric $g$ satisfying

$$\eta(U_1) = g(U_1, \varsigma), \qquad g(\phi U_1, \phi U_2) = g(U_1, U_2) - \eta(U_1)\eta(U_2). \tag{2.2}$$

An almost contact manifold $\mathcal{M}$ together with the almost contact structure $(\phi, \varsigma, \eta)$ is said to be an $f$-Kenmotsu manifold [$f$-KM] or a normal contact metric manifold if

$$[\phi, \phi](U_1, U_2) + 2d\eta(U_1, U_2)\varsigma = 0, \tag{2.3}$$

where $[\phi, \phi]$ represents the Nijenhuis torsion tensor field of $\phi$ and is given by

$$[\phi, \phi](U_1, U_2) = \phi^2[U_1, U_2] + [\phi U_1, \phi U_2] - \phi([\phi U_1, U_2]) - \phi([U_1, \phi U_2]). \tag{2.4}$$

An almost contact metric manifold of odd dimension with stucture $(\phi, \varsigma, \eta)$ is known as an $f$-KM if

$$(\nabla_{U_1}\phi)U_2 = f[g(\phi U_1, U_2)\varsigma - \eta(U_2)\phi U_1], \tag{2.5}$$

where $\nabla$ is the Levi-Civita connection on $\mathcal{M}$ and $f \in C^\infty(\mathcal{M})$ is such that $df \Lambda \eta = 0$ (see [8,16]). An $f$-KM is

1. $\beta$-Kenmotsu manifold [7] when $f = \beta \neq 0$,
2. Kenmotsu manifold [8,14] when $f = 1$,
3. cosymplectic [7] when $f = 0$,
4. regular when $f^2 + f' \neq 0$, where $f' = \varsigma f$.

For an $f$-KM, from (2.1) we have

$$\nabla_{U_1}\varsigma = f[U_1 - \eta(U_1)\varsigma]. \tag{2.6}$$

The condition $df \Lambda \eta = 0$ holds if the dimension of $\mathcal{M}$ is $\geq 5$ and does not hold if dimension of $\mathcal{M}$ is $= 3$ (see [13]).
In an $f$-KM of dimension 3, we have

$$\begin{aligned} R(U_1, U_2)U_3 =& \left(\frac{r}{2} + 2f^2 + 2f'\right)(U_1 \Lambda U_2)U_3 \\ &- \left(\frac{r}{2} + 3f^2 + 3f'\right)[\eta(U_1)(\varsigma \Lambda U_2)U_3 + \eta(U_2)(U_1 \Lambda \varsigma)U_3], \end{aligned} \tag{2.7}$$

$$R(U_2, \varsigma)U_3 = -(f^2 + f')[\eta(U_3)U_2 - g(U_2, U_3)\varsigma], \tag{2.8}$$

$$S(U_1, U_2) = \left(\frac{r}{2} + f^2 + f'\right) g(U_1, U_2) - \left(\frac{r}{2} + 3f^2 + 3f'\right) \eta(U_1)\eta(U_2), \tag{2.9}$$

$$QU_1 = \left(\frac{r}{2} + f^2 + f'\right) U_1 - \left(\frac{r}{2} + 3f^2 + 3f'\right) \eta(U_1).\varsigma \tag{2.10}$$

## 3. Main Results

Let us consider the distribution $D \subset T\mathcal{M}$ defined by

$$\eta(U_2) = g(U_2, \varsigma) = 0. \tag{3.1}$$

The distribution $D$ is nowhere integrable and for any $U_2 \in T\mathcal{M}$. We have $\phi(U_2) = \nabla_{U_2}\varsigma \in D$, since $\varsigma$ is a Killing vector field. Taking the inner product with respect $U_4$, on both side of the Eq.(2.8), we have

$$R(U_2, \varsigma, U_3, U_4) = -(f^2 + f')[\eta(U_3)g(U_2, U_2) - g(U_2, U_3)g(\varsigma, U_4)]. \tag{3.2}$$

Using the distribution condition in (3.1), this reduces to

$$R(U_2, \varsigma, U_3, U_4) = -(f^2 + f')[\eta(U_3)g(U_2, U_4)]. \tag{3.3}$$

In view of (2.2) and (3.3), we get

$$R(U_2, \varsigma, U_3, U_4) = -(f^2 + f')[g(U_3, \varsigma)g(U_2, U_4)]. \tag{3.4}$$

Now, choosing an orthonormal basis vector for $U_2 = U_4 = e_i$ and taking summation over the index $i$, we obtain

$$S(\varsigma, U_3) = -(f^2 + f')2ng(U_3, \varsigma). \tag{3.5}$$

Consider an almost $RYS$ admitted on $f$-KM. From (1.2), we have

$$(\mathfrak{L}_{V_1}g)(\varsigma, U_4) + 2pS(\varsigma, U_4) + (2\lambda - qr)g(\varsigma, U_4) = 0. \tag{3.6}$$

Substituting the value of $S$ from equation (3.5), we obtain

$$(\mathfrak{L}_{V_1}g)(\varsigma, U_4) + [-4np(f^2 + f') + (2\lambda - qr)]g(\varsigma, U_4) = 0. \tag{3.7}$$

This implies that

$$g(\nabla_\varsigma V_1, U_4) + g(\varsigma, \nabla_{U_4}V_1) = [4np(f^2 + f') + (qr - 2\lambda)]g(\varsigma, U_4). \tag{3.8}$$

Setting $U_4 = \varsigma$ in the above equation and simplifying, we obtain

$$g(\nabla_\varsigma V_1, \varsigma) = [4np(f^2 + f') + (qr - 2\lambda)]g(\varsigma, \varsigma). \tag{3.9}$$

Again, $\varsigma$ being a Killing vector field, we have

$$(\mathfrak{L}_\varsigma(\mathfrak{L}_{V_1}g))(U_2, U_3) = 0. \tag{3.10}$$

This implies that,

$$(\mathfrak{L}_\varsigma(\mathfrak{L}_{U_1}g))(U_2, U_3) - \mathfrak{L}_{U_1}g([\varsigma, U_2], U_3) - \mathfrak{L}_{U_1}g(U_2, [\varsigma, U_3]) = 0. \tag{3.11}$$

Using the condition $(\nabla_{U_1}g)(\varsigma, U_3) = 0$ and simplifying, we obtain

$$R(U_1, \varsigma, \varsigma, U_2) + \nabla_{U_2}g(\nabla_\varsigma U_1, \varsigma) + g(\nabla_\varsigma\nabla_\varsigma U_1, U_2) = 0. \tag{3.12}$$

As the vector field $U_2$ is orthogonal to $\varsigma$, in view of (3.1) and (3.5), we have

$$\begin{aligned} R(U_1, \varsigma, \varsigma, U_2) &= -(f^2 + f')g(\varsigma, \varsigma)g(U_1, U_2) - g(U_1, \varsigma)g(\varsigma, U_2) \\ &= -(f^2 + f')g(U_1, U_2). \end{aligned} \tag{3.13}$$

Substituting (3.13) in (3.12), we get

$$-(f^2 + f')g(U_1, U_2) + \nabla_{U_2}g(\nabla_\varsigma U_1, \varsigma) + g(\nabla_\varsigma\nabla_\varsigma U_1, U_2) = 0. \tag{3.14}$$

As well from (3.9) using the value of $g(\nabla_\varsigma U_1, \varsigma)$ then we get

$$\begin{aligned} -(f^2+f')g(U_1,U_2) + \nabla_{U_2}[4np(f^2{+}f') + (qr-2\lambda)] \\ + g(\nabla_\varsigma\nabla_\varsigma U_1, U_2) = 0. \end{aligned} \tag{3.15}$$

Again, almost RYS being GRYS, taking $U_1 = \nabla u$, (3.15) becomes

$$\begin{aligned} -(f^2+f')g(\nabla u,U_2) + \nabla_{U_2}[4np(f^2{+}f') + (qr-2\lambda)] \\ + g(\nabla_\varsigma\nabla_\varsigma U_1, U_2) = 0. \end{aligned} \tag{3.16}$$

From (3.9), we observe that

$$\nabla_\varsigma U_1 = [4np(f^2+f') + (qr-2\lambda)]\varsigma$$

and using the GRYS condition $U_1 = \nabla u$, we have

$$\nabla_\varsigma \nabla u = [4np(f^2+f') + (qr-2\lambda)]\varsigma. \tag{3.17}$$

Using (3.17) in (3.16), we obtain

$$\begin{aligned} -(f^2+f')g(\nabla_u,U_2) + g(\nabla_\varsigma[4np(f^2+f') + (qr-2\lambda)\varsigma], U_2) \\ + \nabla_{U_2}[4np(f^2+f') + (qr-2\lambda)] = 0. \end{aligned} \tag{3.18}$$

Straight forward calculation yields

$$\begin{aligned} -(f^2+f')g(\nabla_u,U_2) + \varsigma[4np(f^2+f') + (qr-2\lambda)g(U_2,\varsigma) \\ + \nabla_{U_2}[4np(f^2+f') + (qr-2\lambda)] = 0. \end{aligned} \tag{3.19}$$

for any vector field $U_2$ in the distribution $D \subset T\mathcal{M}$. Thus, the result is true for all vector field $U_2$ satisfying $g(U_2,\varsigma) = 0$. Hence (3.19) becomes

$$(f^2+f')g(\nabla_u,U_2) = \nabla_{U_2}[4np(f^2+f') + (qr-2\lambda)]. \tag{3.20}$$

If we consider $\lambda = 2np(f^2+f') + \frac{qr}{2}$, then we get $g(\nabla_u,U_2) = 0$, $\forall U_2 \in D \subset T\mathcal{M}$, where $(f^2+f') \neq 0$, i.e., the $f$-KM is regular. Hence we conclude that the potential function $u$ is constant. Thus, we have the following theorem:

**Theorem 3.1** *If the manifold $\mathcal{M}$ is an $f$-KM satisfying the almost $GRYS$ and $D \subset T\mathcal{M}$, then the potential function $u$ is a constant when $\mathcal{M}$ is regular and $\lambda = 2np(f^2+f') + \frac{qr}{2}$.*

The above theorem immediately leads to the following corollary:

**Corollary 3.1** *If the manifold $\mathcal{M}$ is a regular $f$-KM satisfying the almost $GRYS$ with the potential function $u$ is a constant and $D \subset T\mathcal{M}$, then the $RYS$ is*

1. *expanding when $2np(f^2+f') + \frac{qr}{2} > 0$,*
2. *shrinking when $2np(f^2+f') + \frac{qr}{2} < 0$,*
3. *steady when $2np(f^2+f') + \frac{qr}{2} = 0$.*

For a manifold $\mathcal{M}$ satisfying the almost $GRYS$, from (1.3) we can write

$$\nabla_{U_1} Du = -pQU_1 + [\frac{qr}{2} - \lambda]U_1, \tag{3.21}$$

where $Q$ is the Ricci operator such that $g(QU_1,U_2) = S(U_1,U_2)$. Differentiating the above equation covariantly along any vector field $U_2$, yields

$$\begin{aligned} \nabla_{U_2}\nabla_{U_1} Du = -p[(\nabla_{U_2}Q)U_1 + Q(\nabla_{U_2}U_1)] + [\frac{q}{2}(U_2 r) - (U_2\lambda)]U_1 \\ + [\frac{qr}{2} - \lambda]\nabla_{U_2}U_1. \end{aligned} \tag{3.22}$$

Interchanging $U_1$ and $U_2$ in (3.22), we have

$$\begin{aligned}\nabla_{U_1}\nabla_{U_2}Du = -p[(\nabla_{U_1}Q)U_2 + Q(\nabla_{U_1}U_2)] + [\frac{q}{2}(U_1 r) - (U_1\lambda)]U_2 \\ + [\frac{qr}{2} - \lambda]\nabla_{U_1}U_2.\end{aligned} \tag{3.23}$$

Also, replacing $U_1$ by $[U_1, U_2]$ in (3.21), we obtain

$$\nabla_{[U_1,U_2]}Du = -pQ[U_1, U_2] + [\frac{qr}{2} - \lambda][U_1, U_2], \tag{3.24}$$

where $[U_1, U_2]$ denotes the Lie bracket operation. Now, using (3.22)-(3.24) in

$$R(U_1, U_2)Du = \nabla_{U_1}\nabla_{U_2}Du - \nabla_{U_2}\nabla_{U_1}Du - \nabla_{[U_1,U_2]}Du, \tag{3.25}$$

we have

$$\begin{aligned}R(U_1, U_2)Du = &- p[(\nabla_{U_1}Q)U_2 - (\nabla_{U_2}Q)U_1] + \frac{q}{2}[\nabla_{U_1}r(U_2) - \nabla_{U_2}r(U_1)] \\ &+ (U_2\lambda)U_1 - (U_1\lambda)U_2.\end{aligned} \tag{3.26}$$

Thus, we have the following Theorem:

**Theorem 3.2** *If the manifold $\mathcal{M}$ is an $f$-KM satisfying almost GRYS, then for the curvature tensor of $\mathcal{M}$ the following relation holds:*

$$\begin{aligned}R(U_1, U_2)Du = &- p[(\nabla_{U_1}Q)U_2 - (\nabla_{U_2}Q)U_1] + \frac{q}{2}[\nabla_{U_1}r(U_2) - \nabla_{U_2}r(U_1)] \\ &+ (U_2\lambda)U_1 - (U_1\lambda)U_2.\end{aligned}$$

Replace $U_1 = \varsigma$ in (3.26) we have

$$R(\varsigma, U_2)Du = -p[(\nabla_\varsigma Q)U_2 - (\nabla_{U_2}Q)\varsigma] + (U_2\lambda)\varsigma - (\nabla_\varsigma\lambda)U_2 \tag{3.27}$$

Taking inner product with $\varsigma$ to the above equation, we obtain

$$g(R(\varsigma, U_2)Du, \varsigma) = (U_2\lambda) - (\varsigma\lambda)\eta(U_2) \tag{3.28}$$

From (2.8), we have

$$g(R(\varsigma, U_2)Du, \varsigma) = -(f^2 + f')[\eta(U_2)(\varsigma u) - U_2(u)] \tag{3.29}$$

In view of (3.28) and (3.29), we have

$$(U_2\lambda) - (\varsigma\lambda)\eta(U_2) = -(f^2 + f')[\eta(U_2)(\varsigma u) - U_2(u)] \tag{3.30}$$

We have the following:

- If $(f^2 + f') = 0$, then from (2.8) we have $R(U_1, U_2)\varsigma = 0$ and therefore the manifold is flat.
- If $U_2 u = \eta(U_2)(\varsigma u)$, then we have $Du = (\varsigma u)\varsigma$. Substituting this to (3.21), we get

$$QU_1 = \frac{1}{p}\left\{[\frac{qr}{2} - \lambda - (\varsigma u)f]U_1 - [f(\varsigma u)\eta(U_1) - U_1(\varsigma u)]\varsigma\right\} \tag{3.31}$$

Comparing (2.10) and (3.31), we obtain

$$\frac{r}{2} + f^2 + f' = \frac{1}{p}\left[\frac{qr}{2} - \lambda - (\varsigma u)f\right] \tag{3.32}$$

and

$$\left[\frac{r}{2} + 3f^2 + 3f'\right]\eta(U_1) = [f(\varsigma u)\eta(U_1) - U_1(\varsigma u)] \tag{3.33}$$

From (3.33), it follows that $f(\varsigma u) = \frac{r}{2} + 3f^2 + 3f'$ and $U_1(\varsigma u) = 0$. Therefore $\varsigma u$ is a constant. In this case potential vector field $v$ being gradient of $u$ becomes a null vector.

Thus, we have the following theorem:

**Theorem 3.3** *If the manifold $\mathcal{M}$ is an $f$-KM satisfying almost GRYS, then the manifold is flat and $\varsigma u$ is a constant.*

From (3.32), we have $\lambda = p[\frac{r}{2} + f^2 + f'] + \frac{qr}{2} + (\varsigma u)f$ and we can state the following result:

**Theorem 3.4** *If the manifold $\mathcal{M}$ is an $f$-KM satisfying GRYS, then the soliton is*

1. *expanding when* $p[\frac{r}{2} + f^2 + f'] + \frac{qr}{2} + (\varsigma u)f > 0$,
2. *shrinking when* $p[\frac{r}{2} + f^2 + f'] + \frac{qr}{2} + (\varsigma u)f < 0$,
3. *steady when* $p[\frac{r}{2} + f^2 + f'] + \frac{qr}{2} + (\varsigma u)f = 0$.

**Definition 3.1** *A vector field $V$ on a Riemannian manifold $\mathcal{M}$ is said to be a concurrent vector field, if it satisfies*

$$\nabla_{U_1} V = U_1,$$

*for all smooth vector fields $U_1$ on $\mathcal{M}$.*

Let us consider an $f$-KM $\mathcal{M}$ admitting almost $RYS$ with concurrent potential vector field $V$. Then we have

$$(\mathfrak{L}_V g)(U_1, U_2) + 2pS(U_1, U_2) + (2\lambda - qr)g(U_1, U_2) = 0, \tag{3.34}$$

But

$$\begin{aligned}(\mathfrak{L}_V g)(U_1, U_2) &= g(\nabla_{U_1} V, U_2) + g(U_1, \nabla_V U_2) \\ &= 2g(U_1, U_2).\end{aligned} \tag{3.35}$$

Therefore, (3.34) can be written as

$$S(U_1, U_2) = \left[\frac{qr - 2\lambda - 2}{2p}\right] g(U_1, U_2) \tag{3.36}$$

This proves that the manifold is Einstein. Thus we have the following result:

**Theorem 3.5** *If the manifold $\mathcal{M}$ is an $f$-KM admitting an almost RYS with concurrent potential vector field $V$, then the manifold is an Einstein manifold.*

Comparing (2.9) with (3.36) and taking $U_1 = U_2 = \varsigma$, we have

$$\lambda = 2p(f^2 + f') + \frac{qr}{2} - 1 \tag{3.37}$$

This leads to the following result:

**Theorem 3.6** *If the manifold $\mathcal{M}$ is an $f$-KM admitting an almost RYS with concurrent potential vector field $V$, then the RYS is*

1. *expanding when* $2p(f^2 + f') + \frac{qr}{2} - 1 > 0$,
2. *shrinking when* $2p(f^2 + f') + \frac{qr}{2} - 1 < 0$,
3. *steady when* $2p(f^2 + f') + \frac{qr}{2} - 1 = 0$.

Putting $p = 0$ in (3.37), $\lambda = \frac{qr}{2} - 1$ and we have the following corollary to the above theorem:

**Corollary 3.2** *If the manifold $\mathcal{M}$ is an $f$-KM admitting an almost RYS with concurrent potential vector field $V$, then the $q$-Yamabe soliton is*

1. *expanding when* $\frac{qr}{2} - 1 > 0$,

2. *shrinking when* $\frac{qr}{2} - 1 < 0$,

3. *steady when* $\frac{qr}{2} - 1 = 0$.

Putting $q = 0$ in (3.37), $\lambda = 2p(f^2 + f') - 1$ and we have the following corollary

**Corollary 3.3** *If the manifold $\mathcal{M}$ is an $f$-KM admitting an almost RYS with concurrent potential vector field $V$, then the $p$-Ricci soliton is*

1. *expanding when* $2p(f^2 + f') - 1 > 0$,

2. *shrinking when* $2p(f^2 + f') - 1 < 0$,

3. *steady when* $2p(f^2 + f') - 1 = 0$.


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